Geometry of the Pfaff Lattices

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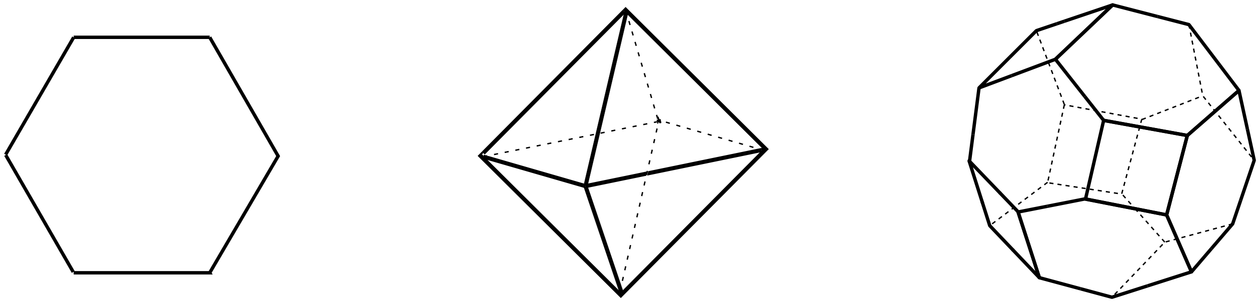

Joint work with V. Pierce

Outline

- **The Pfaff lattice**
  - The Lie algebra splitting $\mathfrak{sl}(2n,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{sp}(n,\mathbb{R})$
  - Lie-Poisson structure on $\mathfrak{sl}(2n,\mathbb{R})^*$

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- **Connections to random matrix models**

The Pfaff Lattice

Example: $\mathfrak{sl}(4,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{sp}(2,\mathbb{R})$

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ * & b_1(t) & a_1(t) & 0 \\ * & * & -b_1(t) & 1 \\ * & * & * & 0 \end{pmatrix}.$$

The **Pfaff lattice hierarchy** in L is

$$\frac{\partial L}{\partial t_j} = -[B_j, L] \quad \text{for} \quad j = 1,2,3.$$

with

$$B_j = \pi_{\mathfrak{k}}(L^j).$$

The Pfaff Lattice

The **Lie algebra splitting**

$$\mathfrak{sl}(2n) \cong \mathfrak{k} \oplus \mathfrak{sp}(n)$$

where $\mathfrak{sp}(n) = \{X \in \mathfrak{sl}(2n) : \sigma(X) = X\}$ with the involution $\sigma(X) := JX^T J$. Here the skew symmetric matrix J is

$$J = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

and

$$\pi_{\mathfrak{k}} X = X_- - J(X_+)^T J + \frac{1}{2}(X_0 - JX_0^T J).$$

where $X_\pm$ is 2×2 upper (lower) block triangular part of X, X_0 is the 2×2 block diagonal part of X.

The Pfaff Lattice

The Lie algebra $\mathfrak{k}$ is the set of matrices of the form,

$$\begin{pmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 & 0 \\ * & * & d_2 & 0 & 0 & 0 \\ * & * & 0 & d_2 & 0 & 0 \\ * & * & * & * & d_3 & 0 \\ * & * & * & * & 0 & d_3 \end{pmatrix} .$$

where $\sum_{j=1}^{n} d_j = 0$. The dimension of $\mathfrak{k}$ is

$$4 \times \frac{n(n-1)}{2} + (n-1) = 2n^2 - n - 1\,.$$

The Pfaff Lattice

With the pairing $\langle x, y\rangle = \text{tr}(xy)$ for $x, y \in \mathfrak{sl}(2n)$,

$$\mathfrak{sl}(2n)^* \cong \mathfrak{sp}(n)^* \oplus \mathfrak{k}^* \,.$$

where $\mathfrak{sp}(n)^* = \mathfrak{k}^\perp$ and $\mathfrak{k}^* = \mathfrak{sp}(n)^\perp$.

The **Lie-Poisson structure** on $\mathfrak{g}^* = \mathfrak{sl}(2n, \mathbb{R})^*$:

$$\{F, G\}_{\mathfrak{g}^*}(L) = \langle L, [\nabla F, \nabla G]_{\mathfrak{g}}\rangle$$

where $[\nabla F, \nabla G]_{\mathfrak{g}} = [R\nabla F, \nabla G] - [\nabla F, R\nabla G]$ with the R-matrix,

$$R = \frac{1}{2}(\pi_{\mathfrak{k}} - \pi_{\mathfrak{sp}})\,.$$

The Pfaff Lattice

The **Pfaff lattice** is defined with an Sp-invariant function H,

$$\frac{dL}{dt} = \{H, L\}_{\mathfrak{g}^*}(L) = -[\pi_{\mathfrak{k}} \nabla H, L],$$

We consider the matrix $L \in \mathfrak{sl}(2n)^*$ in the form,

$$\begin{aligned} L &= \qquad \xi \qquad + \qquad \kappa \\ &= \begin{pmatrix} \alpha_1 & * & 0 & 0 \\ * & -\alpha_1 & 0 & 0 \\ * & * & \alpha_2 & * \\ * & * & * & -\alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_1 & a_1 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ -a_1 & 0 & 0 & -\beta_1 \end{pmatrix}. \end{aligned}$$

with $\xi \in \mathfrak{sp}(n)^*, \ \kappa \in \mathfrak{k}^* = \mathfrak{sp}^\perp$. Note $\langle \kappa, [\mathfrak{sp}^\perp, \mathfrak{sp}^\perp] \rangle = 0$.
dim$(\mathfrak{sp}(n)^*) = 2n^2 + n$ and the number $|\kappa| = 2 \times (n-1)$.

The Pfaff Lattice

The Lax matrix L is given by (Adler-van Moerbeke et al):

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ * & b_1 & a_1 & 0 & 0 & 0 \\ * & * & -b_1 & 1 & 0 & 0 \\ * & * & * & b_2 & a_2 & 0 \\ * & * & * & * & -b_2 & 1 \\ * & * & * & * & * & 0 \end{pmatrix}.$$

Note $[B_k, L]_{2i-1,2i} = 0$ and $\sum_{j=1}^{2i-1} [B_k, L]_{j,j} = 0$ for $i = 1, \ldots, n$.

L has $2n^2 + n - 2$ free variables, i.e.

$$\dim(\mathfrak{sp}(n)^*) + |\kappa| - (2n) = (2n^2 + n) + (2n - 2) - (2n).$$

The Pfaff Lattice

The SR factorization:

Let $L(0)$ be an initial matrix of L, and

$$e^{t_1 L(0)} = Q(t_1)^{-1} P(t_1)\,, \quad \text{with} \quad Q \in G_{\mathfrak{k}}, \quad P \in Sp(n)\,,$$

where $Lie(G_{\mathfrak{k}}) = \mathfrak{k}$. Then the solution $L(t_1)$ is given by

$$L(t_1) = Ad_{Q(t_1)} L(0) = Ad_{P(t_1)} L(0)\,.$$

The Pfaff lattice hierarchy is

$$\frac{\partial L}{\partial t_j} = -[\pi_{\mathfrak{k}} \nabla H_j, L]\,, \quad H_j = \frac{1}{j+1} \mathrm{tr}(L^{j+1})\,,$$

for $j = 1, 2, \ldots, 2n-1$.

Integrability

An Sp-**invariant curve** on $\mathbb{CP}^2$:

$$F_L(x,y,z) = \text{det}[(x-y)L + yJL^TJ - zI] = 0\,.$$

$F_L(1,0,z)$ is the characteristic polynomial generating H_j.

The curve has an involution $\iota : (x,y,z) \to (x, x-y, -z)$.

$$F_L(x,y,z) = \sum_{r=0}^{2n} \sum_{k=0}^{[r/2]} F_{r,k}(L)\varphi^{(r,k)}(x,y,z)\,.$$

$$\begin{cases} \varphi^{(2r,k)}(x,y,z) = x^{2(r-k)}(y(x-y))^k z^{2(n-r)}, \\ \varphi^{(2r+1,k)}(x,y,z) = x^{2(r-k)}(2y-x)(y(x-y))^k z^{2(n-r)-1}. \end{cases}$$

Those are invariant under the involution.

Integrability

The total number of $F_{r,k}(L)$ is $n^2 + 2n - 1$.

Integrability

The total number of $F_{r,k}(L)$ is $n^2 + 2n - 1$. However they are not all independent. Define an algebraic variety,

$$\mathcal{C}(L) = \{[x : y : z] : F_L(x, y, z) = 0\} \subset \mathbb{C}P^2$$

The variety $\mathcal{C}(L)$ is singular at $x = 0$,

$$\begin{aligned} F_L(0, 1, z) &= \text{det}\left(-L + JL^T J - zI\right) \\ &= \left[\text{pf}\left(-JL - L^T J - zJ\right)\right]^2 . \end{aligned}$$

For generic L, $\mathcal{C}(L)$ has n double points over $x = 0$, and there are n relations among $\{F_{k,[k/2]}(L) : k = 1, \ldots, 2n\}$.

Integrability

The solution $L(t)$ is given by the coadjoint action $Ad_g^*(L^0)$,

$$L(t) = Ad^*_{g(t)}(L^0) = \pi_{\mathfrak{sp}^*}(P\xi^0 P^{-1}) + \pi_{\mathfrak{k}^*}(Q\kappa^0 Q^{-1})\,,$$

with $g(t) = e^{tL^0} = Q(t)^{-1}P(t)$ and $L^0 = \xi^0 + \kappa^0$. From this, one finds that there are Casimirs $C_k(L)$ for $k = 1, \ldots, n$:

$$F_L(2,1,z) = z^{2n} + \sum_{k=1}^{n} C_k(L) z^{2n-2k}\,.$$

There are n Casimirs,

$$C_k(L) = \sum_{j=0}^{k} 2^{2j} F_{2k,j}(L) \qquad k = 1, \ldots, n\,.$$

Integrability

Proposition:

We have $n^2 - 1$ independent Hamiltonians in $\{F_{r,k}\}$, i.e.

$$n^2 - 1 = (n^2 + 2n - 1) - (n) - (n)\,.$$

Theorem:

The Pfaff lattice is a completely integrable Hamiltonian system with $n^2 - 1$ Hamiltonians and n Casimirs.

Proof: Construct the angle variables conjugate to $F_{r,k}$. The curve $\mathcal{C}(L)$ has genus $g(L) = 2n^2 - 4n + 1$. The angles are defined by the differentials on the curve.

(Deift, Li and Tomei (1989) for the generalized Toda lattice)

Matrix Factorizations and τ-Functions

Let us define

$$\theta(t,X)=\sum_{j=1}^{2n-1} t_jX^j\,.$$

With the SR-factorization $e^{\theta(t,L^0)}=Q(t)^{-1}P(t)$, we define

$$M(t):=e^{\theta(t,L^0)}Je^{\theta(t,L^0)^T}=Q(t)^{-1}JQ(t)^{-T}\,.$$

Skew-symmetric matrix M is called the **moment matrix**, and $M=Q^{-1}JQ^{-T}$ is a Cholesky-type factorization.

Using $L^0\Psi_0=\Psi_0C_\gamma$ with $\Psi_0\in G_{\mathfrak{k}}$, and C_γ is the companion matrix of L^0, we write

Matrix Factorizations and τ-Functions

$$
\begin{aligned}
M(t) &= e^{\theta(t,L^0)} J e^{\theta(t,L^0)^T} \\
&= \Psi_0 e^{\theta(t,C_\gamma)} \Psi_0^{-1} J \Psi_0^{-T} e^{\theta(t,C_\gamma)} \Psi_0^T \\
&= \Psi_0 \tilde{M}(t) \Psi_0^T
\end{aligned}
$$

where

$$
\tilde{M}(t) = e^{\theta(t,C_\gamma)} \tilde{B} e^{\theta(t,C_\gamma)^T}, \qquad \tilde{B} = \Psi_0^{-1} J \Psi_0^{-T}
$$

Alos note $C_\gamma V = V\Lambda$ with the Vandermonde matrix V, and $\Lambda = \text{diag}(z_1, \dots, z_{2n})$,

$$
e^{\theta(t,C_\gamma)} = V e^{\theta(t,\Lambda)} V^{-1} = E(t,\Lambda) V^{-1}.
$$

where $e^{\theta(t,\Lambda)} = \text{diag}(E_1(t), \dots, E_{2n}(t))$ with $E_k = e^{\theta(t,z_k)}$.

Matrix Factorizations and τ-Functions

Then $E(t,\Lambda)$ is given by the Wronskian matrix,

$$E(t,\Lambda) = \begin{pmatrix} E_1 & E_2 & \cdots & E_{2n} \\ E_1' & E_2' & \cdots & E_{2n}' \\ \vdots & \vdots & \ddots & \vdots \\ E_1^{(2n-1)} & E_2^{(2n-1)} & \cdots & E_{2n}^{(2n-1)} \end{pmatrix}.$$

The moment matrix $M = \Psi_0 \tilde{M} \Psi_0^T$ is given by

$$\tilde{M}(t) = E(t,\Lambda) B E(t,\Lambda)^T .$$

where $B = V^{-1}\tilde{B}V^{-T}$, and gives the initial data for the Pfaff lattice.

Matrix Factorizations and τ-Functions

Summary: With $L(0)\Psi_0 = \Psi_0 C_\gamma$ and $e^{\theta(t,L(0))} = Q^{-1}P$,

$$\begin{array}{ccc} L(0) & \xrightarrow{c_\gamma} & \tilde{B} = \Psi_0^{-1} J \Psi_0^{-T} \\ {\scriptstyle Ad_{Q(t)}}\Big\downarrow & & \Big\downarrow \\ L(t) & \xrightarrow{c_\gamma} & \tilde{M}(t) = e^{\theta(t,C_\gamma)} \tilde{B} e^{\theta(t,C_\gamma)^T} \end{array}$$

The map c_γ is the companion embedding.

Now we define the τ-functions,

$$\tau_{2k} = \text{pf}(\tilde{M}_{2k}) \,,$$

where $\tilde{M}_{2k}$ is the principal $2k \times 2k$ submatrix of $\tilde{M}$.

Matrix Factorizations and τ-Functions

The $QMQ^T = J$ defines skew-orthogonal polynomials: Write

$$M = V\mathcal{M}V^T$$

Then each entry $m_{i,j}$ of M can be written by

$$m_{i,j} = \langle z^{i-1}, z^{j-1} \rangle_{\mathcal{M}} = \sum_{1 \le k,l \le 2n} z_k^{i-1} z_l^{j-1} \mu_{k,l} \,.$$

The skew-othogonal polynomials $p_i(z)$ of $\deg(p_i(z)) = i$ are

$$P\mathcal{M}P^T = J, \qquad P := QV = (p_{i-1}(z_j))_{1 \le i,j \le 2n} \,.$$

Those polynomials are given by Pfaffians.

Matrix Factorizations and τ-Functions

$$\tau_{2k} = \mathsf{pf}(\tilde{M}_{2k}) = \sum_{I_k} (-1)^{\sigma} m_{i_1,j_1} m_{i_2,j_2} \cdots m_{i_k,j_k} \, .$$

where $I_k = \{1 = i_1 < i_2 < \cdots < i_k, i_s < j_s, s = 1, \ldots, k\}$ and the $\sigma(i_1, j_1, \ldots, i_k, j_k)$ is the length of the permutation. E.g.

$$\begin{aligned} \tau_2 &= m_{1,2}, \\ \tau_4 &= m_{1,2} m_{3,4} - m_{1,3} m_{2,4} + m_{1,4} m_{2,3}. \end{aligned}$$

From the bi-vector $\Omega_2 := \sum_{1 \le i < j \le 2n} m_{i,j} e_i \wedge e_j$,

$$\tau_{2k} = \left\langle \wedge^k \Omega_2, e_1 \wedge \cdots \wedge e_{2k} \right\rangle .$$

Matrix Factorizations and τ-Functions

Lemma: (Ishikawa-Wakayama, 1995)

$$\tau_{2k} = \sum_{1 \le i_1 < \cdots < i_{2k} \le 2n} \mathrm{pf}(i_1, \ldots, i_{2k}) E(i_1, \ldots, i_{2k})\,.$$

where $E(i_1, \ldots, i_{2k})$ is the det of $2k \times 2k$ submatrix of $E(t, \Lambda)$, $\mathrm{pf}(i_1, \ldots, i_{2k})$ is the Pfaffian of the submatrix of the B-matrix.

Matrix Factorizations and τ-Functions

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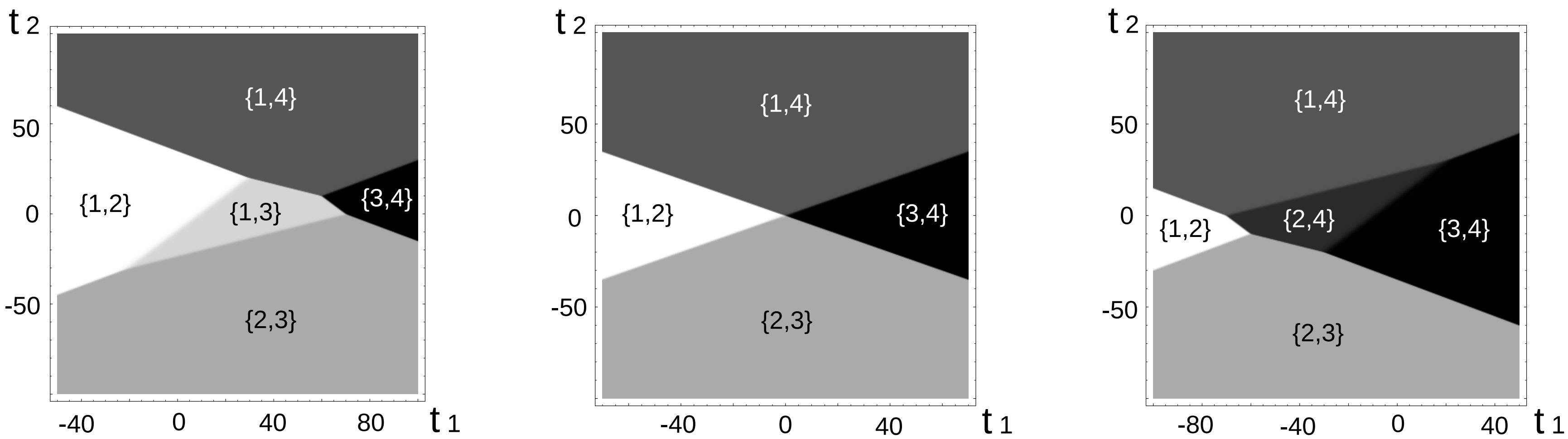
Remark: For all real and distinct eigenvalues $\{z_i\}$,

- $\{E_i = e^{\theta(t,z_i)} : i = 1, \ldots, 2n\}$ gives a basis for $\mathbb{R}^{2n}$.
- $\{E(i,j) : 1 \le i < j \le 2n\}$ forms a basis for $\wedge^2\mathbb{R}^{2n}$.
- In general, $\{E(i_1, \ldots, i_{2k})\}$ forms a basis for $\wedge^{2k}\mathbb{R}^{2n}$.
- So τ_{2k} represents a vector in $\wedge^{2k}\mathbb{R}^{2n}$.

Matrix Factorizations and τ-Functions

Example: $n = 2$ with $b_{i,j} = 1$ and $(z_1, \ldots, z_4) = (-2, -1, 0, 3)$
Graphs of $b_1 = \frac{\partial}{\partial t_1} \ln(\tau_2)$ in the $t_1 t_2$ plane for $t_3 = -10, 0, 10$.



t_3=-10 t_3= 0 t_3=10

Each set $\{i, j\}$ represents the dominant exponential in that region, that is,

$$b_1 = z_i + z_j = \{i, j\}\,.$$

Matrix Factorizations and τ-Functions

The τ-functions satisfy a coupled KP (or DKP) equation,

$$\left(-4D_1D_3 + D_1^4 + 3D_2^2\right)\tau_{2k} \cdot \tau_{2k} = 24\tau_{2k-2}\tau_{2k+2}\,,$$

where $\tau_0 = 1$, and D_k is the Hirota derivative (Hirota-Ota, 1991). If the RHS=0, it gives the KP equation.

Remark:

- The DKP equation appeared (implicitly) in (Jimbo-Miwa, 1983).
- The partition functions for GOE and GSE random matrix models (Adler-van Moerbeke, Kakei, 1999-).
- The charged BKP equation (Kac-van de Leur, 1999).

Fixed points of the Pfaff flow

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- Here $\text{diag}_2(\hat{Q}) = \text{diag}(I_2, \dots, I_2)$ (cf. $Q \in G_{\mathfrak{k}}$), and

$$H = \text{diag}\,(h_1 I_2, \dots, h_n I_2) \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $\prod_{k=1}^n h_k = 1$. Note $\tau_{2k} = \prod_{j=1}^k h_j^2$.

Fixed Points of the Pfaff flow

Define

$$\hat{L} = HLH^{-1} = \hat{Q}L(0)\hat{Q}^{-1}\,.$$

For the case with $n = 2$,

$$\hat{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ * & b_1(t) & 1 & 0 \\ * & * & -b_1(t) & 1 \\ * & * & * & 0 \end{pmatrix}$$

We have assumed that $a_k(0) = 1$ for all k. Note that

$$a_k = \frac{\sqrt{\tau_{2k-2}\tau_{2k+2}}}{\tau_{2k}}\,, \qquad b_k = \frac{\partial}{\partial t_1}\ln \tau_{2k}\,.$$

Fixed Points of the Pfaff flow

Entries $q_{i,j}$ of $\tilde{Q} := \hat{Q}\Psi_0$ are given by (Adler-van Moerbeke)

$$\begin{cases} q_{2k,2k-j}(t) = \dfrac{S_j(-\tilde{\partial})\tau_{2k}(t)}{\tau_{2k}(t)} \\ q_{2k+1,2k-j}(t) = \dfrac{[S_{j+1}(-\tilde{\partial}) + \partial_1 S_j(-\tilde{\partial})]\tau_{2k}(t)}{\tau_{2k}(t)} \end{cases}.$$

where $S_k(-\tilde{\partial})$ denotes $S_k(-\partial_1, -\frac{1}{2}\partial_2, -\frac{1}{3}\partial_3, \ldots)$, and

$$\exp\left(\sum_{j=1}^{\infty} t_j z^j\right) = \sum_{k=0}^{\infty} S_k(t_1, \ldots, t_k) z^k .$$

$\hat{L} = \tilde{Q} C_\gamma \tilde{Q}^{-1}$, and $\tilde{Q}$ is invariant under the scaling of τ_{2k}.

Fixed Points of the Pfaff flow

As $t_1 \to -\infty$ the $\hat{L}(t)$ approaches $\hat{L}^-$ which has the following block form:

$$\hat{L}^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -z_1 z_2 & z_1 + z_2 & 1 & 0 \\ 0 & 0 & z_3 + z_4 & 1 \\ 0 & 0 & -z_3 z_4 & 0 \end{pmatrix}$$

Note that the diagonal blocks have eigenvalues (z_1, z_2) and (z_3, z_4), i.e.

$$b_1 \to z_1 + z_2 \quad \text{as} \quad t_1 \to -\infty \,.$$

(Compare with the figure for $n = 2$.)

Fixed Points of the Pfaff flow

As $t_1 \to \infty$ the order is reversed:

$$\hat{L}^{+} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -z_3 z_4 & z_3 + z_4 & 1 & 0 \\ 0 & 0 & z_1 + z_2 & 1 \\ 0 & 0 & -z_1 z_2 & 0 \end{pmatrix}$$

Theorem: As $t_1 \to \pm\infty$, $\hat{L} \to \mathrm{diag}(\hat{L}^{\pm}_{0,0}, \ldots, \hat{L}^{\pm}_{n-1,n-1}) + \epsilon$,

$$\hat{L}^{-}_{k,k} = \begin{pmatrix} -\sigma_1(2k) & 1 \\ -\left(\sigma_1(2k) + z_{2k+1}\right)\left(\sigma_1(2k) + z_{2k+2}\right) & \sigma_1(2k+2) \end{pmatrix}$$

$\hat{L}^{+}_{k,k} = \hat{L}^{-}_{n-k-1,n-k-1}$ with $\sigma_1(m) = \sum_{j=1}^{m} z_j$ and $\sigma_1(2n) = 0$.

Fixed Points of the Pfaff flow

The set of the fixed points can be represented by

$$Fix(L) = \{ \ (\{z_{i_1}, z_{i_2}\}, \ldots, \{z_{i_{2n-1}}, z_{i_{2n}}\}) \ : \ i_{2k-1} < i_{2k} \ \} \, .$$

We have an isomorphism,

$$Fix(L) \cong W^P = S_{2n}/W_P \, , \quad W_P = \langle s_1, s_3, \ldots, s_{2n-1} \rangle \, .$$

and

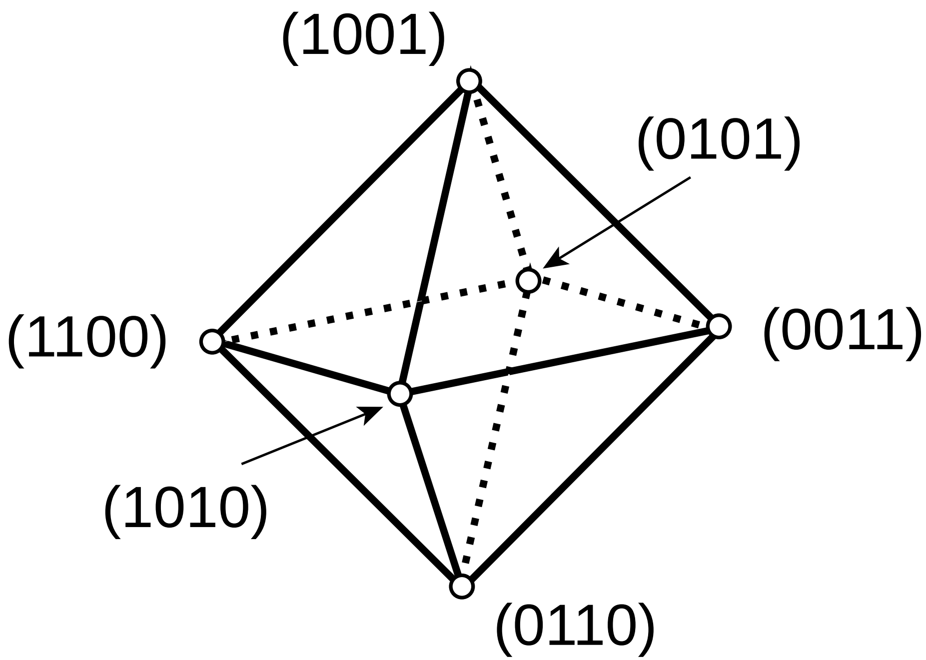
$$|Fix(L)| = \frac{(2n)!}{2^n} \, .$$

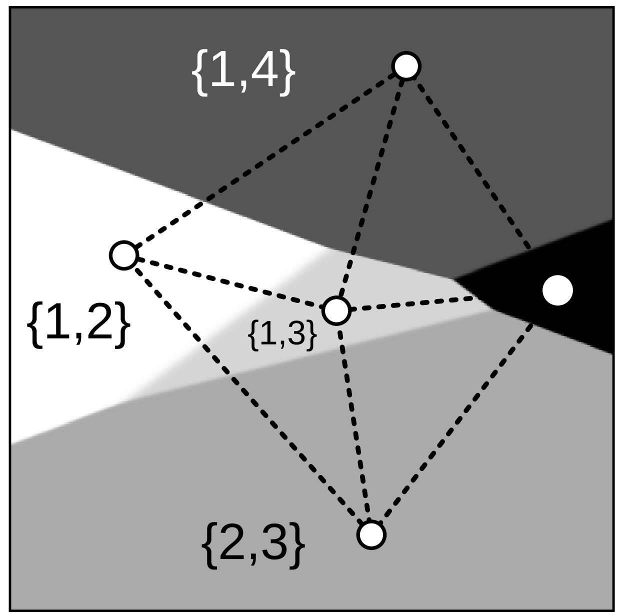
$Fix(L)$ is parametrized by the set of minimal reps of W^P.

$$W^P = \{e, \ s_2, \ s_1 s_2, \ s_3 s_2, \ s_1 s_3 s_2, \ s_2 s_1 s_3 s_2\}, \quad (n = 2).$$

Fixed Points of the Pfaff flow

The fixed points for the case of $n = 2$:





$e = (1100), s_2 = (1010), s_1 s_2 = (0110), s_3 s_2 = (1001),$
$s_1 s_3 s_2 = (0101), s_2 s_1 s_3 s_2 = (0011).$

Geometry of the Pfaff lattice

- Recall Ishikawa-Wakayama formula,

$$\tau_{2k} = \sum_{1 \le i_1 < \cdots < i_{2k} \le 2n} \mathrm{pf}(i_1, \ldots, i_{2k}) E(i_1, \ldots, i_{2k}) \,.$$

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- The set $\{E(i_1, \ldots, i_{2k})\}$ gives a basis for $\bigwedge^{2k} \mathbb{R}^{2n}$.

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- The set $\{E(i_1, \ldots, i_{2k})\}$ gives a basis for $\bigwedge^{2k} \mathbb{R}^{2n}$.
- τ_{2k} is a vector in this basis.

Geometry of the Pfaff lattice

- Recall Ishikawa-Wakayama formula,

$$\tau_{2k} = \sum_{1 \le i_1 < \cdots < i_{2k} \le 2n} \mathrm{pf}(i_1, \ldots, i_{2k}) E(i_1, \ldots, i_{2k}) \,.$$

- The set $\{E(i_1, \ldots, i_{2k})\}$ gives a basis for $\bigwedge^{2k} \mathbb{R}^{2n}$.
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- Consider a moment map:

$$\mu:\mathbb{P}(\wedge^{2k}\mathbb{R}^{2n})\longrightarrow \mathfrak{h}^*_{\mathbb{R}}\cong\mathbb{R}^{2n-1}$$

The Moment Map

This is a convex map (Gel'fand-Sarganova, 1987),

$$\mu(\tau_{2k}) = \frac{\sum_{I_k} |\mathsf{pf}(i_1, \ldots, i_{2k}) E_{i_1} \cdots E_{i_{2k}}|^2 (\mathcal{L}_{i_1} + \cdots + \mathcal{L}_{i_{2k}})}{\sum_{I_k} |\mathsf{pf}(i_1, \ldots, i_{2k}) E_{i_1} \cdots E_{i_{2k}}|^2} .$$

where $I_k = \{1 \leq i_1 < \cdots < i_{2k} \leq 2n\}$. Here $\mathfrak{h}^*_{\mathbb{R}}$ is the dual space of Cartan subalgebra of $\mathfrak{sl}(2n, \mathbb{R})$,

$$\mathfrak{h}^*_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}} \left\{ \mathcal{L}_1, \ldots, \mathcal{L}_{2n} : \sum_{i=1}^{2n} \mathcal{L}_i = 0 \right\} ,$$

with the weights $\mathcal{L}_k$.

The Moment Map

The moment map can be extended to

$$\begin{aligned} \mu: \quad \mathbb{RP}^{2n-1} \times \cdots \times \mathbb{P}(\wedge^{2n-2}\mathbb{R}^{2n}) \quad &\rightarrow \quad \mathfrak{h}^*_{\mathbb{R}} \\ (\tau_2, \ldots, \tau_{2n-2}) \quad &\mapsto \quad \sum_{k=1}^{n-1} \mu(\tau_{2k}) \end{aligned}$$

Theorem: The image of the map is a convex polytope whose vertices are the fixed points of the Pfaff lattice.

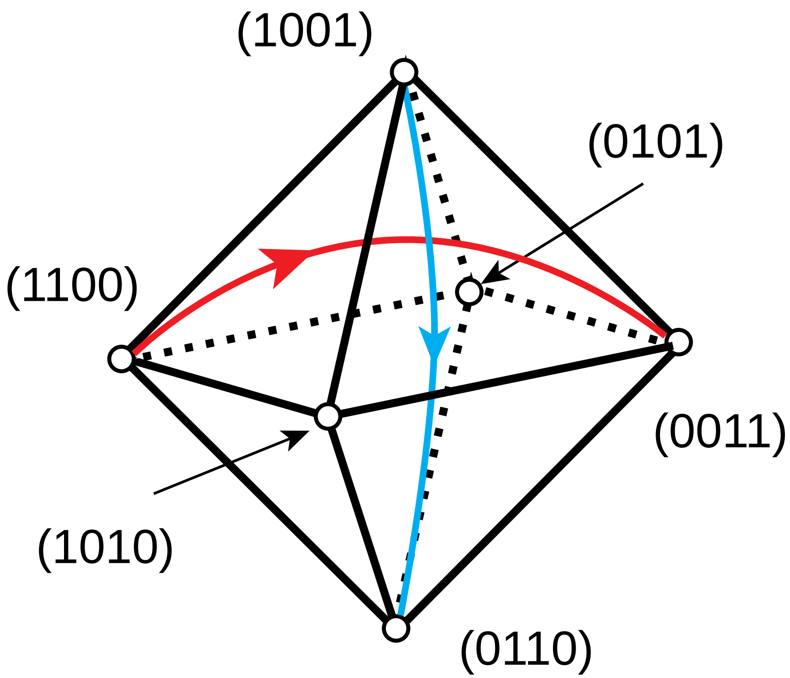
Remark: The polytope is a tensor product rep of $SL(2n)$, i.e. $\Gamma_{0,1,0,\ldots,1,0}$, and each fixed point is marked by the weight

$$(\alpha_1, \ldots, \alpha_{2n}) := \sum_{k=1}^{2n} \alpha_k \mathcal{L}_k \qquad \alpha_k \in \{0, 1, \ldots, n-1\}\,.$$

The Moment Polytope

Example of $n = 2$: The τ-function is given by

$$\tau_2 = \sum_{1\le i,j\le 4} b_{i,j} E(i,j).$$



Each vertex $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ represents the weight vector $\sum_{j=1}^{4} \alpha_j \mathcal{L}_j$.

The Moment polytope in this case is the representation with highest weight $\mathcal{L}_1 + \mathcal{L}_2 = (1100)$.

The Moment Polytope

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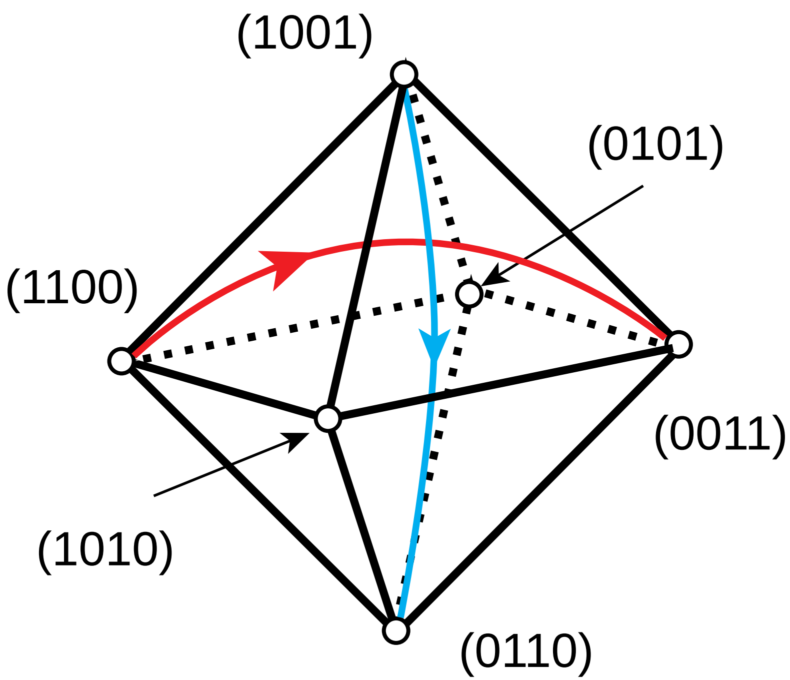


The Pfaff orbit in the generic case (i.e. all $b_{k,l} \neq 0$) can be described by a curve inside of the polytope approaching the vertex (1100) as $t_1 \to -\infty$ and (0011) as $t_1 \to \infty$.

A non-generic flow with only $b_{1,4} b_{2,3} \neq 0$ is given by a curve in blue.

Foliation by the integrals $F_{r,k}(L)$

The isospectral variety of the Pfaff lattice is

$$Z_{\mathbb{R}}(\gamma) = \left\{ L \in \mathfrak{sl}(2n)^* : H_j = \frac{1}{j+1}\mathrm{tr}(L^{j+1}) = \gamma_j \in \mathbb{R} \right\}$$

The variety is foliated by additional integrals:

Proposition: With $\mathrm{pf}(B) = 1$,

$$F_L(x, y, z) = \det\left((x-y)\Lambda B - yB\Lambda - zB\right) .$$

where $\Lambda = \mathrm{diag}(z_1, \ldots, z_{2n})$. Thus

- $F_{r,k}(L)$ are expressed in terms of $\{z_k\}$ and the B-matrix.
- $\{F_{r,0}(L)\}$ gives $\{H_j\}$ and they do not depend on B.

Foliation by the integrals $F_{r,k}(L)$

Other integrals $F_{r,k}(L)$ with $k \neq 0$ give a foliation of $Z_{\mathbb{R}}(\gamma)$:

Example of $n = 2$: With tr$(L) = z_1 + z_2 + z_3 + z_4 = 0$,

- $F_{2,1} = 2(b_{12}b_{34}(z_1 + z_2)^2 - b_{13}b_{24}(z_1 + z_3)^2 + b_{14}b_{23}(z_1 + z_4)^2)$
- $F_{3,1} = 0$
- $F_{4,1} = -b_{12}b_{34}(z_1 + z_2)^2(z_1 z_2 + z_3 z_4) + b_{13}b_{24}(z_1 + z_3)^2(z_1 z_3 + z_2 z_4) - b_{14}b_{23}(z_1 + z_4)^2(z_1 z_4 + z_2 z_3)$
- $F_{4,2} = F_{2,1}^2/2$

For example, if only $b_{14}b_{23} \neq 0$, then the flow contains a non-generic leaf (cf. previous figure).

Random matrix models

The partition function of the GOE matrix model has the form:

$$Z_N = \int_{\mathcal{S}_N} \exp\left(\mathrm{tr}(V(X))\right) dX$$
$$= \int \cdots \int \left(\prod_{i<j}(z_i - z_j)\right) \exp\left(\sum_{k=1}^{N} V(z_k)\right) dz_1 \ldots dz_N$$

where $\mathcal{S}_N$ is the set of real symmetric matrices, and

$$V(z) = -\frac{1}{2}z^2 + \sum_{i=1}^{2m} t_i z^i$$

Random matrix models

The partition function of the GSE matrix model has the form:

$$Z_N = \int_{\mathcal{Q}_N} \exp\left(\mathrm{tr}(V(X))\right) dX$$

$$= \int \cdots \int \left(\prod_{i<j} (z_i - z_j)^4 \right) \exp\left(\sum_{k=1}^{N} V(z_k) \right) dz_1 \ldots dz_N$$

where $\mathcal{Q}_N$ is the set of self-dual Hermitian matrices with quartanion entries.

GOE Example

Take $b_{k,l} = \mathrm{sgn}(l-k)$. Then the moment matrix is

$$m_{i,j} = \sum_{1\le k<l\le 2n} (z_k z_l)^{i-1}(z_l^{j-i} - z_k^{j-i}) E_k E_l \, .$$

The τ-function ($2k < 2n$) is given by

$$\begin{aligned}\tau_{2k} &= \text{pf}(M_{2k\times 2k}) \\ &= \sum_{1\le i_1<\cdots<i_{2k}\le 2n} \left(\prod_{j<l}(z_{i_l} - z_{i_j})\right) E_{i_1}\ldots E_{i_{2k}} \, .\end{aligned}$$

This is a finite version of the GOE partition function.

GSE Example

Take

$$b_{2k-1,2k} = -b_{2k,2k-1} = \frac{1}{z_{2k} - z_{2k-1}}$$

and $b_{i,j} = 0$ for all other (i,j).

Consider the limit $z_{2k} \to z_{2k-1}$, and relabel $z_{2k-1} \to z_k$.

The moment matrix is:

$$m_{ij} = (j-i)\sum_{k=1}^{n} z_k^{i+j-3} E_k^2 \,.$$

GSE Example

The τ-functions are given by

$$\tau_{2k} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \left(\prod_{j<l} (z_{i_l} - z_{i_j})^4 \right) E_{i_1}^2 \ldots E_{i_k}^2 \, .$$

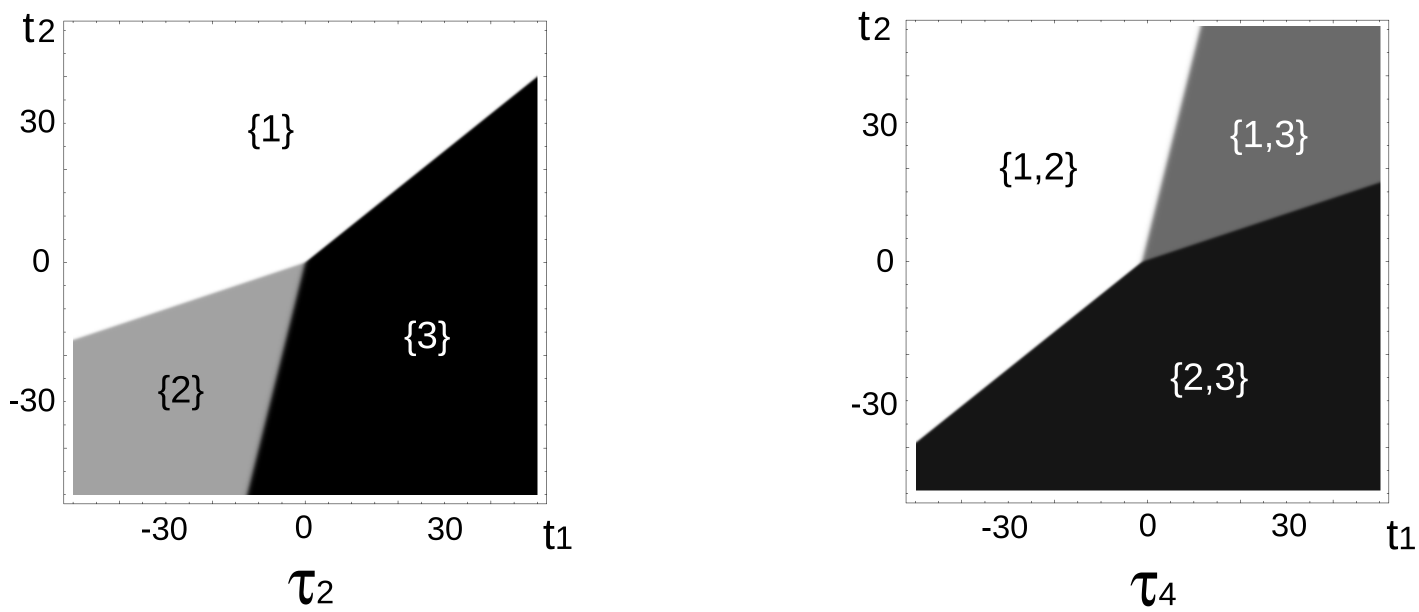
This is a finite version of the GSE partition function.

Example: $n = 3$

$$\begin{aligned} \tau_2 &= E_1^2 + E_2^2 + E_3^2 \\ \tau_4 &= (z_1 - z_2)^4 E_1^2 E_2^2 + (z_1 - z_3)^2 E_1^2 E_3^2 + (z_2 - z_3)^4 E_2^2 E_3^2 \end{aligned}$$

GSE Example

Graphs of $b_k(t_1, t_2, t_3) = \frac{\partial}{\partial t_1} \ln \tau_{2k}$ for $k = 1, 2$.

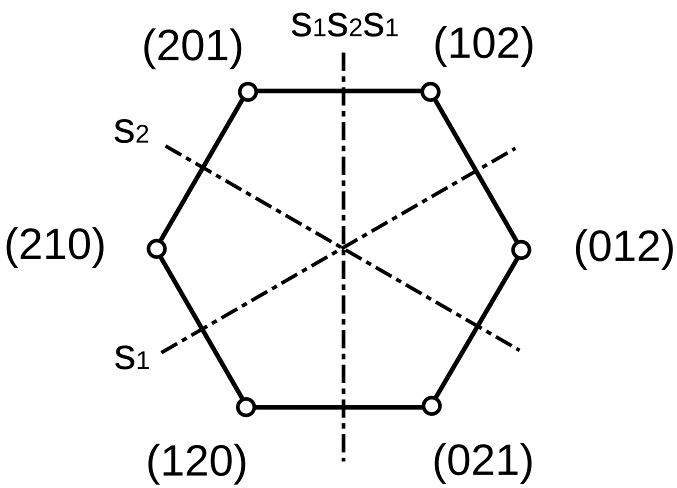


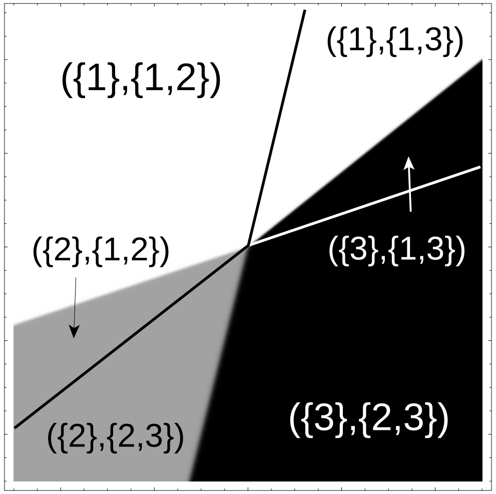
The sets $\{i\}$ represent the values of b_1, i.e. $b_1 = z_j$.

The sets $\{i, j\}$ represent the values of b_2, i.e. $b_2 = z_i + z_j$.

GSE Example

The moment polytope is given by the irreducible rep of $SL(3)$ with the highest weight $2\mathcal{L}_1 + \mathcal{L}_2 = (210)$:





The right figure is the combined graphs of b_1 and b_2.

The sets $(\{i\}, \{j, k\})$ represent the pair of values (b_1, b_2), and the boundaries correspond to the permutations (i.e. the polytope is the permutahedron of S_3).

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