Differential Galois Theory and Spectral Theory
(Work in Progress)

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(joint work with

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Motivation: Known shape invariant potentials in Quantum Mechanics

Potential	**Name**
$\frac{1}{2}m\omega^2\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2$	Shifted H. O.
$\frac{1}{2}m\omega^2r^2+\frac{l(l+1)\hbar^2}{2mr^2}-\left(l+\frac{3}{2}\right)\hbar\omega$	3D H.O.
$-\frac{e^2}{r}+\frac{l(l+1)\hbar^2}{2mr^2}-\frac{me^4}{2(l+1)^2\hbar^2}$	Coulomb
$A^2+B^2e^{-2ax}-2B\left(A+\frac{a\hbar}{2\sqrt{2m}}\right)e^{-ax}$	Morse 1
$A^2+\frac{B^2-A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\cosh^2 ax}+\frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\sinh ax}{\cosh^2 ax}$	Morse 2

$A^2 + \frac{B^2}{A^2} + 2B\tanh ax - A\frac{A+\frac{a\hbar}{\sqrt{2m}}}{\cosh^2 ax}$	Rosen-Morse 1
$A^2 + \frac{B^2+A^2+\frac{Aa\hbar}{\sqrt{2m}}}{\sinh^2 ar} - \frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\cosh ar}{\sinh^2 ar}$	Rosen-Morse 2
$A^2 + \frac{B^2}{A^2} - 2B\coth ar + A\frac{A-\frac{a\hbar}{\sqrt{2m}}}{\sinh^2 ar}$	Eckart 1
$-A^2 + \frac{B^2+A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\sin^2 ax} - \frac{B\left(2A-\frac{a\hbar}{\sqrt{2m}}\right)\cos ax}{\sin^2 ax}$	Eckart 2
$-(A+B)^2 + \frac{A\left(A-\frac{a\hbar}{\sqrt{2m}}\right)}{\cos^2 ax} + \frac{B\left(B-\frac{a\hbar}{\sqrt{2m}}\right)}{\sin^2 ax}$	Pöschl-Teller 1
$(A-B)^2 - \frac{A\left(A+\frac{a\hbar}{\sqrt{2m}}\right)}{\cosh^2 ar} + \frac{B\left(B-\frac{a\hbar}{\sqrt{2m}}\right)}{\sinh^2 ar}$	Pöschl-Teller 2

(R. Dutt, A. Khare, U.P. Sukhatme, Am.J.Phys. 56(1988)163-168)

Preliminaries and notations

The One-dimensional Stationary Schrödinger Equation (SSE) is given by

$$S_{(u,x)}\Psi = \lambda\Psi, \quad S_{(u,x)} = \frac{d^2}{dx^2} - u(x). \tag{1}$$

In quantum mechanics x is the **cartesian or radial coordinate** , Ψ is the **wave function**, the eigenvalue λ is the **energy** level, $u(x)$ is the **potential or potential energy** and the solutions of (1) are the **eigenfunctions** of the particle.

Notation. Denote by $\Lambda \subseteq \mathbb{C}$ the set of eigenvalues λ such that (1) is Picard-Vessiot integrable.

Galois groups of SSE

Denotes by Card(Λ) the cardinality of Λ and by $\mathbf{G}_\lambda = \mathbf{G}_\lambda(1)$ the Galois group (over $\mathbf{K}$) of SSE (1) for λ.

Remark. Λ can be $\emptyset$, i.e., $\mathbf{G}_\lambda = SL(2, \mathbb{C})$ $\forall \lambda \in \mathbb{C}$. On the other hand, if $\lambda_0 \in \Lambda$ then $\mathbf{G}_{\lambda_0}^0 \subseteq \mathbb{C}^* \ltimes \mathbb{C}^+$.

Examples. Given $\mathbf{K}$ defined as follows (see [1])

1. $\mathbf{K} = \mathbb{C}(x)$, if $u(x) = P_{2n+1}(x)$, then $\Lambda = \emptyset$.
2. $\mathbf{K} = \mathbb{C}(x)$, if $u(x) = P_{2n}(x)$, $n > 1$, then either, $\Lambda = \{\lambda_0\}$, with $\mathbf{G}_{\lambda_0} = \mathbb{C}^* \ltimes \mathbb{C}^+$ or $\Lambda = \emptyset$.
3. $\mathbf{K} = \mathbb{C}(e^{ix})$, if $u(x) = b \sin(x)$, $b \in \mathbb{C}^*$ then $\Lambda = \emptyset$.

Proposition 1. Given SSE (1) with $u(x) \in \mathbb{C}(z(x), z'(x))$. Let be $u(x) = g(z(x))$, $z = z(x)$, and $(z')^2 = \alpha(z)$, we can obtain $S_{(v,z)}\Psi = \lambda\Psi$, with $v(z) \in \mathbb{C}(z)$, provided

$$\alpha(z), \quad g(z) \in \mathbb{C}(z).$$

Remark. The above proposition is an adapted version to this context of the so-called *algebrization* procedure (it is taken from P. Acosta-Humanez & D. Blazquez-Sanz, *Non-Integrability of some hamiltonian systems with rational potential*, preprint 2006). In particular, using a result about the invariance of the identity component of the Galois group by changes of variable given by finite ramified coverings in the independent variable (M-R and Ramis, 2001), it is possible to prove that the identity component of the Galois group is preserved in the algebrization mechanism. Then

Proposition 2. Assume that the SSE has coefficient field $K = \mathbb{C}(z(x), z'(x))$ and it is possible to algebrize it (ie, the conditions of Proposition 1 are verified), then only one of the following cases are possible

- $\Lambda = \emptyset$
- $\mathrm{Card}\Lambda = 1$
- Λ is an infinite discrete set

- $\Lambda = \mathbb{C}$

In the quantum mechanics bound states context we are interested in the case of Λ a infinite discrete set. Furthermore if we apply *in a direct way* the algebrization procedure and the Kovacic algorithm to the shape invariants list of potentials, (ie , to the corresponding SSE equation) it is possible **to handle all of them** by means of the Picard-Vessiot theory and as a by-product to obtain the eigenfunctions and the corresponding energy spectrum Λ:

Proposition 3. The complete list of known SSE with shape invariant potentials is integrable in the Picard-Vessiot sense for some discrete energy spectrum.

Darboux Transformation (DT)

Theorem (Darboux 1887). Let be $\Lambda \neq \emptyset$, suppose that $\Psi_\lambda = \Psi_\lambda(x)$ is an eigenfunction of SSE (1) for $\lambda \in \Lambda$, $\Psi_{\lambda_0} = \Psi_{\lambda_0}(x)$ an eigenfunction for $\lambda_0 \in \Lambda$. Then we can obtain the SSE

$$S_{(v,x)}\Phi = \lambda\Phi, \quad v(x) = \Psi_{\lambda_0}\left(\frac{1}{\Psi_{\lambda_0}}\right)'' - \lambda_0. \qquad (2)$$

with the same Λ.

Furthermore, if $\lambda_1 \neq \lambda_0$ then $\Phi_{\lambda_1} = \Phi_{\lambda_1}(x)$ (eigenfunction of (2) of eigenvalue $\lambda_1 \in \Lambda$) is given by

$$\Phi_{\lambda_1} = \Psi'_{\lambda_1} - \frac{\Psi'_{\lambda_0}}{\Psi_{\lambda_0}}\Psi_{\lambda_1} = \left(\frac{d}{dx} - \frac{\Psi'_{\lambda_0}}{\Psi_{\lambda_0}}\right)\Psi_{\lambda_1}.$$

We observe that (factorization method in quantum mechanics: Schrödinger, Infeld-Hull,...):

1. If we denote $S_u := S_{(u,x)}$ y $S_v := S_{(v,x)}$, we have

$$S_u = A^+ A^-, \quad S_v = A^- A^+,$$

$$A^- = \frac{d}{dx} - \frac{\Psi'_{\lambda_0}}{\Psi_{\lambda_0}}, \quad A^+ = \frac{d}{dx} + \frac{\Psi'_{\lambda_0}}{\Psi_{\lambda_0}}$$

   (in order to obtain these factorizations we have to solve a Riccati equation associated to the SSE!)

2. $[A^+, A^-] = \frac{\Psi'_{\lambda_0}}{\Psi_{\lambda_0}}$

3. If Ψ_n is an eigenfunction of S_u of eigenvalue λ_n, then $A^+\Psi_n$ is an eigenfunction with eigenvalue λ_n.

Proposition 3. Let be Gal_0 the Galois group of the equation (1) for $\lambda_0 \in \Lambda$ and Gal_1 the Galois group of the DT equation (2) for $\lambda_1 \in \Lambda$. Then the identity components of these two groups are the same, provided we can algebrize them.

Shape-invariant potentials

Question: Is it possible by iteration of the DT to solve the espectral problem of the SSE, *solving the equation only for a eigenvalue*, say λ_0?

It is known that the answer is positive for the shape-invariant potentials:

A potential $u(x) = u(x; \mathbf{a})$ ($\mathbf{a}$ is a set of parameters) is shape-invariant if in the DT

$$u(x; \mathbf{a_0}) = v(x; \mathbf{a_1}) + R(\mathbf{a_0}, \mathbf{a_1}),$$

(R does not depend on x), being $\mathbf{a_1} = f(\mathbf{a_0})$.

This property of the shape-invariant potentials explains the importance of the method of factorization (equivalent to DT) in order to solve the SSE: we solve for the ground state and we get the complete solution of the Sturm-Liouville bound states spectral problem. From a Galoisian point of view, we only need to study the integrability of the linear differential equation corresponding to the bound state.

Projects and future work

1. Obtain a Galoisian classification of the shape invariant and other quantum mechanics solvable potentials. Try to contribute with new potentials of these types.
2. To clarify the relationship between Interwining operators (a generalization of Darboux transformations) and Eigenrings.
3. Applications to solve specials partial differential equations such as KdV an KP.