Singular limits of Hamiltonian PDES

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- Singular limits of Hamiltonian PDES $\nearrow$ hyperbolic equations, $\searrow$ elliptic equations

- Characterize the behavior of solutions near critical points.
  Painleve equations

- Universality result: the behavior of the solution near a critical point does not depend on the initial data (as in Random Matrix Theory)

- As a prototype we consider the Korteweg de Vries (KdV) equation and the focusing nonlinear Schroedinger equation

- B. Dubrovin, On Hamiltonian perturbations of hyperbolic systems of conservation laws, II: universality of critical behaviour, Comm. Math. Phys. 267 (2006) 117 - 139.
- T.Grava, C.Klein, Numerical solution of the small dispersion limit of Korteweg de Vries and Whitham equations. ArXiv:math-ph0511011, to appear in Comm. Pure Appl. Math., 2007.
- B. Dubrovin, T. Grava and C. Klein, On universality of critical behaviour in the focusing nonlinear Schroedinger equation, elliptic umbilic catastrophe and the tritronquee solution to the Painleve-I equation. ArXiv: 0704.0501.

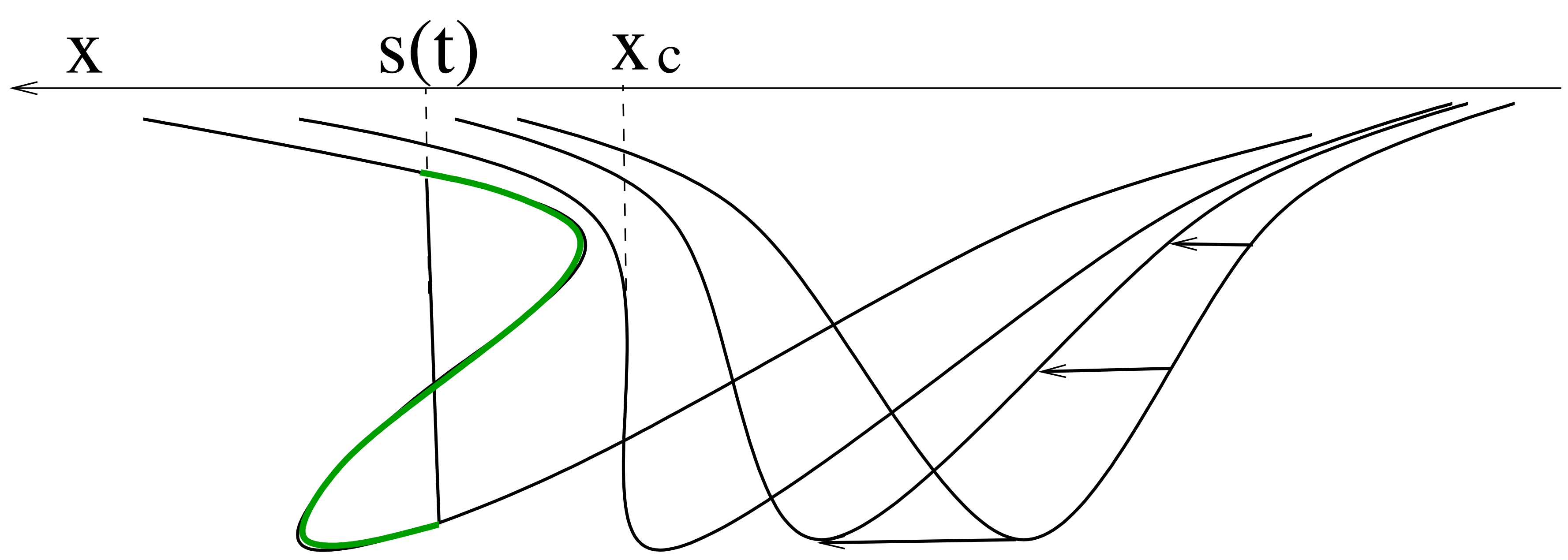
Hopf equation

$$u_t + uu_x = 0, \quad x, t, u \in \mathbb{R},$$

For generic initial data $u(x, t = 0) = u_0(x)$, the solution is obtained by the method of characteristics

$$u(x, t) = u_0(\xi), \qquad x = tu_0(\xi) + \xi.$$

The solution of the Cauchy problem exists till the point (x_c, t_c, u_c) of gradient catastrophe: $u_x \to \infty$ for $x = x_c, t = t_c$. For $t > t_c$ a shock wave develops.



Teorem 1. Up to shifts, Galilean transformation and reascalings the solution of the Hopf equation behaves near the point of gradient catastrophe as the root $u = u(x,t)$ of the cubic equation

$$x = ut - \frac{u^3}{6}$$

(universal unfolding of A_2 singularity.)

Proof: it is sufficient to expand the solution in Taylor series near the critical point (x_c, t_c, u_c):

$$\tilde{x} - u_c\tilde{t} = \tilde{u}\tilde{t} - \frac{k}{6}\tilde{u}^3 + O(\tilde{u}^4)$$

$\tilde{x} = x - x_c$, $\tilde{t} = t - t_c$, $\tilde{u} = u - u_c$. Then do the rescalings

$$\tilde{x} \to \lambda\tilde{x}, \quad \tilde{t} \to \lambda^{\frac{1}{3}}\tilde{t}, \quad \tilde{u} \to \lambda^{\frac{1}{3}}\tilde{u}$$

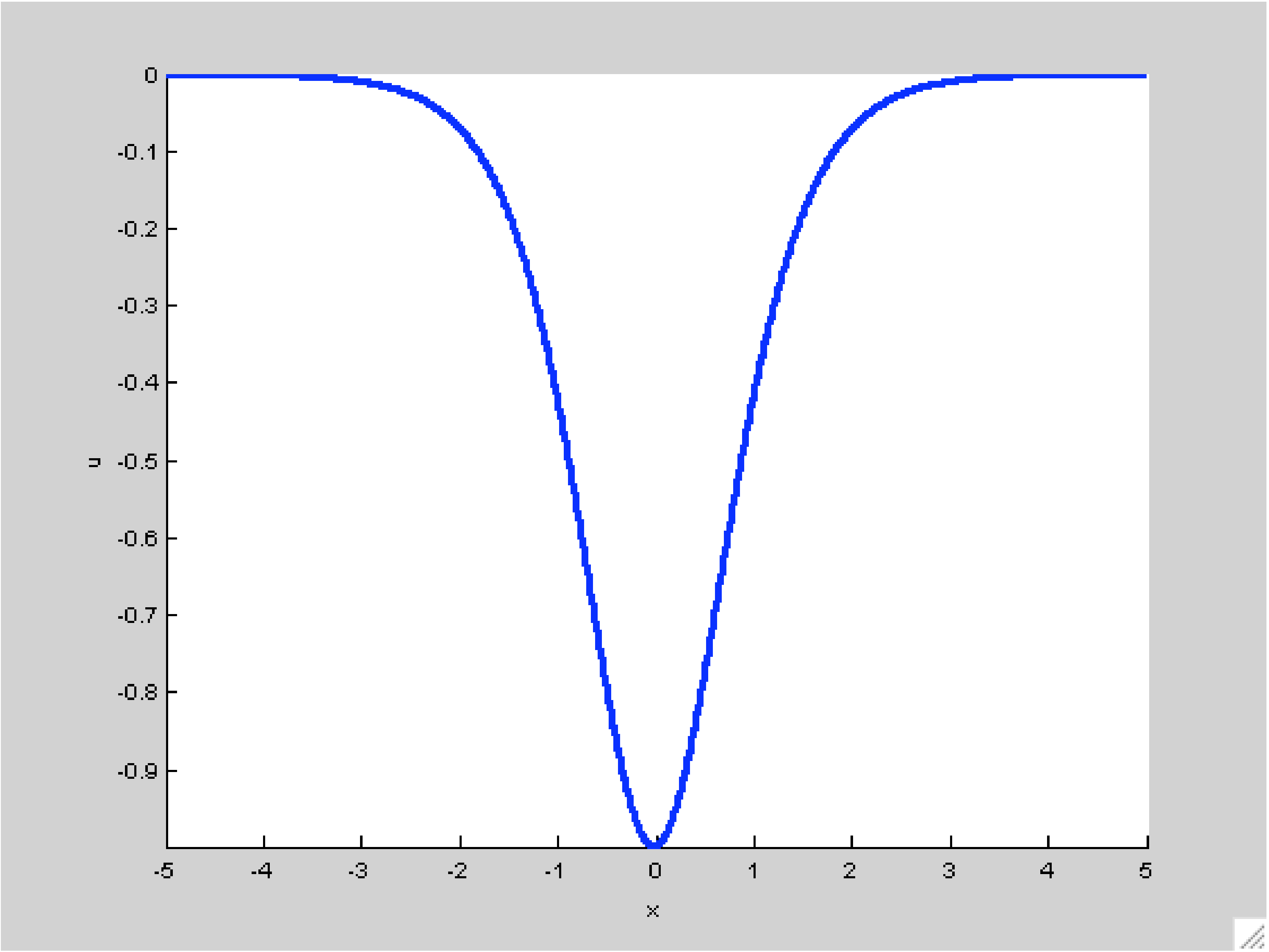
and then let $\lambda \to 0$.

Perturbations: two scenarios

1. Dissipative perturbations: shock waves
2. Hamiltonian perturbations: oscillations

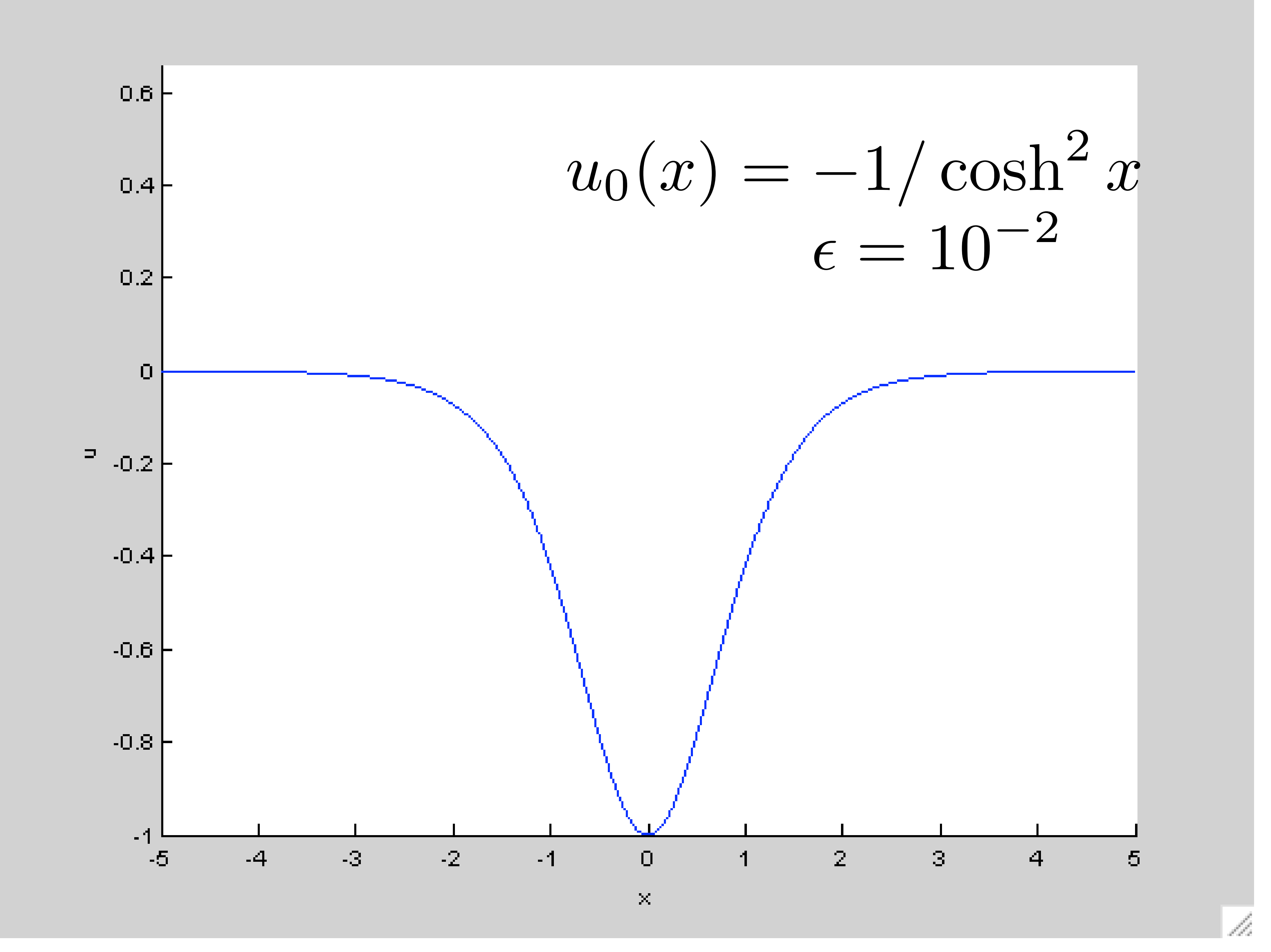
Dissipative regularization

$$u_t + uu_x = \epsilon u_{xx}$$

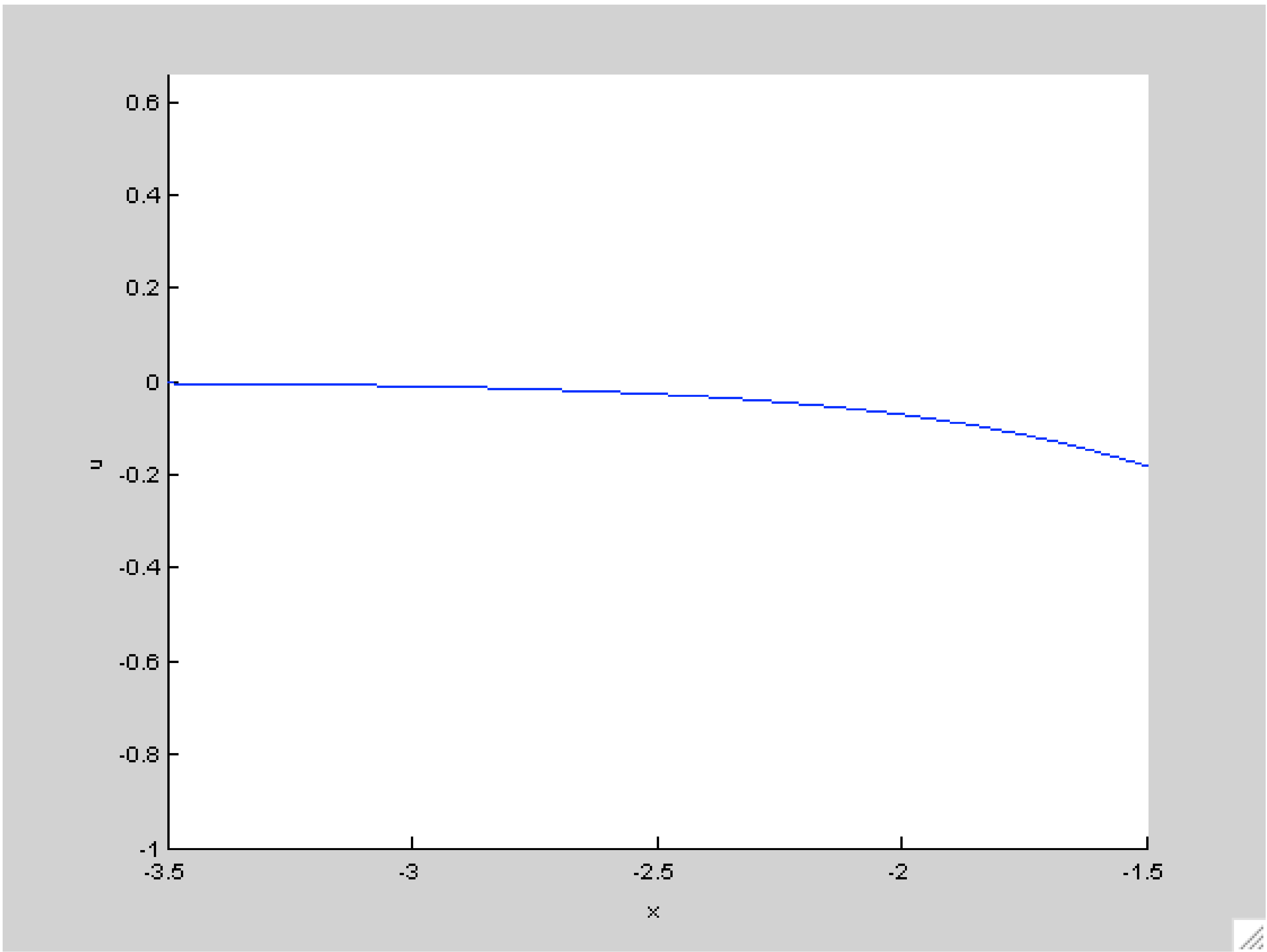


Hamiltonian regularization: Korteweg de Vries equation

$$u_t + uu_x + \epsilon^2 u_{xxx} = 0$$



Oscillatory zone



Hamiltonian perturbation of the Hopf equation

The Hopf equation $u_t + uu_x = 0$ is an Hamiltonian PDE

$$u_t + \{u, H_0\} = u_t + \partial_x \frac{\delta H_0}{\delta u(x)} = 0, \quad H_0 = \int \frac{u^3}{6} dx$$

with the G-FZ Poisson bracket $\{u(x), u(y)\} = \delta'(x - y)$.

Introduce slow variables $x \to \epsilon x, \quad t \to \epsilon t$ and classifiy perturbations up to Miura-type transformations

$$u \to u + \sum_{k=1}^{\infty} \epsilon^k F_k(u; u_x, \ldots, u^{(k)}),$$

with F_k graded homogeneuos polynomial in $u_x, u_{xx}, \ldots$ and $\deg F_k = k$.

Definition. A perturbation is trivial if it can be eliminated by a Miura transformation.

Hamiltonian perturbations:

1. Rigidity of the G-FZ bracket $\{u(x), u(y)\} = \delta'(x-y)$ (follows from triviality of the Poisson cohomology, Getzler, Magri et all. 2001).

2. It suffices to classifies perturbation of the Hamiltonian

$$H_0 = \int \frac{u^3}{6} dx, \quad H_\epsilon = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \dots$$

up to canonical transformations

$$u \to u + \epsilon\{u, F\} + \frac{\epsilon^2}{2}\{\{u, F\}, F\} + \dots,$$

$$H_\epsilon \to H_\epsilon + \epsilon\{H_\epsilon, F\} + \dots.$$

Theorem 2. [Dubrovin 2005] Any Hamiltonian perturbation of the Hopf equation up to terms of order ϵ^5 is equivalent to

$$u_t + u_x + \frac{\epsilon^2}{24}[2cu_{xxx} + 4c'u_xu_{xx} + c''u_x^3] + \epsilon^4[2pu_{xxxxx} +$$
$$+ 2p'(5u_{xx}u_{xxx} + 3u_xu_{xxxx}) + p''(7u_xu_{xx}^2 + 6u_x^2u_{xxx} + 2p'''u_x^3u_{xx}] = 0$$

where $c = c(u)$ and $p = p(u)$ are arbitrary functions. The coresponding Hamiltonian

$$H_\epsilon = \int \left[\frac{u^3}{6} - \frac{\epsilon^2}{24}c(u)u_x^2 + \epsilon^4 p(u)u_{xx}^2\right] dx + O(\epsilon^5).$$

Furthermore, any Hamiltonian perturbation is integrable up to order $O(\epsilon^4)$.

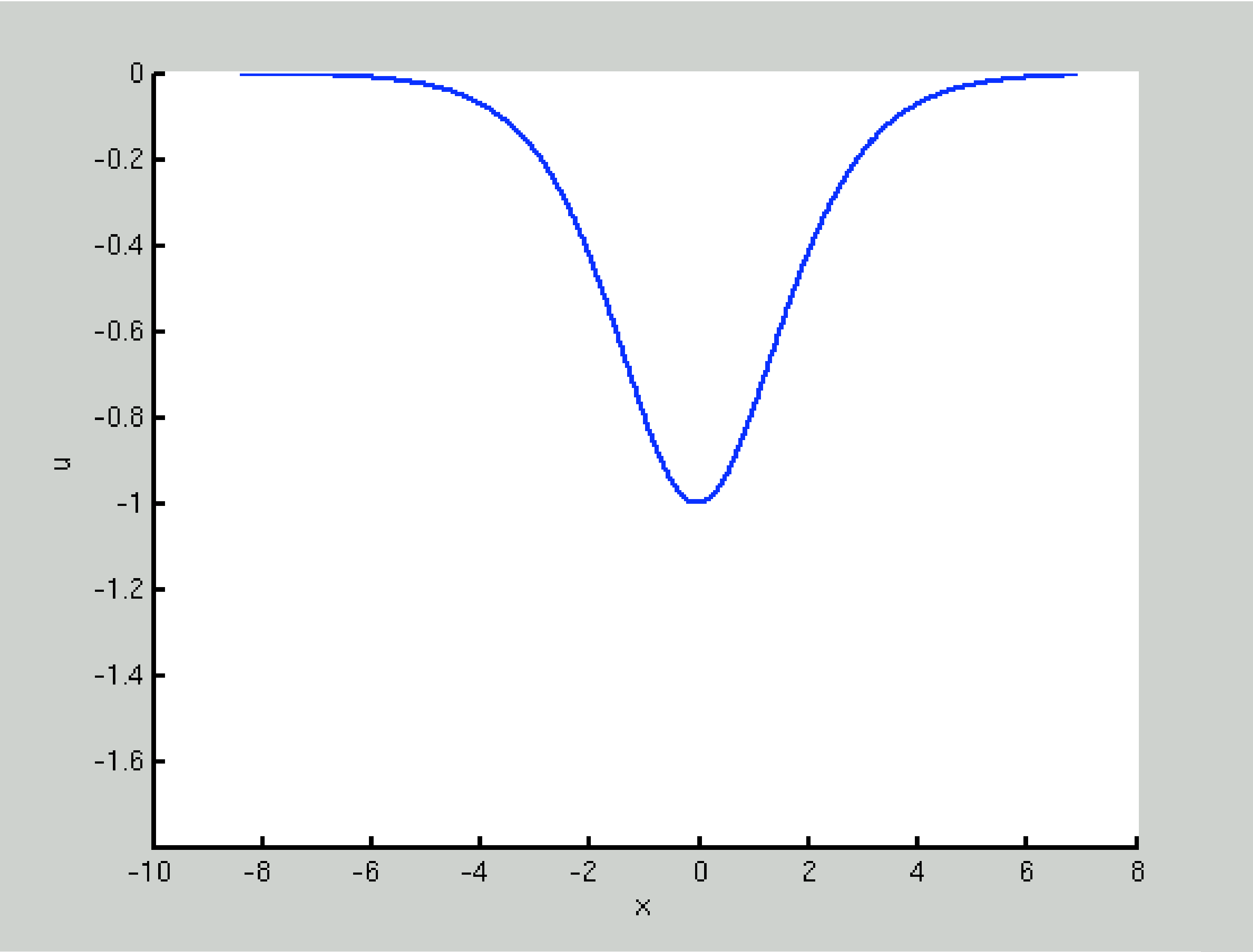
Examples

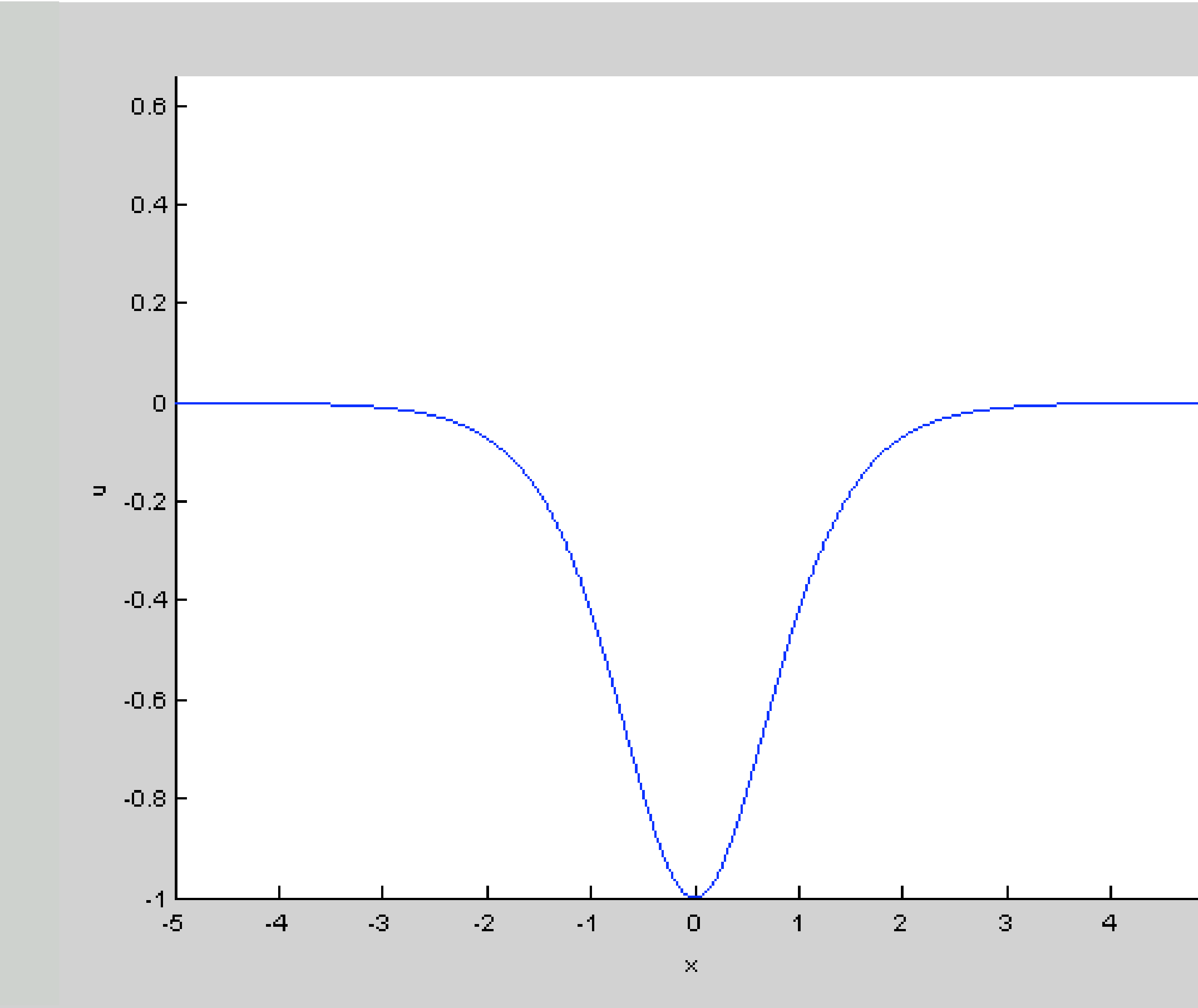
- $c(u) = 24,\ p(u) = 0$ corresponds to KdV $u_t + uu_x + \epsilon^2 u_{xxx} = 0$;
- $c(u) = 8u,\ p(u) = \dfrac{u}{3}$ corresponds to the equation (Camassa-Holm, Fokas-Fuchsteiner)

$$v_t - \epsilon^2 v_{xxt} + 3vv_x = \epsilon^2(2v_x v_{xx} + v_{xxx})$$

with $u = (v - \epsilon^2 v_{xx})^{\frac{1}{2}}$.

Numerical solution of CH-FF and KdV equations





Behaviour of the solution of the perturbed equation near the critical point (x_c, t_c, u_c)

Conjecture 1. [Dubrovin 2005] The solution of the Cauchy problem of any Hamiltonian perturbation of the Hopf equation, near the critical point (x_c, t_c, u_c) does not depend on the initial data and it is described (locally)

$$u(x,t,\epsilon) = u_c + \left(\frac{\epsilon^2 c_0}{k^2}\right)^{1/7} U\left(\frac{x - x_c - u_c(t-t_c)}{(\epsilon^6 k c_0^3)^{1/7}}, \frac{t-t_c}{(\epsilon^4 k^3 c_0^2)^{3/7}}\right) + O(\epsilon^{\frac{4}{7}}$$

where $U(X,T)$ is the unique real analytic solution of the Painlevé-I2 equation (T.Claeys, M.Vanlessen 2006)

$$X = TU - (U^3 + \frac{1}{2}U_X^2 + UU_{XX} + \frac{1}{10}U_{XXXX}).$$

(second member of the Painlevé I hierarchy) with boundary conditions

$$U(X) = \pm X^{\frac{1}{3}}, \quad X \to \mp\infty.$$

“Proof”

The solution of the Hopf equation near the point of gradient catastrophe (x_c, t_c, u_c) takes the form

$$\tilde{x} \simeq \tilde{t}(u - u_c) + \frac{1}{3!} f'''(u_c)(u - u_c)^3$$

where $\tilde{x} = x - x_c - 6t_c(u - u_c)$ and $\tilde{t} = t - t_c$. Let $h_k = \delta H_k / \delta u(x)$ be the KdV Hamiltonians such that $h_k = u^{k+2}/(k+2)! + O(\epsilon^2)$. For example

$$h_0 = \frac{u^2}{2} + \epsilon^2 u_{xx}, \quad h_1 = \frac{1}{6}(u^3 + \frac{\epsilon^2}{2}(u_x^2 + 2uu_{xx}) + \frac{\epsilon^4}{10} u_{xxxx}).$$

Then

$$x = tu + a_0 h_0 + a_1 h_1 + \dots a_k h_k,$$

is a symmetry of the KdV equation.

Setting $a_0 = 0$, $a_1 = |f'''(u_c)|/6$ and $a_{k>2} = 0$, $x \to \tilde{x}$, $t \to \tilde{t}$ and $u \to u - u_c$

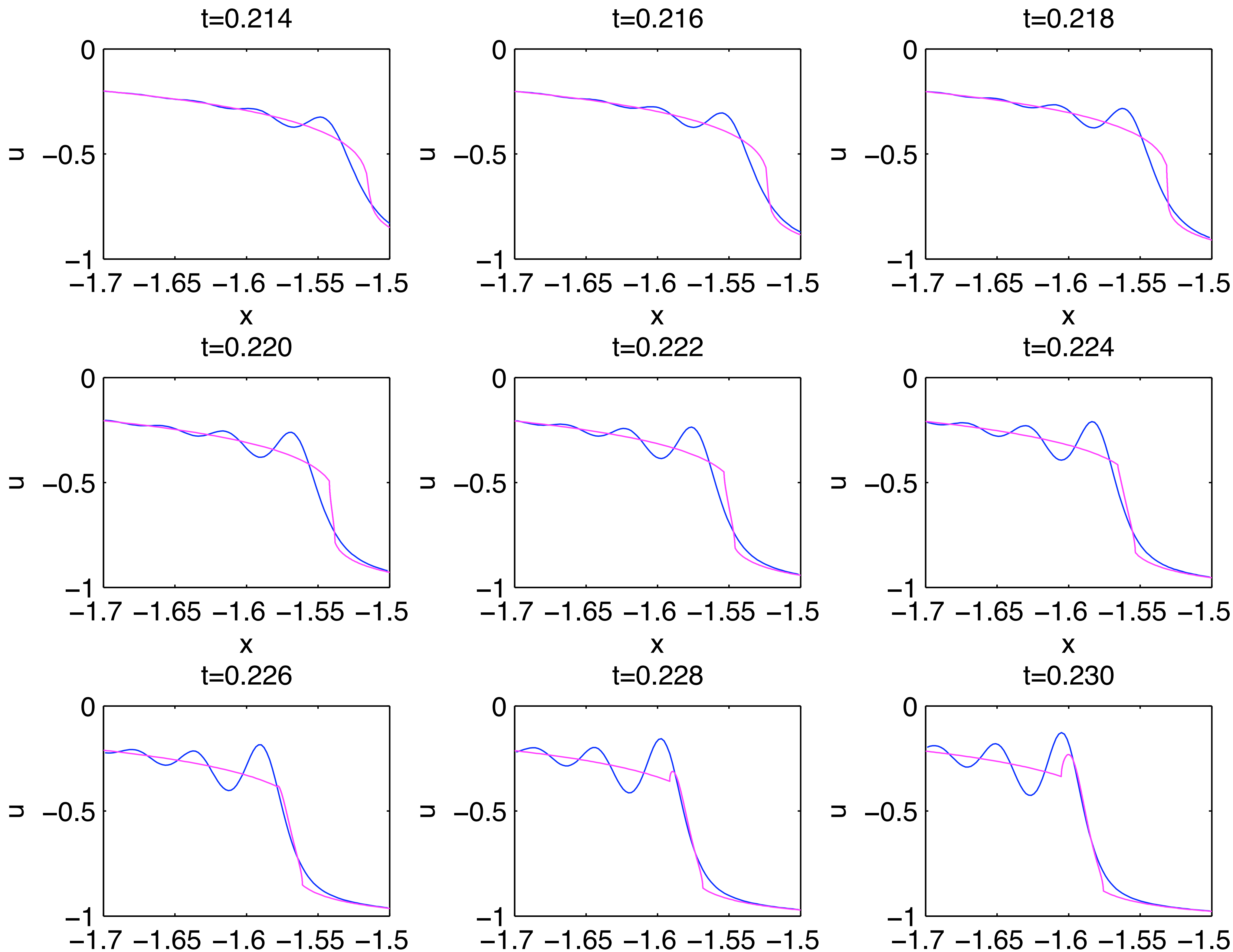
$$\tilde{x} = \tilde{t}(u - u_c) - \frac{|f'''(u_c)|}{6}[(u-u_c)^3 + \frac{\epsilon^2}{2}(u_x^2 + 2(u-u_c)u_{xx}) + \frac{\epsilon^4}{10}u_{xxxx}],$$

is still a symmetry of the KdV equation and it is equivalent up to shifts, rescalings and Galileian transformation to the ODE of Painlevé type ($PI2$, Brezin, Marinari Parisi),

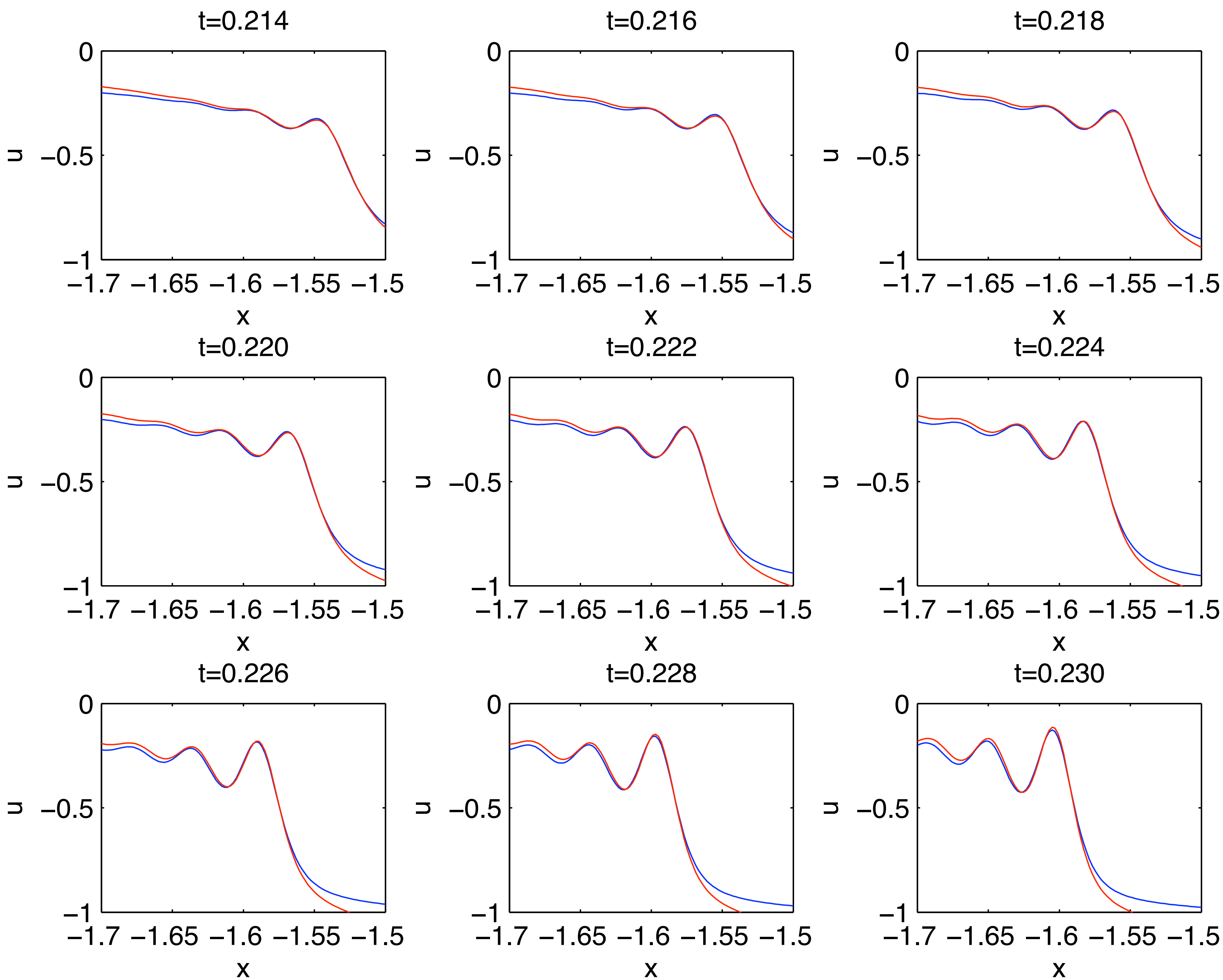
$$X = UT - \left[U^3 + \frac{U_X^2}{2} + UU_{xx} + \frac{1}{10}U_{XXXX}\right].$$

Example 1 (with C. Klein): $u_t + uu_x + \epsilon^2 u_{xxx} = 0$

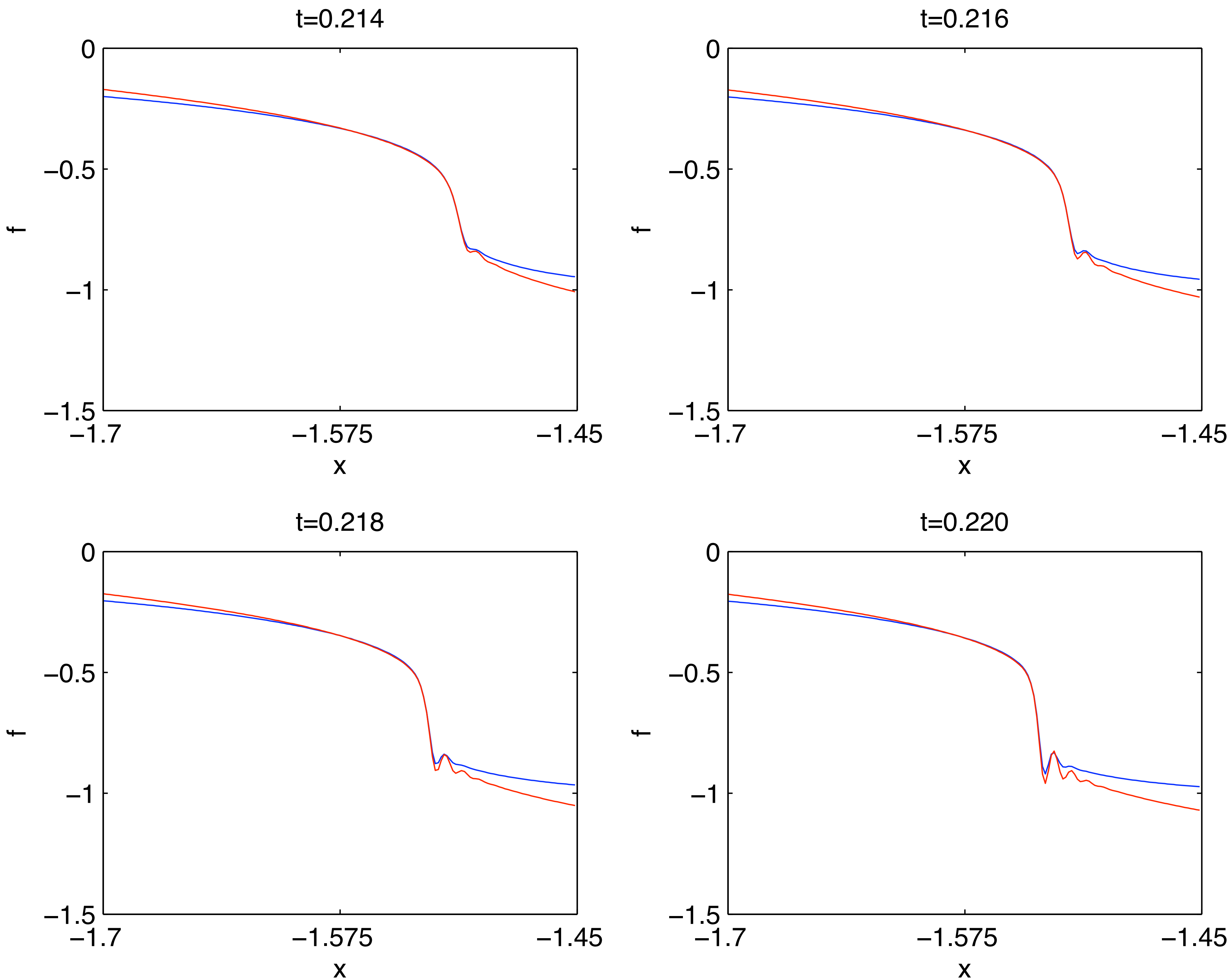
Comparison of KdV and asymptotic solution given by Lax-Levermore-Venakides theory at breakup time for ϵ=0.01



NUMERICAL SOLUTION OF KDV AND PI2 AT THE BREAK TIME (T.G. &C.Klein)



NUMERICAL COMPARISON of CH-FF equation and PI2



Nonlinear Schrödinger equation

- NLS equation

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} - \rho|\psi|^2\psi = 0.$$

- $\rho = -1$ focusing case, $\rho = 1$ defocusing case.
- Applications: nonlinear optics (fiber optics), hydrodynamic....
- Completely integrable (Zakharov-Shabat).
- Focusing case: modulation instability. Namely slow modulations of plane wave solutions develop fast oscillations in finite time.

Semiclassical limit $\epsilon \to 0$

Initial data:

$$\psi(x, t = 0, \epsilon) = A(x) e^{\frac{i}{\epsilon} S(x)}.$$

Introduce the new variables

$$u = |\psi|^2, \quad v = \frac{\epsilon}{2i} \left(\frac{\psi_x}{\psi} - \frac{\bar{\psi}_x}{\bar{\psi}} \right).$$

Then NLS is equivalent to

$$u_t + (uv)_x = 0$$

$$v_t + vv_x + \rho u_x + \frac{\epsilon^2}{4} \left(\frac{u_x^2}{2u^2} - \frac{u_{xx}}{u} \right)_x = 0.$$

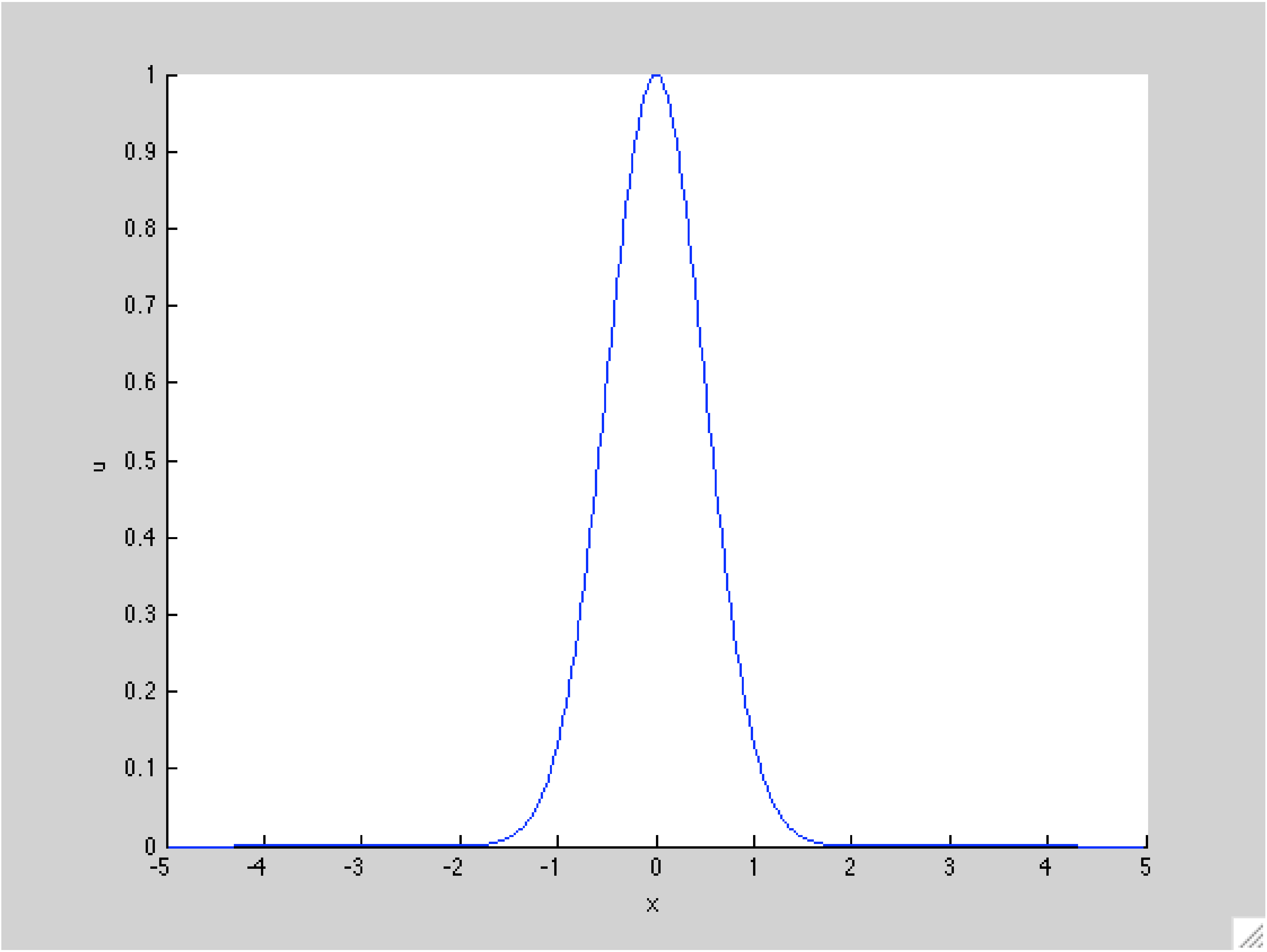
Defocusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\epsilon = 0.5,$$

$$0 \leq t \leq 1,$$

$$u = |\psi|^2$$



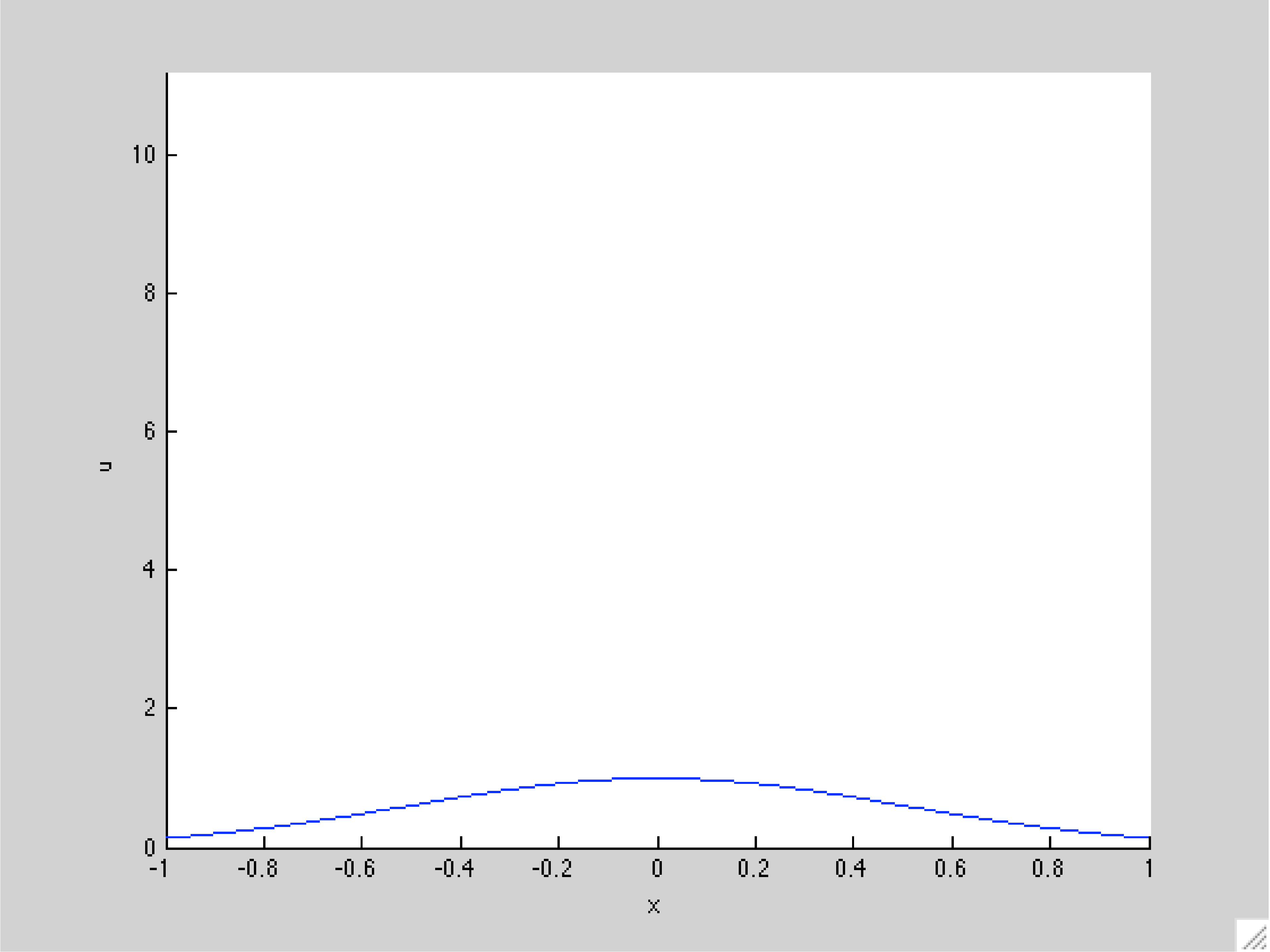
Focusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\epsilon = 0.1,$$

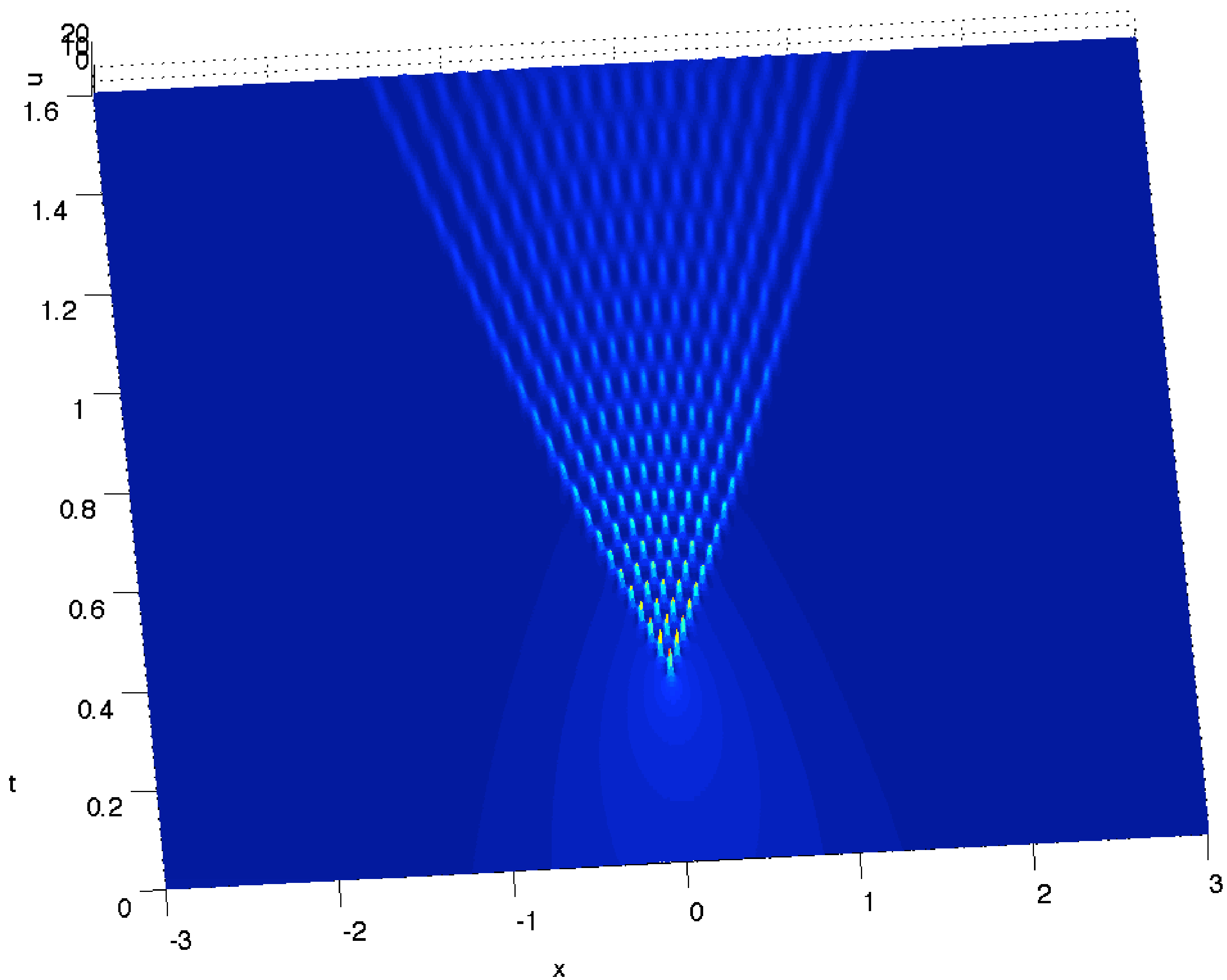
$$0 \leq t \leq 0.8,$$

$$u = |\psi|^2$$



Large Times

$$\psi_0 = -\mathrm{sech} x \exp(-i\mu \ln \cosh x), \quad t_c = 1/(2+\mu)$$



$\mu = 1$, solitonic

$\epsilon = 0.04$

- Semiclassical limit $\epsilon \to 0$ gives the hydrodynamic system

$$u_t + (vu)_x = 0$$
$$v_t + \rho u_x + v v_x = 0$$

- coefficient matrix $\begin{pmatrix} v & u \\ \rho & v \end{pmatrix}$, eigenvalues $v \pm \sqrt{\rho u}$
- semiclassical limit of defocusing NLS is similar to KdV, zeroth order equations are hyperbolic (Jin Levermore, McLaughlin 1999)
- semiclassical limit of focusing NLS is studied only for a special class of analytic initial data:
  1. Kamvissis, McLaughing and Miller (2003): $v(x, 0) = 0$, $u(x, 0)$ a bump.
  2. Tovbis, Venakides and Zhou, 2004, $u(x, t = 0) = \operatorname{sech}^2 x$, $v(x, 0) = -\mu \tanh x$

Zeroth order solution (with B. Dubrovin and C. Klein)

$$u_t + (vu)_x = 0$$
$$v_t + \rho u_x + v v_x = 0, \quad \rho = \pm 1.$$

Method of characteristics

$$x = vt + f_u(u, v)$$
$$0 = ut + f_v(u, v)$$

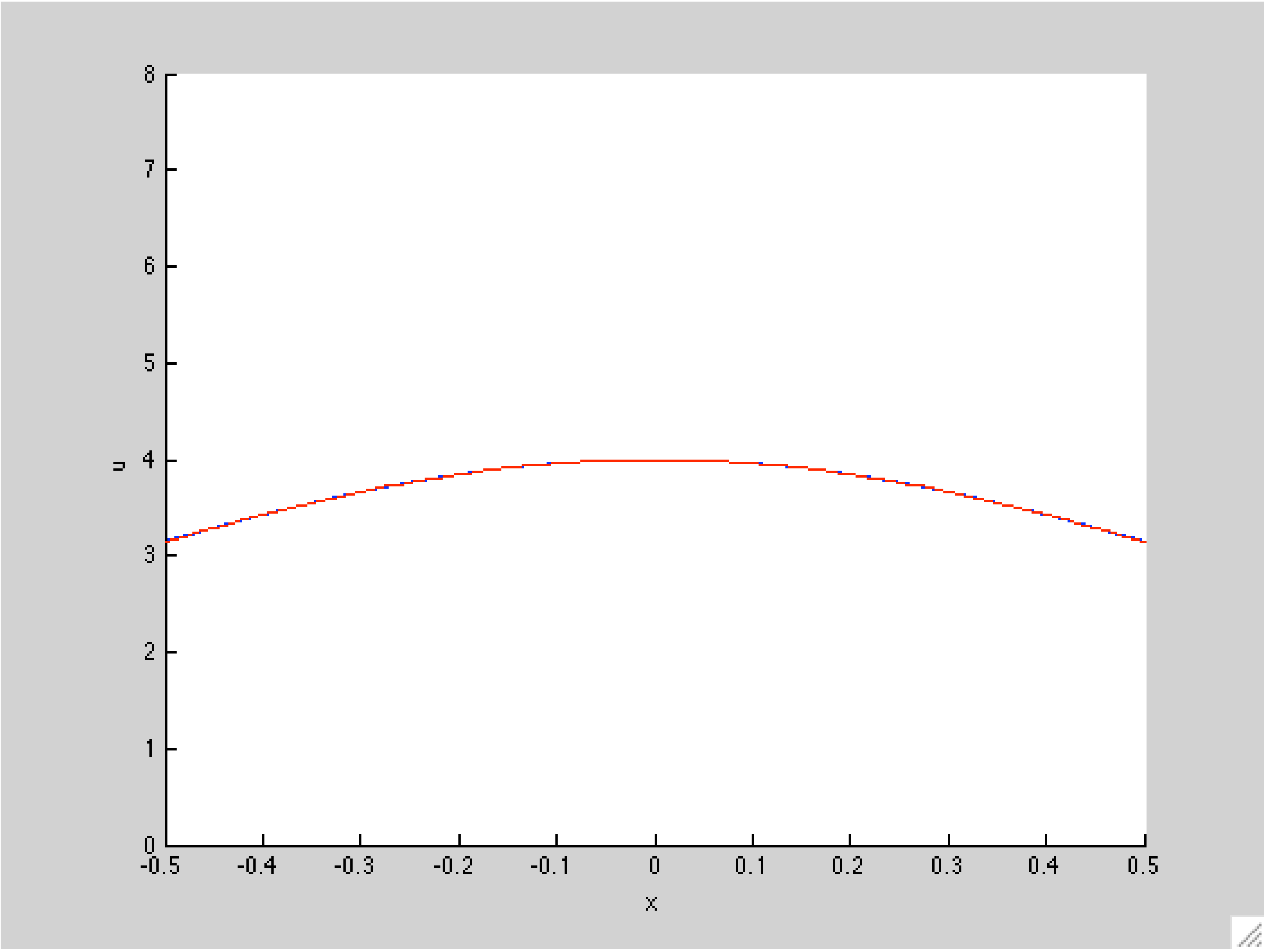
where $f = f(u, v)$ satisfies the equation

$$f_{vv} - \rho u f_{uu} = 0.$$

The solution of the above system exists provided applicability of the implicit function theorem. This condition fails to fulfill at the critical point (x_c, t_c, u_c, v_c).

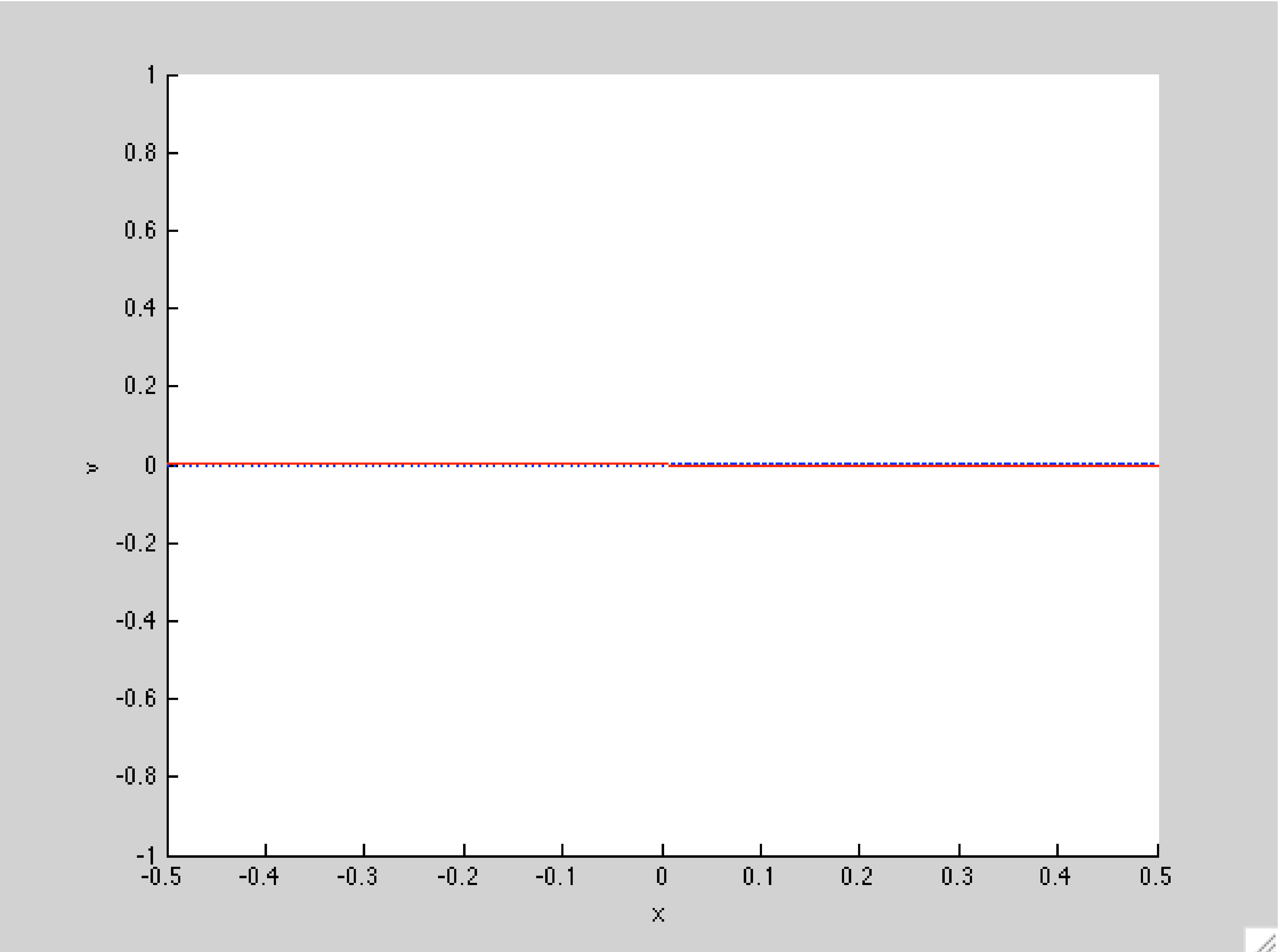
Example

$$u(x,0) = sech^2(x), v(x,0) = 0$$



$\epsilon = 0.04$

Phase



$$\epsilon = 0.04$$

Critical point (x_c, t_c, u_c, v_c) of FNLS

$$x_c = v_c t_c + f_u(u_c, v_c), \quad 0 = u_c t_c + f_v(u_c, v_c)$$

$$f_{uu}(u_c, v_c) = f_{vv}(u_c, v_c) = 0, \;\; f_{uv}(u_c, v_c) = -t_c,$$

and $f_{uuv}(u_c, v_c) \neq 0$. The solution of the elliptic system near the critical point at $t = t_c$ can be written in the form

$$\frac{1}{2}(\tilde{u}^2 - \tilde{v}^2) + a_+ = 0, \quad \tilde{u}\tilde{v} + a_- = 0$$

$\tilde{u} = u - u_c$, $\tilde{v} = v - v_c$ and $a_\pm = a_\pm(x - x_c, r, \phi)$, and $1/re^{-i\phi} = f^c_{uuv} + i\sqrt{u_c} f^c_{uuu}$.

Equivalent form: Euler Lagrange equations $\delta S = 0$

$$S = \int \mathcal{L}(\tilde{u}, \tilde{v}) dx, \quad \mathcal{L} = \frac{1}{6}(\tilde{u}^3 - 3\tilde{u}\tilde{v}^2) + a_+\tilde{u} + a_-\tilde{v}$$

The function $\mathcal{L}$ has an isolated singularity of type D_4 also called elliptic umbilic singularity according to R. Thom.

Comment: singularity of defocusing NLS

- Local behaviour: Whitney singularity

$$x_+ = r_+$$

$$x_- = r_+ r_- - \frac{1}{6} r_-^3$$

(by a nonlinear/linear change of dependent/independent variables $r_\pm = r_\pm(u - u_c, v - v_c)$, $x_\pm = a_\pm(x - x_c) + b_\pm(t - t_c)$).

- The behaviour of the solution of defocusing NLS at the critical point is decribed by the particular solution of the PI2 equation.

Solution of FNLS near the critical point

Conjecture 2a. The generic solution of FNLS near the critical point (x_c, t_c, u_c, v_c) behaves as follows

$$u(x,t,\epsilon)+i\sqrt{u_c}v(x,t,\epsilon) = u_c+i\sqrt{u_c}v_c+2\epsilon^{\frac{2}{5}}(3r\sqrt{u_c})^{\frac{2}{5}}e^{\frac{2i\phi}{5}}\Omega(\xi)+O(\epsilon^{\frac{4}{5}}),$$

$$\xi = \left(\frac{3r}{u_c^2}\right)^{\frac{1}{5}} e^{\frac{i\phi}{5}}\left[\frac{-u_c\tilde{t}+i\sqrt{u_c}(\tilde{x}-v_c\tilde{t})+\frac{1}{2}re^{i\phi}\tilde{t}^2}{\epsilon^{\frac{4}{5}}}\right]$$

and $\Omega(\xi)$ is the tritronquée solution of the Painlevé-I equation

$$\Omega_{\xi\xi} = 6\Omega^2 - \xi,$$

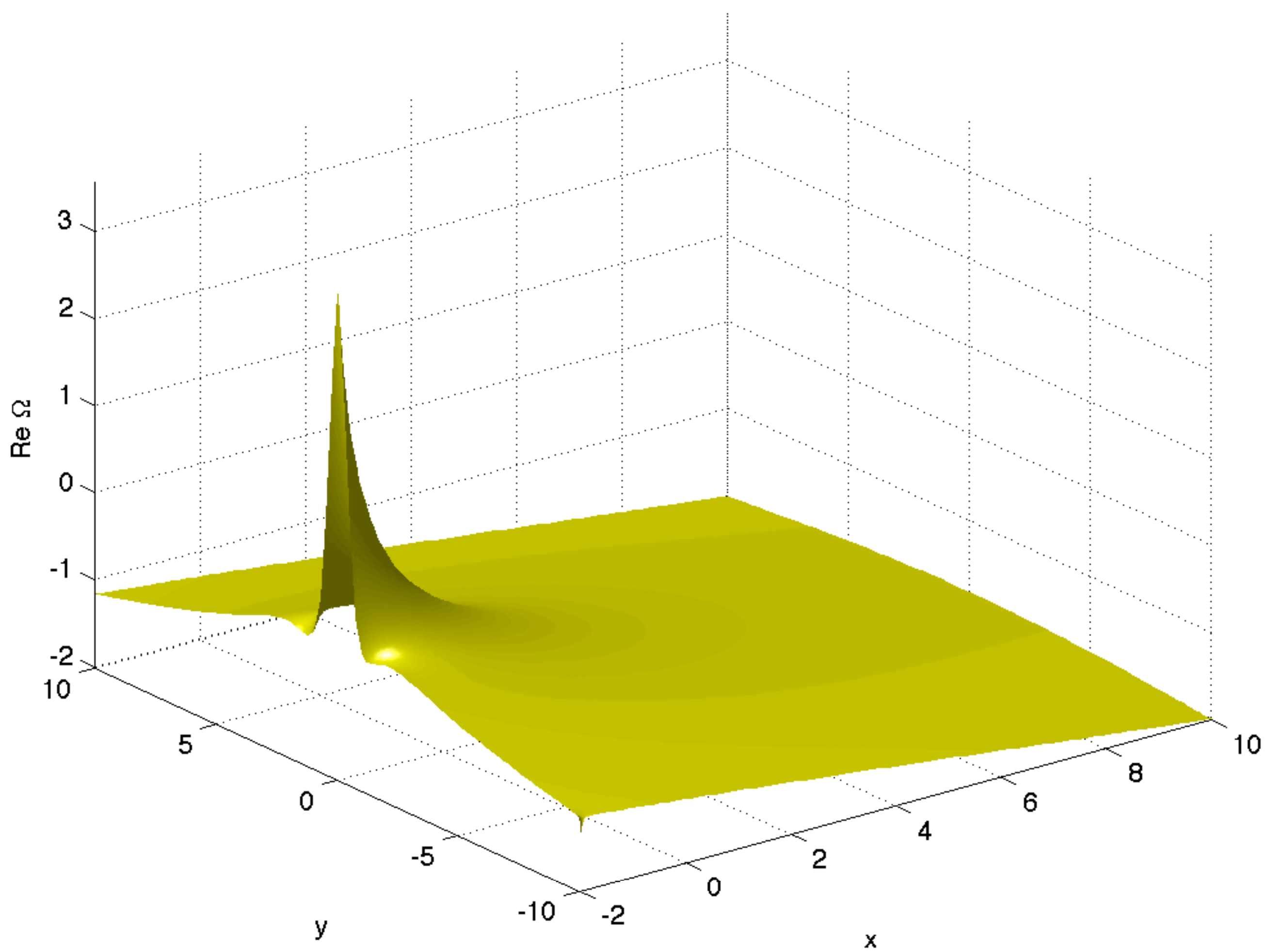
such that

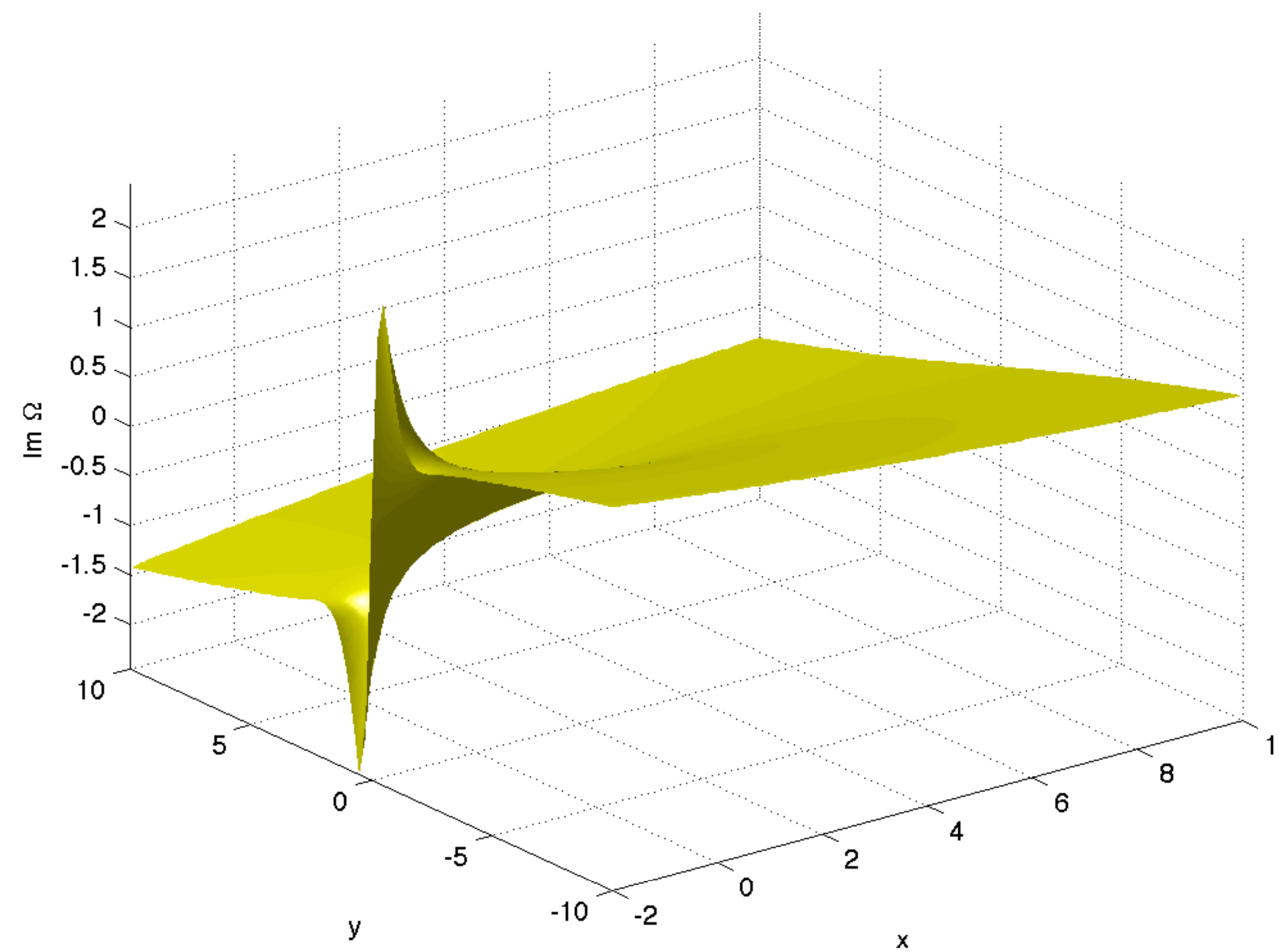
$$\Omega(\xi) = -\left(\frac{\xi}{6}\right)^{\frac{1}{2}}\left[1+O(\xi^{-\frac{3}{4}})\right] \quad |\xi| \to \infty.$$

Remark. The tritronquée solution of PI has no poles for large ξ and $|\arg \xi| < \dfrac{4}{5}\pi$. (Boutroux (1913), Joshi-Kitaev (2001)).

Conjecture 2b. The tritronquée solution of PI has no poles for $|\arg \xi| < \dfrac{4}{5}\pi$.

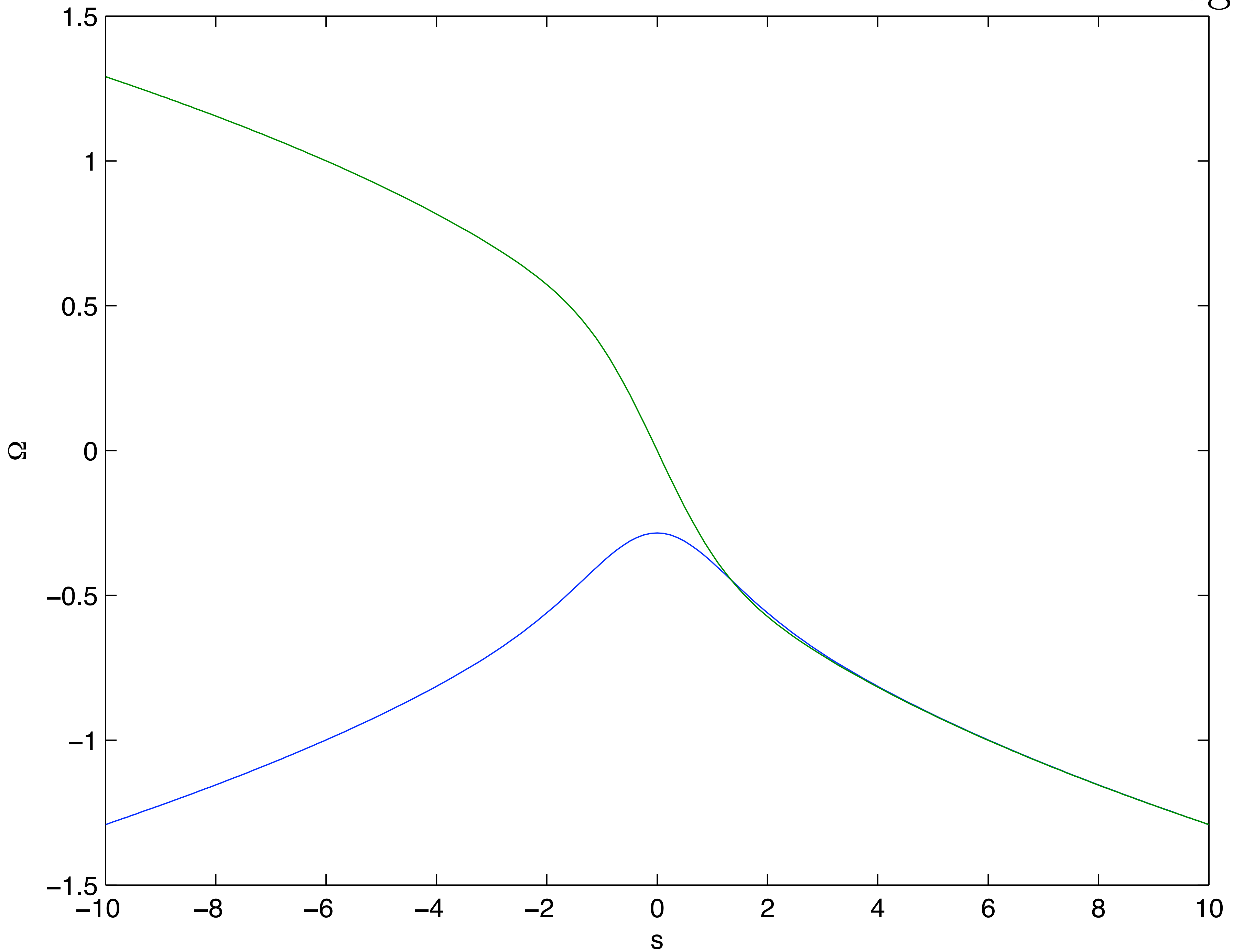
tritronquée solution of PI





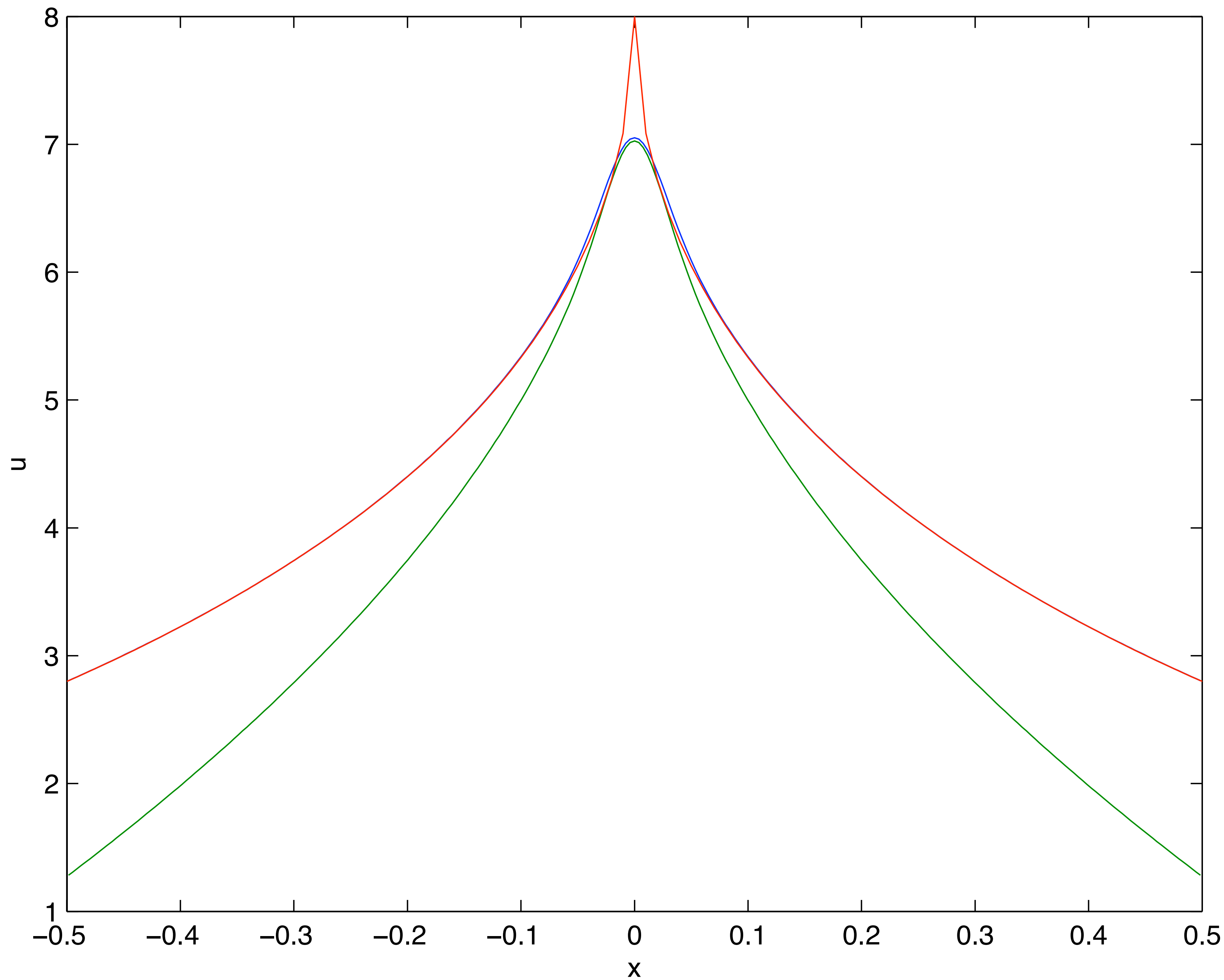
tritronquée solution

solution on the imaginary axis
real part blue,
imaginary part green



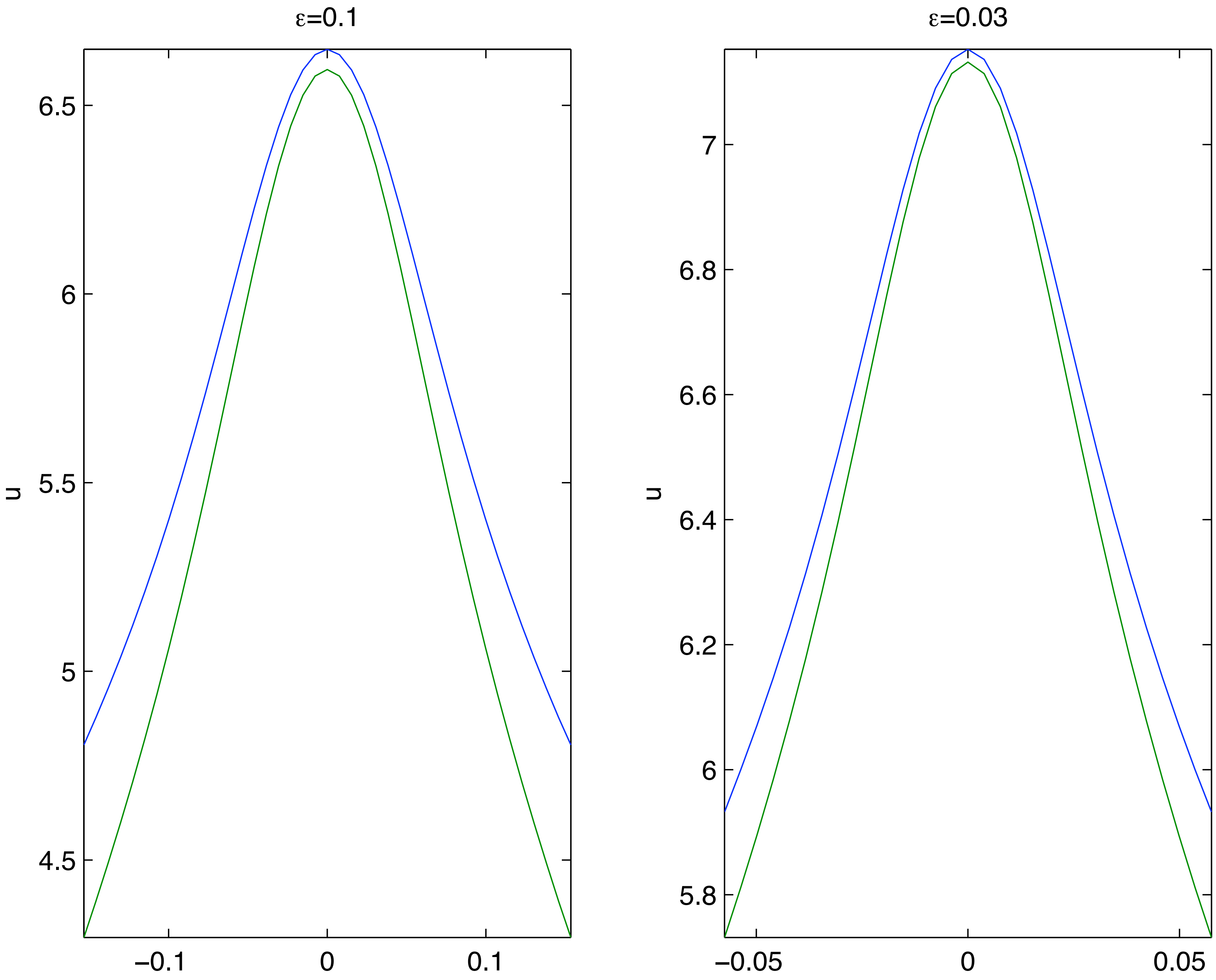
Critical point for the initial data

$$u(x,0) = sech^2(x), v(x,0) = 0$$



ϵ dependence

$$|u - u_{tri}| \propto \epsilon^{4/5}$$

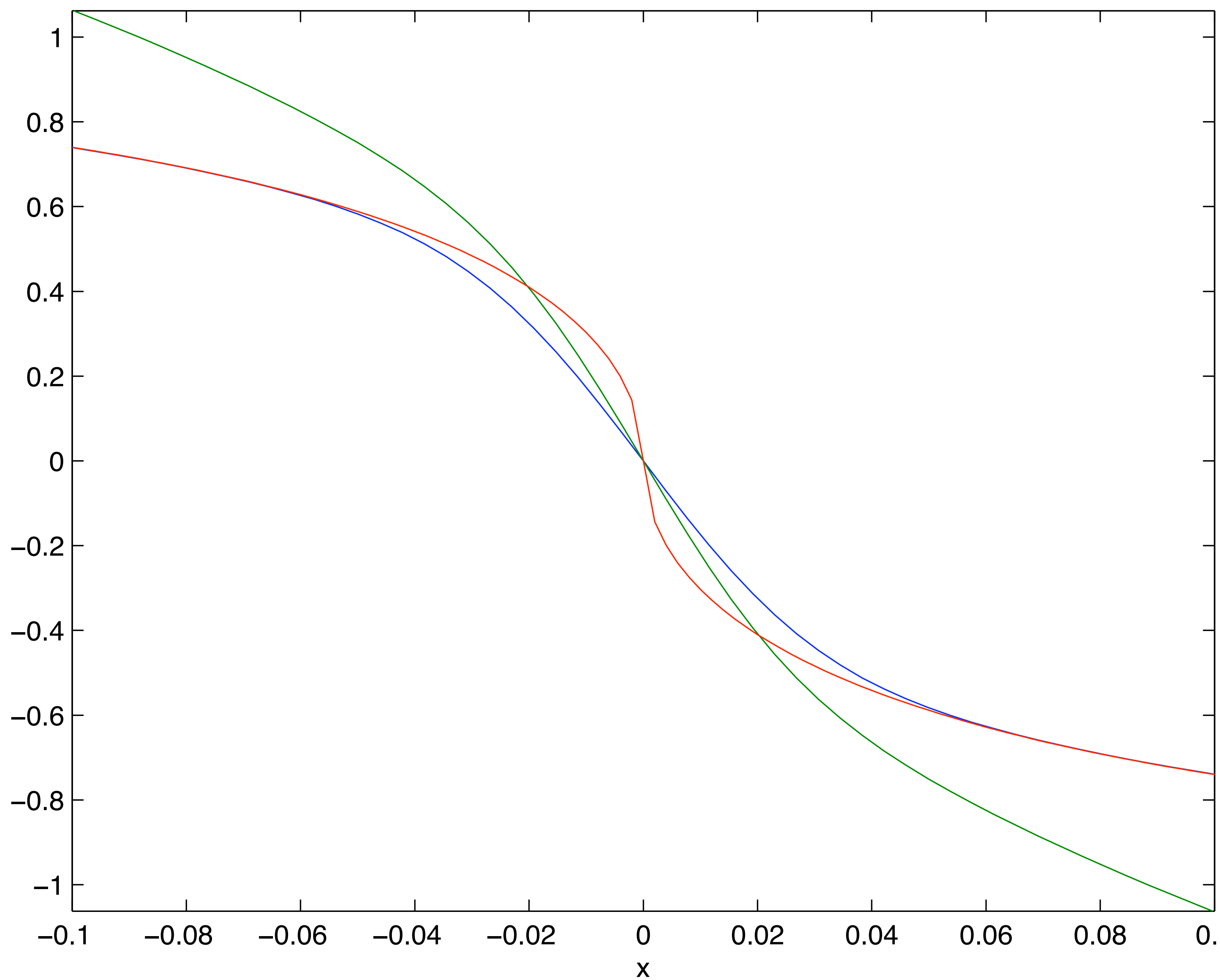


Summary

- The behaviour of solutions of Hamiltonian PDEs at critical points is universal in the sense that it does not depend on the initial data.
- This behaviour is described by a particular solution of an ODE.

Thank you for your attention!

Critical point



Numerical methods

- task: resolve steep gradients in rapid oscillations
- Fourier series for spatial coordinates, method of lines
- fourth-order time stepping to avoid aliasing, integrating factor method (fourth-order Runge-Kutta), exponential time differencing, sliders (Driscoll), time splitting
- Krasny filtering (modulational instability) or more than double precision ($\epsilon>0.025$), dealiasing

- Fourier space: equation of the form

$$U_t = cU + N[U]$$

  here: U vector (1+1) or matrix (2+1), c array, $N[U]$ convolution, steep gradients: high frequency terms in c lead to large absolute values despite small ϵ

- exponential time differencing: time discretization and integration with integrating factor

$$U(t_n + h) = e^{ch}U(t_n) + \int_0^h d\tau e^{c(h-\tau)} N[U(t_n + \tau)]$$

  fourth-order Runge-Kutta scheme (Cox-Matthews), coefficients via contour integrals (Kassam-Trefethen)

- integrating factor, fourth-order Runge-Kutta (e.g. Trefethen):

$$\left(e^{-ct}U\right)_t = e^{-ct}N[U]$$

Comment: singularity of defocusing NLS

- Local behaviour: Whitney singularity

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- The behaviour of the solution of defocusing NLS at the critical point is decribed by the particular solution of the PI2 equation.