Poisson pencils, Integrability, and Separation of Variables

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Outlook

- Main aim: use tools from the theory of bihamiltonian manifolds/structure to solve integrable systems (finite-dimensional ones)
- Solve = provide methods to separate Hamilton-Jacobi equations, i.e., to reduce the solution of H-J eq. to the solution of ODEs.
- Secondary aim: discuss some related items in the theory of (bi)hamiltonian systems/manifolds.
- The talk is based on joint work with F. Magri, M. Pedroni and other collaborators.

Plan of the talk:

- Definitions and remarks
- Integrability, H-J equations and separation of variables.
- Criteria of separability for “regular” (or symplectic) bihamiltonian manifolds.
- The bihamiltonian scheme for SoV of a “Gel’fand-Zakharevich” finite-dimensional system.
- Example: periodic Toda lattices.
- Time permitting: a “symmetric” so(4) rigid body.

Bihamiltonian structures on manifolds

- The notion was introduced by F. Magri in the study of the KdV equation,

$$u_t = u_{xxx} + 6\,u\,u_x = \{u(x), H_2\}_1 = \{u(x), H_1\}_2$$

$$\{u(x), u(y)\}_1 = \delta'(x-y),$$

$$\{u(x), u(y)\}_2 = \frac{1}{2}\delta'''(x-y) + 2u(x)\,\delta'(x-y) + u_x(x)\delta(x-y)$$

$$H_1 = \int u^2\,dx, \qquad H_2 = \int \left(u^3 - \frac{u_x^2}{2}\right) dx$$

Definitions

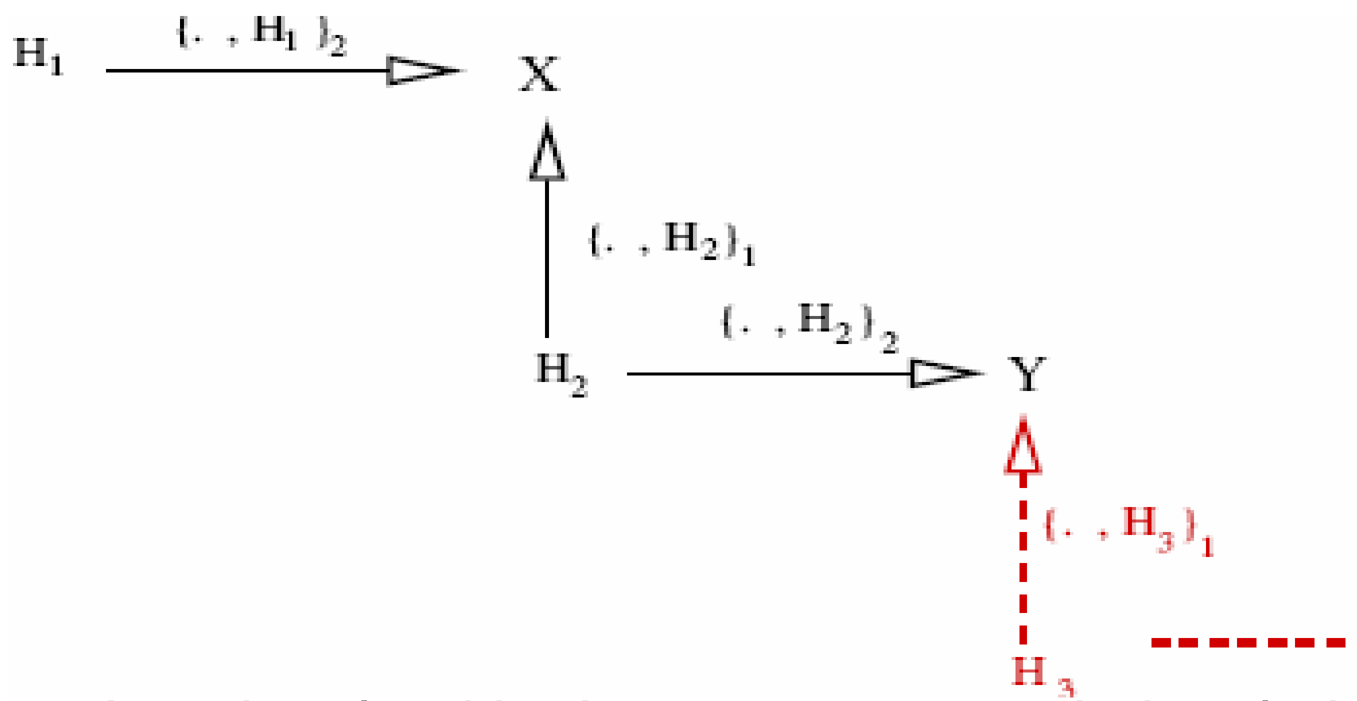
- A bihamiltonian manifold is a manifold endowed with a linear pencil of Poisson brackets (tensors):

$$\{f,g\}_\lambda = \{f,g\}_2 - \lambda\{f,g\}_1 = \langle df, P_2\, dg\rangle - \lambda\langle df, P_1\, dg\rangle = \langle df, P_\lambda dg\rangle$$

- i.e. $[P_i, P_j]_{\mathrm{Schout}} = 0, \quad i,j = 1,2.$
- Bihamiltonian vector field:

$$\{\cdot, H_1\}_2 = P_2\, dH_1 = X = P_1\, dH_2 = \{\cdot, H_2\}_1$$

Lenard-Magri Sequences



- All functions involved in the sequence are in involution (w.r.t. both brackets).
- Basically thanks to compatibility, the sequence can be iterated.
- This yields (or, at least, indicates) “integrability” of the system.

Gel'fand-Zakharevich (finite dim.) Systems

- Anchored sequences: they start with a Casimir of one bracket and end with a Casimir of the other. Hamiltonians can be grouped in GZ polynomials:

$$H^{(a)}(\lambda) = \sum_{i=1}^{N_a} \lambda^i H_i^{(a)}, \text{ with } \left(P_2 - \lambda P_1\right) d\, H^{(a)}(\lambda) = 0$$

- All Hamiltonians $H^{(a)}_j$ commute between each other.

- If $N_1+N_2+\ldots+N_k-k=\frac{1}{2}(\mathrm{rank}(P_1))$ the restriction to generic symplectic leaves of these system is Liouville-integrable, provided the Hamiltonians are functionally independent.

SoV- Formulation of the problem

- (M,ω) be a symplectic 2n-dimensional manifold, equipped with canonical coordinates (**p**,**q**).
- Stationary Hamilton-Jacobi (HJ) equation:

$$H(q_1, \ldots, q_n; \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}) = E$$

i.e, a PDE for S=S(**q**), obtained setting **p**=∂S/∂**q** in the Hamiltonian function (E is a constant here)

SoV standard definition(s)

- H is separable in the coordinates (**p**,**q**) if HJ admits a complete integral S=S(**q**;**α**), depending on n constants **α** (s.t. det $\partial^2 S/\partial \mathbf{q}\, \partial \boldsymbol{\alpha} \neq 0$) having the additive form:

$$S(\mathbf{q}; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} S_i(q_i; \alpha_1, \ldots, \alpha_n)$$

- Suppose $H=H_1$ is integrable, i.e., there are n-1 other mutually commuting integrals of the motion $H_2, \ldots. H_n$. Then a second – more “constructive” definition of Separability can be given.

Separation relations

- An Integrable System (H_1,.....H_n) is separable in the canonical coordinates (**p**,**q**) iff there exist n non trivial relations $\Phi_i(q_i, p_i; H_1, \ldots, H_n) = 0,$ tying single pairs of canonical coordinates with the “Hamitonians”.
- Separation relations allow to reduce to quadratures the problem of finding an additively separated solution of HJ as follows:

$$S(\mathbf{q}; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \int^{q_i} p_i(q_i'; H_1, \ldots, H_n)\Big|_{H_i = \alpha_i} dq_i' \,.$$

A *voil d'oiseau* retrospective of the SoV problem

- Separating HJ equations involves clever coordinate choices, and ingenuity in finding the separation relations (i.e., in determining the additively separated complete integral). This was known from the classical works of Jacobi and Stäckel.
- E.g.: Stäckel's Thm (end of XIX century):

  H (quadratic in the momenta) is separable in the coordinates (**p**,**q**) if there exists a matrix M(q), and a column vector U(q), s.t.:

  a) the i-th row of M and the i-th element of U depend only on the i-th coordinate, one column of M is normalized to 1;

  b) H is among the solutions of $\sum_i M_{ij}(q_i) H_j = p_i^2 + U_i(q_i)$

- Eisenhart (30'ies): first look at the free problem:separate geodesic motion through the theory of orthogonal nets, and them determine which potentials preserve the separation property.

More recently (>80): Lax setting

- Let us suppose that

$$\frac{d}{dt}L(\lambda) = [L(\lambda), M(\lambda)]$$

  is a Lax representation with “spectral parameter” for our system, such that the spectral invariants of L(λ) provide a complete set of mutually commuting “Hamiltonians”. Then – under some circumstances – the spectral curve itself provides Jacobi separation relations.

Sklyanin's "magic recipe"

- Separation coordinates are projections on – respectively - the (λ,μ)-plane of a suitable set of points P_i belonging to the spectral curve
$$\mathrm{Det}\,(\mu - L(\lambda)) = 0$$
or suitable functions thereof.

- In the worked-out cases, these points are the poles of suitably normalized Baker-Akhiezer vector-function ψ, i.e. a suitable solution of
$$L(\lambda)\Psi = \mu\Psi\,.$$

- Further instance of a fallback on the classical case of notions introduced to solve the corresponding quantum problem (Bethe Ansatz equations).

The “Bihamiltonian” setting

- Two steps:

A) Define suitable coordinates in terms of the geometry of a “regular” bihamiltonain manifold and characterize separability in these coordinates

B) Consider Gel’fand-Zakharevich systems and devise methods to apply the “recipes” of point A.

Regular bihamiltonian manifolds

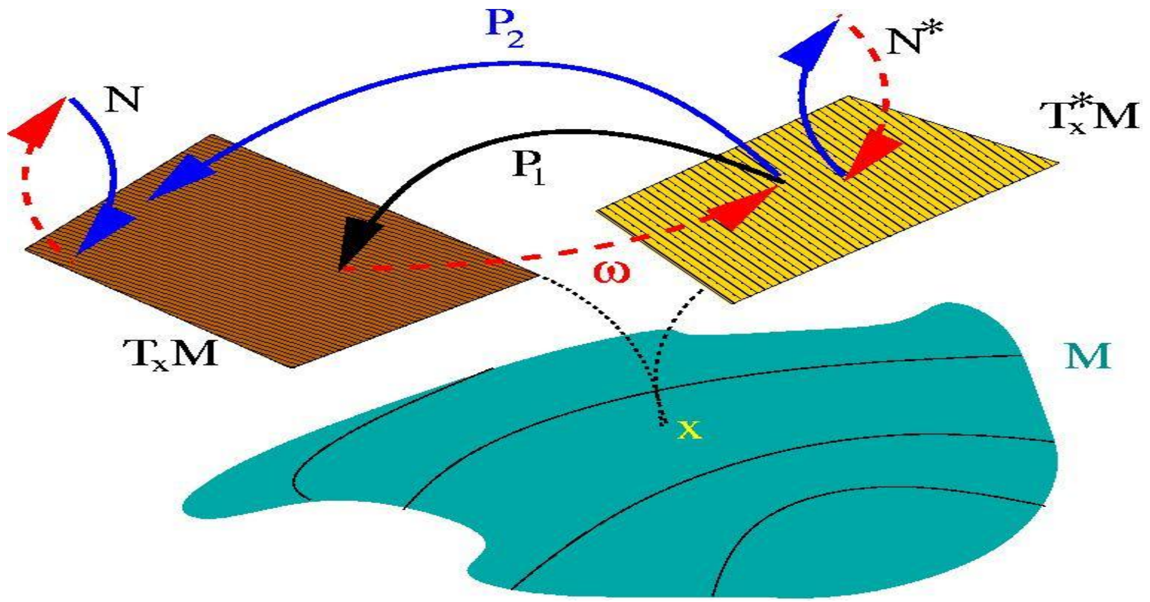
Let (M, P_1, P_2) be a 2n-dimensional bihamiltonian manifold.

It is called regular when one of the two Poisson tensors (say, P_1) is "symplectic", that is, is the inverse of a symplectic form on M,

$$\omega = (P_1)^{-1}$$

The “Nijenhuis” tensor

- One can define the tensors $N = P_2 \circ \omega$, (and $N^* = \omega \circ P_2$).



- Thanks to compatibility of the Poisson tensors, the Nijenhuis torsion of N vanishes.

Darboux-Nijenhuis (DN) Coordinates

- Defined basically by the eigenspaces of N*:
- N* has pointwise at most n=½(dim M) different eigenvalues (they are those of N)
- We assume they are (generically) distinct: $\lambda_1,\ldots\ldots,\lambda_n$.
- (Thm) There are n pairs of functions $f_1,g_1,\ldots,f_n,g_n$ that diagonalize N*:

$$N^* df_i = \lambda_i\, df_i, \quad N^* dg_i = \lambda_i\, dg_i$$

and provide canonical coordinates:

$$\{f_i, f_j\} = \{g_i, g_j\} = 0, \quad \{f_i, g_j\} = \delta_{ij}$$

“Semi-simple” bihamiltonian manifolds

- In the majority of worked-out cases half of the DN coordinates can be extracted from the Nijenhuis tensor itself.
- Thm: if the “eigenvalues” of N*, $\lambda_1, \ldots, \lambda_n$, are functionally independent they provide DN coordinates, since it holds

$$N^* d\lambda_i = \lambda_i \, d\lambda_i$$

- For the remaining half of canonical coordinates, no algorithms, but a few recipes.
- When no functional independence, ad hoc techniques.

Separability condition(s)

- Back to our basic questions:

A) When an integrable system $(H_1,\ldots..H_n)$ on a regular bihamiltonian manifold is separable in DN coordinates?

B) When an Hamiltonian H, defined on a regular bihamiltonian manifold is separable (and, a fortiori, integrable) in DN coordinates?

Question B

- This is more fundamental; it can be answered on general grounds, although few examples are known (and in the rest of the talk I will concentrate on question A).
- Basically the idea is the following: a Hamiltonian system on a regular bihamiltonian manifold consists of the datum of three objects:

A) The symplectic structure ω;

B) The Nijenhuis operator N;

C) The Hamilton function H.

The path to discuss the separability of H in the coordinates defined by N consists in combining the different geometrical objects that can be formed with these basic ingredients.

- H and ω give rise to the Hamiltonian vector field X_H, s.t.

$$\omega(\,\cdot\,, X_H) = -\langle dH, \,\cdot\, \rangle.$$

- This vector field can be iterated by means of N, and gives rise to the (at most n-dimensional) distribution

$$\mathcal{D}_H := \langle X_H, N\cdot X_H, N^2\cdot X_H, \ldots\ldots \rangle$$

- H and N produce the 2-form $\omega_H = d(N^*\cdot dH)$
- Proposition: H is separable in the Darboux - Nijenhuis coordinates iff ω_H vanishes on $\mathcal{D}_H$.
- Proof: these vanishing conditions are equivalent to the classical Levi-Civita conditions for separability in the given set of DN coordinates.
- However, remark that the conditions are tensorial (can be checked a-priori in any set of coordinates).

Classical (Mechanical) integrable systems

- M is a cotangent bundle T^*Q, with the canonical symplectic form $\omega = d(\mathbf{p}\, d\mathbf{q}) = d\theta$.
- N is obtained from a torsionless tensor on Q, $L: TQ \to TQ$ via $\omega' = \omega \circ N = d(\mathbf{p}\, L^* d\mathbf{q})$.
- The torsion (on T^*Q) of N vanishes as a consequence of the vanishing of the torsion (on Q) of L.
- The eigenvalues of L provide half of the separating coordinates.
- The second half are their conjugate momenta.

Example: the Neumann System

- Particle on S^n, with harmonic potential:

$$\bar{H} = \frac{1}{2}\sum_{i=1}^{n+1} y_i^2, + \frac{1}{2}\sum_{i=1}^{n+1} \alpha_i {x_i}^2,$$

$$\alpha_i \neq \alpha_j, \quad (\mathbf{x},\mathbf{x}) = 1, \quad (\mathbf{y},\mathbf{x}) = 0$$

- L-tensor on S^n is the restriction of

$$\hat{L}\frac{\partial}{\partial x_i} = \alpha_i \frac{\partial}{\partial x_i} + \frac{x_i}{r^4}\sum_{j,k}(\alpha_k - \alpha_j - \alpha_i){x_k}^2 x_j \frac{\partial}{\partial x_j}$$

- Eigenvalues of L are elliptic coordinates on S^n,

$$\frac{\prod_{a=1}^{n}(\lambda-\lambda_a)}{\prod_{i=1}^{n+1}(\lambda-\alpha_i)} = \sum_{i=1}^{n=1} \frac{x_i^2}{\lambda-\alpha_i}, \quad \text{with} \ \sum_{i=1}^{n=1} x_i^2 = 1.$$

- A sufficient condition for the vanishing of

$$dd_N(\bar{H})\big|_{\mathcal{D}_{\bar{H}}} \ \text{is} \ d\left(N^* d\bar{H} - \frac{1}{2}\mathrm{Tr}(N)\, d\bar{H}\right) = 0$$

- Using this formula, one can retrieve the Uhlenbeck integrals solving the iterative equations

$$dH_{i+1} = N^* dH_i - c_i dH_1,$$

where H_1 is the Neumann Hamiltonian, and the c_i's are the coefficients of the characteristic polynomial of L.

Question A: (Separability of Integrable systems)

- We are given a integrable system $(H_1,\ldots..H_n)$ on a regular bihamiltonian manifold

$$(M,\omega,N)=(M,\ P_1=\omega^{-1},\ P_2=N\circ\omega^{-1})$$

$$\{H_i,H_k\}_1=\langle dH_i,P_1\, dH_k\rangle=0, \text{ all } i,k\text{'s.} \quad (*)$$

- Theorem: The system is separable in the Darboux Nijenhuis coordinates defined by N if the (co)distribution generated by the dH_i is invariant under N^*.

- Equivalently: …. if the hamiltonians H_j are involution also w.r.t the second bracket, i.e., along with (*) it holds

$$\{H_i,H_k\}_2=\langle dH_i,P_2\, dH_k\rangle=0, \text{ all } i,k\text{'s.}$$

Geometric interpretation of point 2:

- Integrability: lagrangian foliation of a symplectic manifold.

- Separability (in DN coordinates): “bilagrangian” foliation of a symplectic manifold, endowed with a Nijenhuis (compatible) tensor.

Remarks on Point 1

- Invariance of $\mathscr{D}^*_H$:=span of dH_1, dH_2,…. means that there is a matrix valued function ***F*** s.t.

$$N^* dH_i = \sum_{k=1}^{n} F_{ik}(x) dH_k \quad \underbrace{\Longleftrightarrow}_{(N^*=P_1^{-1}\cdot P_2)} \quad P_2 dH_i = \sum_{k=1}^{n} F_{ik} P_1 dH_k$$

- This entails that, in general, the system needs not to be bihamiltonian in the strict sense, since these relations in general are not the Lenard-Magri relations.

- The bihamiltonian scheme requires only that Hamiltonians are in involutions w.r.t. a pair of brackets, and not strictly that they obey “standard” recursion relations.

Applications to GZ systems

- Recall that a GZ system is a bihamitonian system defined on a bi-Poisson (non symplectic) manifold.
- Liouville-Arnol'd integrable systems are obtained by restriction to the symplectic leaves of one (say, the first) Poisson structures.

- More in details GZ systems are characterized by “GZ polynomials” that start with a Casimir of one bracket and end with a Casimir of the other and are formed by “Hamiltonians” that commute between each other w.r.t. both brackets.
- What is lacking here is a “honest” symplectic manifold and the Nijenhuis tensor.
- A natural question is whether we can provide, out of these data what we need to separate variables, that is,

-) A symplectic manifold with a compatible Nijenhuis tensor (i.e. a regular bihamiltonian manifold) such that

-) the given - or better, a suitable (=complete in the Liouville-Arnol’d sense) subset of - Hamiltonians satisfy the SoV requirements discussed above.

“Bihamiltonian” reduction: one non common Casimir

- We consider the non-common Casimir H_N of the selected (“first”) Poisson tensor P_1, and the vector field

$$X=P_2\, d\, H_N$$

- We seek for a vector field Z s.t.:

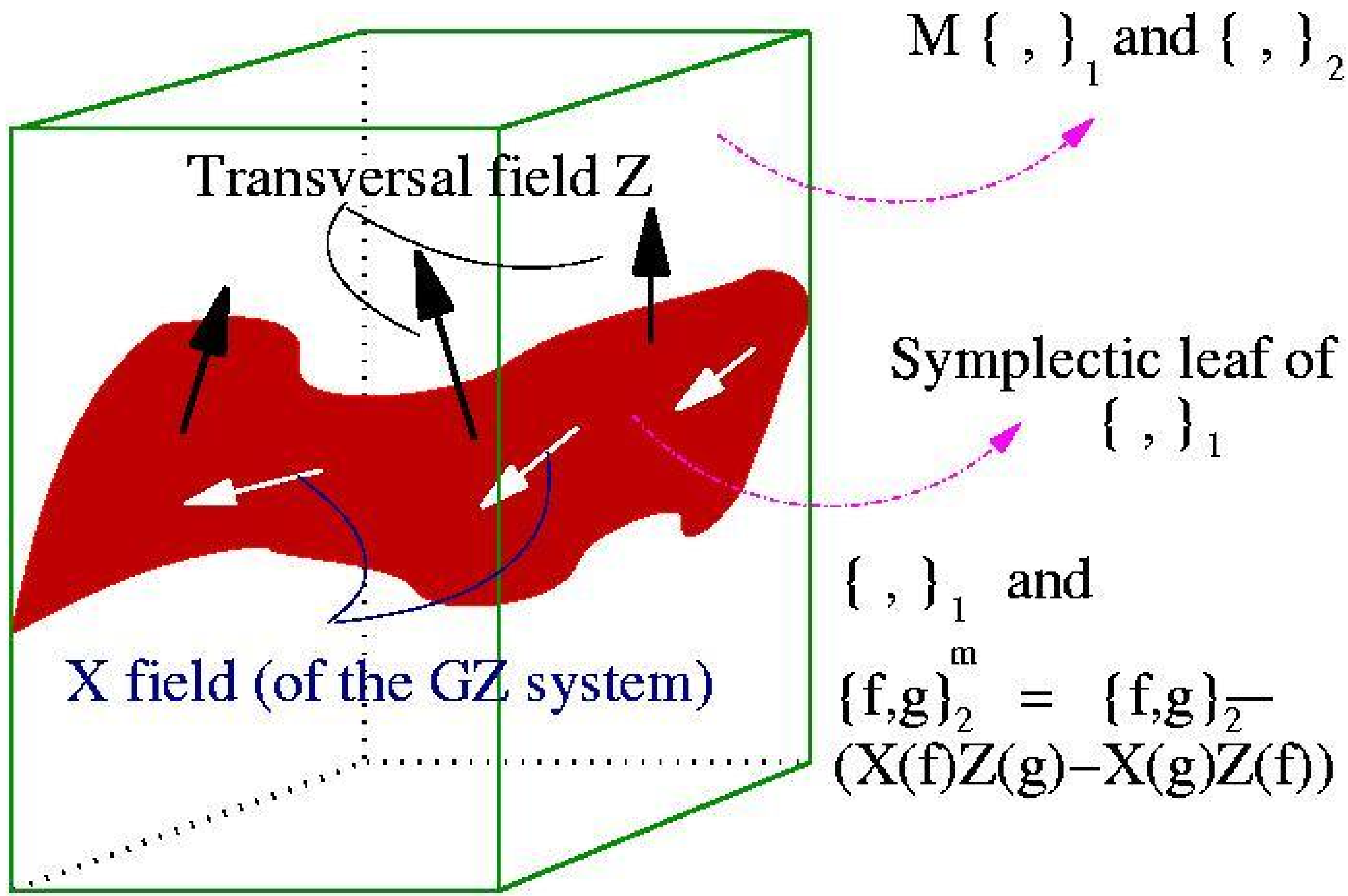
A) It is transversal to the symplectic leaves of P_1

B) It leaves P_1 invariant ($Lie_Z(P_1)=0$)

C) It leaves P_2 “almost” invariant, $Lie_Z(P_2)=Y \wedge X$.

- Then the “modified” bivector $P^{(m)}=P_2 - X \wedge Z$ has the desired properties.

Reduction: in a picture (one non common Casimr)



The periodic Toda lattice revisited

- N particles on the line (circle), interacting with a exponential potential,

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \exp(q_i - q_{i+1})\, modN$$

- Flaschka “coordinates”:

$$a_i = \exp(q_i - q_{i+1})\, modN, \quad b_i = p_i$$

- The Hamiltonian reads $H = \frac{1}{2}\sum_{i=1^N} b_i^2 + \sum_{i=1}^{N} a_i.$

Equations of motion

- In standard coordinates

$$\dot{q}_i = p_i, \qquad \dot{p}_i = \mathrm{e}^{(q_{i-1}-q_i)} - \mathrm{e}^{(q_i-q_{i+1})} \quad Mod\, N$$

- In Flaschka "coordinates"

$$\dot{a}_i = a_i(b_i - b_{i+1}), \quad \dot{b}_i = a_{i+1} - a_i \quad Mod\, N$$

Lax representation

$$L(\mu) = \begin{bmatrix} b_1 & \mu & 0 & & \frac{a_n}{\mu} \\ \frac{a_1}{\mu} & b_2 & \mu & \ddots & \\ 0 & \frac{a_2}{\mu} & \ddots & \ddots & 0 \\ & \ddots & \ddots & b_{N-1} & \mu \\ \lambda & 0 & & \frac{a_{N-1}}{\mu} & b_N \end{bmatrix}$$

- The "companion" matrix M(μ) is the Laurent tail of L(μ), i.e. in the Flaschka coords, the equations of motion read

$$\frac{d}{dt}L(\mu) = [L(\mu), M(\mu)]$$

(Bi)Hamiltonian formulation

- The canonical Poisson brackets in Flaschka coordinates are

$$\{a_i, a_j\}_1 = \{b_i, b_j\}_1 = 0, \qquad \{a_i, b_j\}_1 = a_i(\delta_{i,j} - \delta_{i,j-1})$$

- The Toda lattice admits a second Poisson formulation via the “quadratic” brackets

$$\{a_i, a_j\}_2 = a_i a_j(\delta_{i-1,j} - \delta_{i+1,j}) \quad \{b_i, b_j\}_2 = a_j(\delta_{i,j+1} - \delta_{i,j-1}),$$

$$\{a_i, b_j\}_2 = a_i b_j(\delta_{i,j} - \delta_{i,j-1}), \;\; \text{with "second" Hamiltonian } H_0 = \sum_{i=1}^{N} b_i.$$

GZ structure

The characteristic polynomial of the Lax matrix(the spectral curve of the problem) is given by

$$\mathrm{Det}(L(\mu) - \lambda) = \mu^N + \mathcal{H}(\lambda) + \frac{K}{\mu^N}, \quad \text{with } K = \prod_{i=1}^{N} a_i.$$

K is a Casimir of both brackets, while

$$\mathcal{H}(\lambda) = \lambda^N + H_0\lambda^{N-1} + (1/2H_0^2 - H)\lambda^{N-2} + \cdots$$

is the GZ polynomial collecting the non-trivial integrals of the motion.

BH SoV- results

- Transversal vector field e.g.: $Z = \frac{\partial}{\partial b_N}$
- "Half" of the DN coordinates are the roots λ_a of the degree (N-1) polynomial

$$\Delta(\lambda) = L_Z(\mathcal{H}(\lambda))$$

- The other half are (logarithms of) the values μ_a satisfying the spectral curve relation

$$\mathrm{Det}(L(\mu_a) - \lambda_a) = 0$$

that can be read as separation relations.

SO(4) rigid body

- Hamiltonian system on the (dual of) the Lie Algebra so(4), (coordinates m_{ik}, $i<k=1..4$), and Hamiltonian,

$$H_{\mathcal{E}} = \frac{1}{2} \sum_{i<j=1}^{4} a_{ij} m_{ij}^2$$

- where

$$a_{ij} = J_l^2 + J_k^2, \quad \text{with } \{i, j, l, k\} \text{ a permutation of } \{1, 2, 3, 4\}.$$

- Lax matrix: $L(\lambda) = \lambda\, \mathsf{J}^2 + M$

$$\mathsf{J} := \mathrm{diag}(J_1, J_2, J_3, J_4)$$

- Second Poisson brackets associated with the deformed commutator

$$[M_1, M_2]_{\mathsf{J}^2} :== M_1 \mathsf{J}^2 M_2 - M_2 \mathsf{J}^2 M_1$$

- One common Casimir function, and 3 more constants of the motion collected in a second order polynomial

$$\rho^2 \mathcal{H}_0 + \rho\, \mathcal{H}_1 + \mathcal{H}_2$$

- Change of coordinates (so(4)=so(3)xso(3)) , from $\{m_{ik}\}$ to $\{x_1,x_2,y_1,y_2,z_1,z_2\}$
- The Hamiltonian (up to the common Casimir) is

$$H_{\mathcal{E}} = 2\mu_4 x_1 x_2 + 2\mu_3 y_1 y_2 + 2\mu_2 z_1 z_2$$

- with

$$J_1{}^2 = -\mu_4 + \mu_1 - \mu_3 - \mu_2, \qquad J_2{}^2 = \mu_3 - \mu_4 + \mu_1 + \mu_2,$$

$$J_3{}^2 = \mu_1 - \mu_3 + \mu_4 + \mu_2, \qquad J_4{}^2 = -\mu_2 + \mu_1 + \mu_3 + \mu_4$$

- Remark: it is the Hamiltonian of the two site (classical) Heisenberg XYZ model.

Additional Symmetry: $\mu_4 = \mu_3$

- The Hamitonian reads

$$H_{XXZ} = 2\,\mu_3(x_2x_1 + y_1y_2) + 2\mu_2 z_1\, z_2$$

- With a further change of coordinates,

$$u_k = x_k + iy_k, \quad v_k = x_k - iy_k, \quad k = 1, 2.$$

one can show that the needed transversal vector field can be given by

$$Z = \frac{1}{2u_1}\frac{\partial}{\partial v_1} + \frac{1}{2u_2}\frac{\partial}{\partial v_2}$$

Final results

DN coordinates:

$$(\lambda_1 = z_2 - z_1, \xi_1), (\lambda_2 = \mu_1 - \mu_2 + \mu_3 \left(\frac{u_1}{u_2} + \frac{u_2}{u_1} \right), \xi_2)$$

Separation relations.

Since λ_1 is a further commuting constant of the motion (due to the extra symmetry) one finds

$$\alpha \zeta_1^2 + \mathcal{H}_1 + \beta \mathcal{H}_2 + \gamma_1 \mathcal{H}_0 + \gamma_2 C_2 = 0 \text{ with}$$

$$\alpha = 2\,\frac{{\mu_3}^2 - {\mu_2}^2}{\mu_1 + \mu_2}, \quad \beta = \frac{1}{\mu_1 + \mu_2}, \gamma_1 = \mu_1 + \mu_2, \gamma_2 = 0$$

- The second one is more “ordinary”,

$$p(\lambda_2)\xi_2^2 + q_1(\lambda_2)\mathcal{H}_1 + q_2(\lambda_2)\mathcal{H}_2 - \Psi(\lambda_2, \mathcal{H}_0, C_2) = 0$$

but, as opposed to the usual (=already worked out cases) cases, it is an algebraic, rather than polynomial function of the coordinate

$$\lambda_2 = \mu_1 - \mu_2 + \mu_3 \left(\frac{u_1}{u_2} + \frac{u_2}{u_1} \right)$$

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