Action variables near resonant equilibria

Holger Dullin
(Boulder, Colorado, on leave from Loughborough, UK)
joint work with
Sven Schmidt,
Heinz Hanßmann & Richard Cushman (on 1:-2)

Overview

- Introduction
- Resonant Equilibria, Normal Form
- Singular Reduction
- Image of EM
- Actions, Frequencies, Rotation number

IHS & Momentum Map

Integrable Hamiltonian System:

- Symplectic manifold P, dim P = 4
- Two smooth a.e. independent functions F_1, F_2 on P
- Vanishing Poisson bracket $\{F_1,F_2\} = 0$
- $F=(F_1,F_2) : P \to R^2$ is (energy-) momentum map
- Liouville-Arnold: Generic fibres of F are Tori
- Study the critical values of F, and singular fibres
  critical points e.g. when $\nabla F_1 = 0$

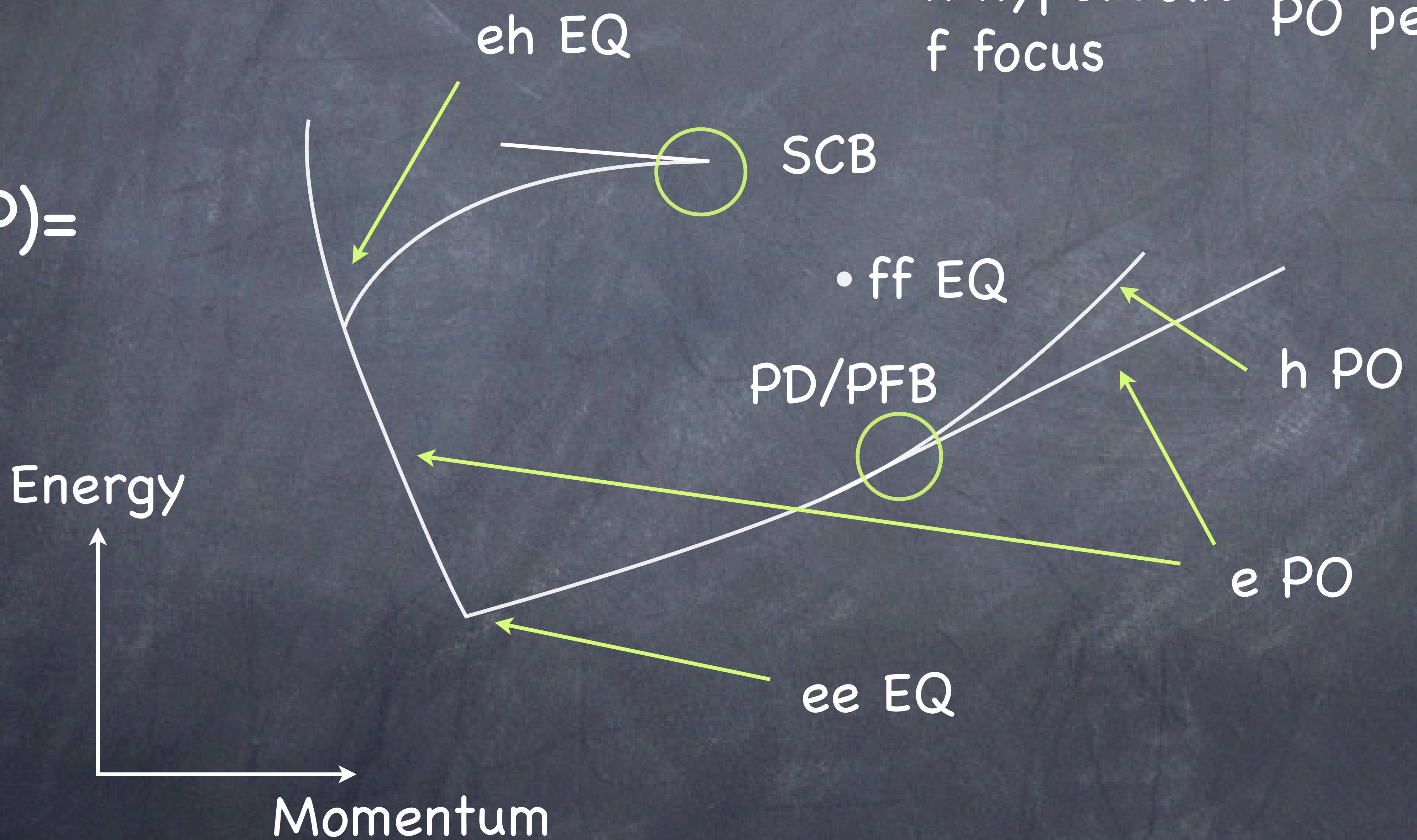
The Image of F

Critical values of the EM-map

notation:
e elliptic
h hyperbolic
f focus

EQ equillibrium
PO periodic orbit

F(P)=



Resonant Equilibria

- Hamiltonian H with 2 degrees of freedom
- Equilibrium of X_H, shift to origin,
  local symplectic coordinates x_1, x_2, y_1, y_2, $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$
- Assume eigenvalues of linearisation of X_H imaginary
- Assume linearisation semisimple
- quadratic part $H_2 = \omega_1(x_1^2+y_1^2) \pm \omega_2(x_2^2+y_2^2)$
- resonant if $\omega_1 : \omega_2 = p : q$; $p, q \in \mathbb{N}$ (notation $p:\pm q$)

Resonant Equilibria

- H_2 definite ⇒ Lyapunov stable (Dirichlet)
- H_2 indefinite
  not 1:-2, 1:-3 and transversality condition on frequency map ⇒ Lyapunov stable (Arnold)
- H_2 p:-q resonant: instability is possible

Normal Form

- H_2 semisimple by assumption, so is ad_{H2}
- $z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$, and c.c.
  $2 H_2 = p |z_1|^2 - q |z_2|^2$
- Unremoveable terms from symplectic spectrum ⇒
  Birkhoff Normal form $H = H(|z_1|^2, |z_2|^2)$
- additional unremovable terms $z_1^q z_2^p$ and c.c.
- for 1:-2 order $p+q < 4$!

Non-resonant Equilibria (EE)

- The foliation is that of the Harmonic Oscillator
- $H(I_1,I_2) = \omega_1 I_1 - \omega_2 I_2 + aI_1^2+bI_1I_2+cI_2^2 + \ldots$
- Frequencies: $\Omega_i(I) = \partial H/\partial I_i$
- Frequency map non-degenerate: det $D^2H \neq 0$ (Kolmogorov non-degeneracy condition for KAM)
- Similarly frequency ratio map R at fixed energy (isoenergetic non-degeneracy condition for KAM)

Resonant Equilibria: IHS

- $H = p|z_1|^2 - q|z_2|^2 + \mathrm{Re}(c\, z_1^q z_2^p) = H_2 + H_{p+q}$
  symplectic form $i(dz_1 \wedge dz_1 + dz_2 \wedge dz_2)/2$
- Flow of H_2: $G^\Phi(z_1, z_2) = (e^{ip\Phi} z_1, e^{-iq\Phi} z_2)$
  S^1 action, not free at origin
- Leaves H_{p+q} invariant, 2π periodic flow, action H_2
- $\{H_2, H_{p+q}\} = 0$, independent: Liouville Integrable Hamiltonian System, defines Liouville foliation
- $F = (H_2, H_{p+q})$ is momentum map of p:-q resonance

Singular Reduction

- Invariants of the S^1 action G: $\pi_1 = |z_1|^2$, $\pi_2 = |z_2|^2$, $\pi_3+i\pi_4 = z_1^q z_2^p$
- Relation: $\pi_3^2+\pi_4^2 = \pi_1^q \pi_2^p$, also $\pi_1 \geq 0$, $\pi_2 \geq 0$
- Poisson structure for invariants π_i
- Relation is Casimir, additionally $H_2 = p\pi_1 - q\pi_2$
- reduced phase space: $P(h_2) = H_2^{-1}(h_2) / G$
- Reduced Poisson structure π_1, π_3, π_4: $\{\pi_1, \pi_3\}=4\pi_4$, ...

Reduced phase space

Lemma: Singular reduced phase space

$P(h_2) = \{\pi_1, \pi_3, \pi_4 : \pi_3^2+\pi_4^2 = q^{-p}\pi_1^q(p\pi_1-h_2)^p,\ \pi_1>0,\ p\pi_1-h_2>0\}$

- $h_2 > 0$: $P(h_2)$ has singular point of order $p-1$
- $h_2 = 0$: $P(h_2)$ has singular point of order $p+q-1$
- $h_2 < 0$: $P(h_2)$ has singular point of order $q-1$

order 0: smooth
order 1: conical singularity
order > 1: higher order degenerate

$\pi_3^2 = \pi_1^2(\pi_1 - h_2)$

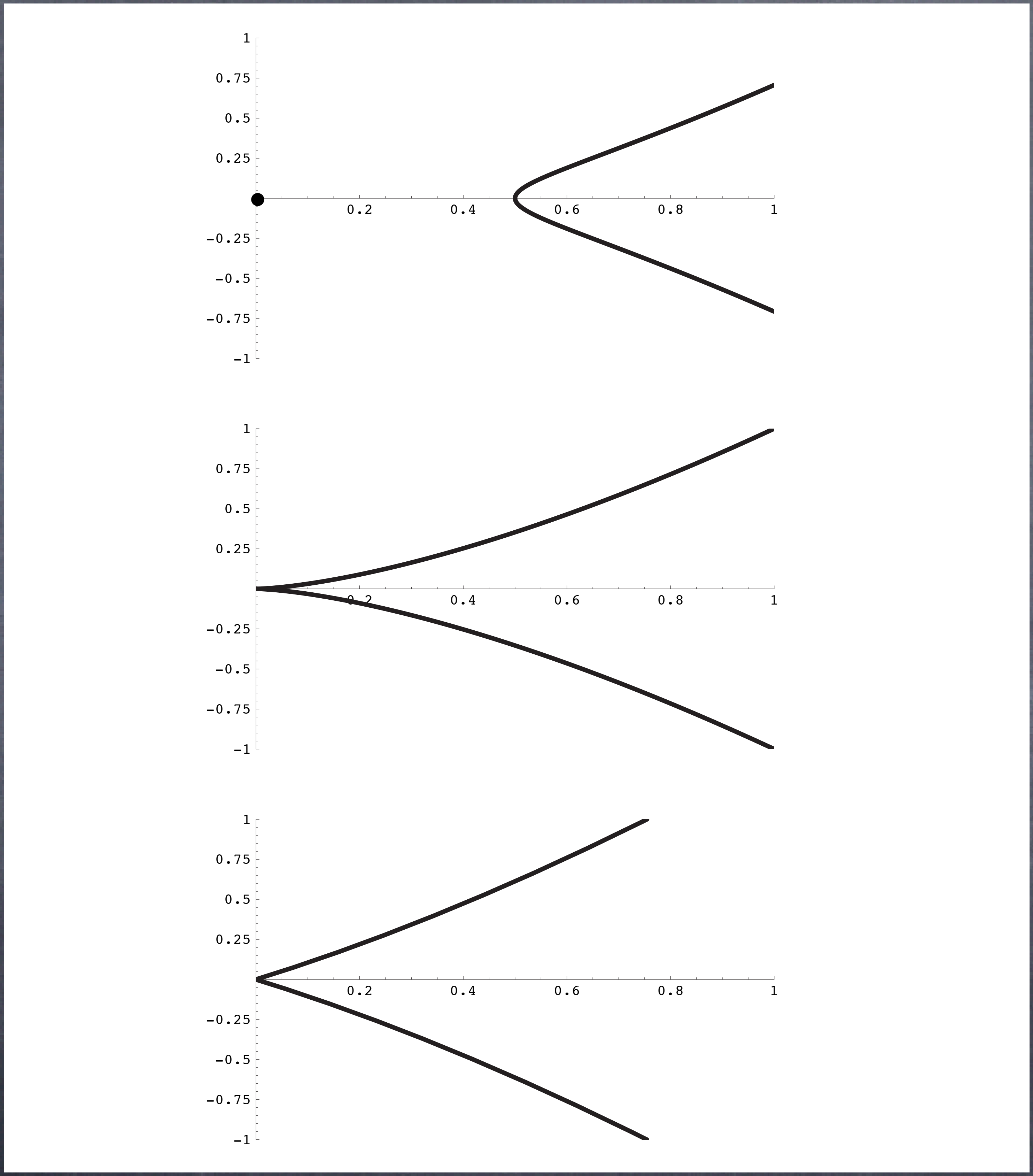
$P(h_2)$, $\pi_4=0$

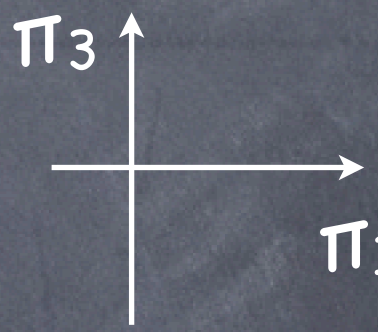
1:-2

$h_2 > 0$

$h_2 = 0$

$h_2 < 0$





Reduced phase space 1:-q

Lemma: Singular reduced phase space

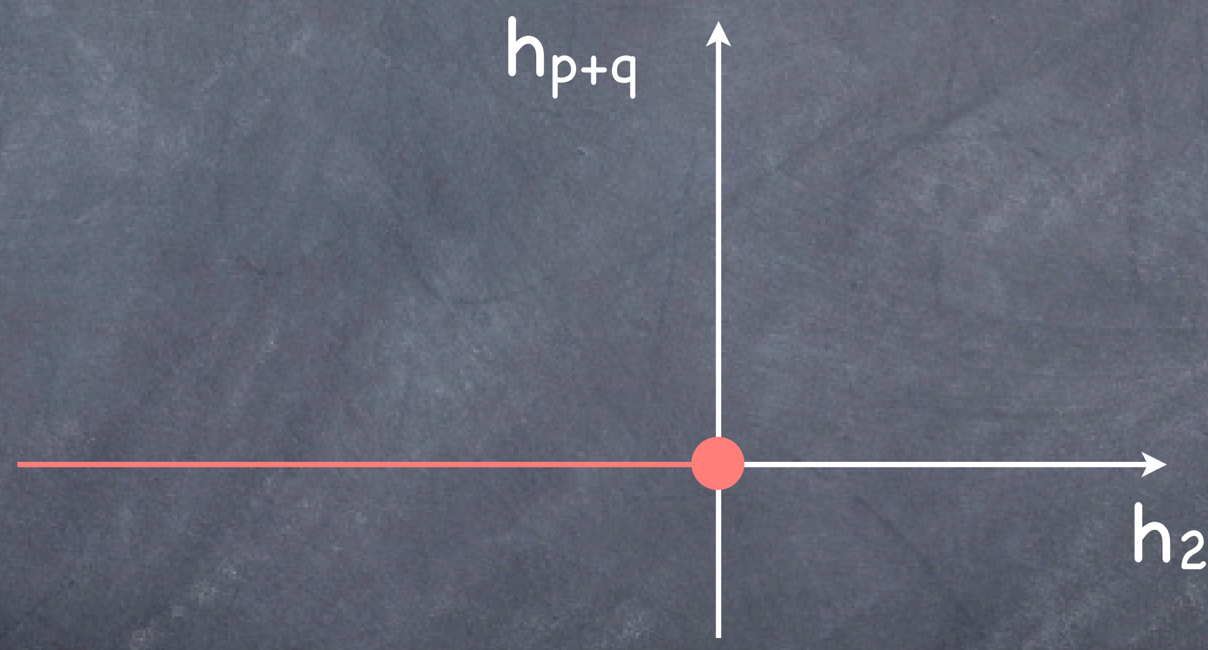
$P(h_2) = \{\pi_1, \pi_3, \pi_4 : \pi_3^2+\pi_4^2 = q\pi_1^q(\pi_1-h_2), \pi_1>0, \pi_1-h_2 >0\}$

- $h_2 > 0$: $P(h_2)$ is smooth
- $h_2 = 0$: $P(h_2)$ has singular point of order q
- $h_2 < 0$: $P(h_2)$ has singular point of order q-1

order 0: smooth
order 1: conical singularity
order > 1: higher order degenerate

Image of Energy-Momentum Map (1:-q)

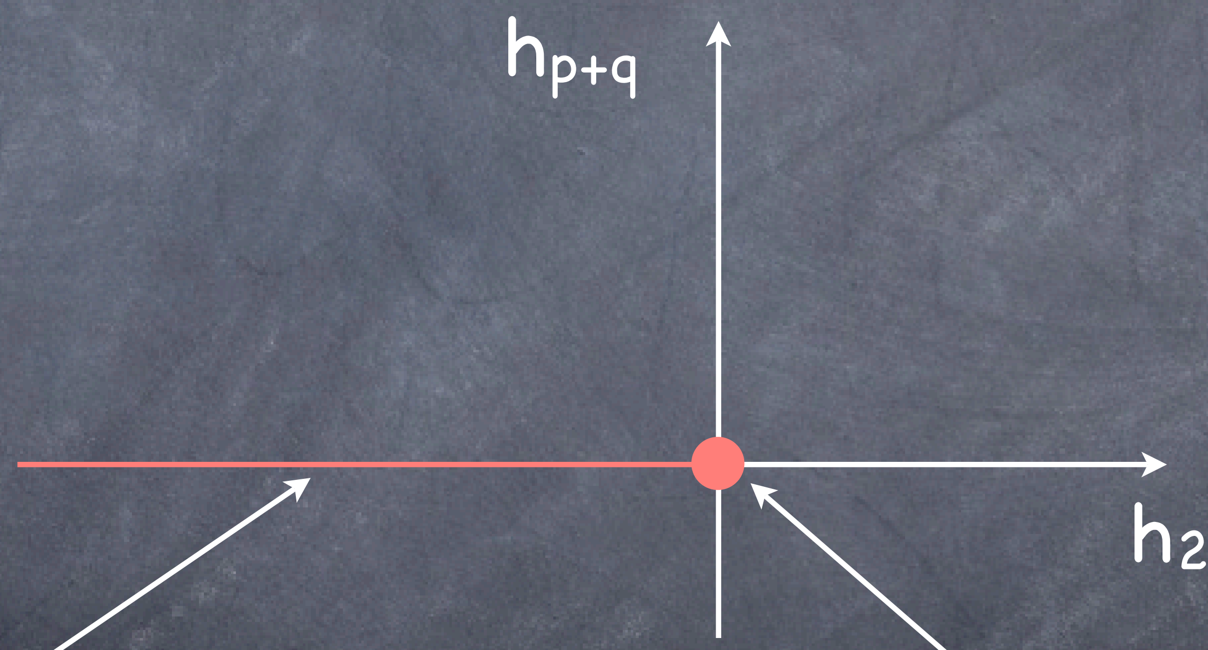
$F = (H_2, H_{p+q})$



: critical values

Image of Energy-Momentum Map (1:-2)

$F = (H_2, H_{p+q})$

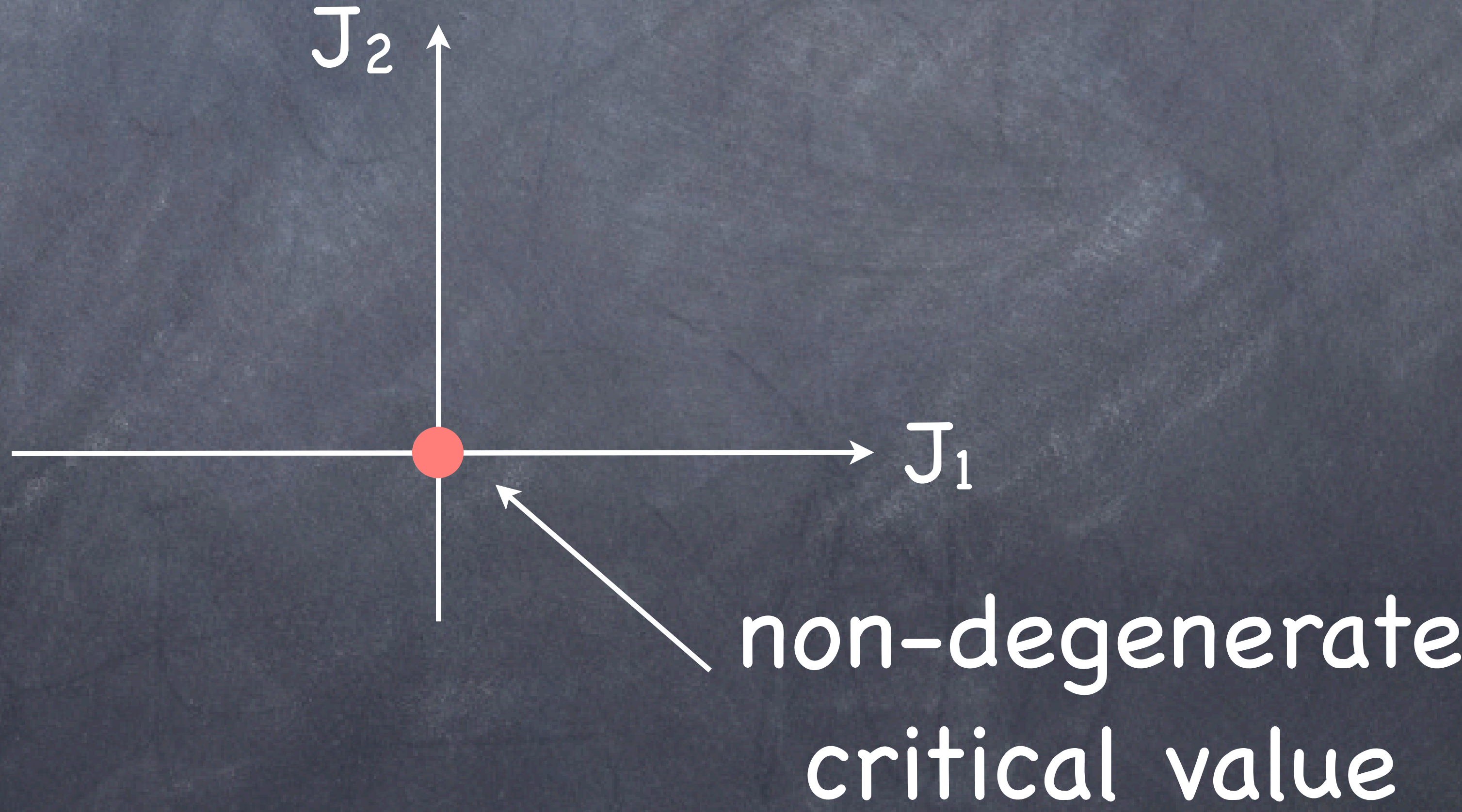


non-degenerate critical values

degenerate critical value: Hessian of H_{p+q} vanishes

Desired Result: Like for Focus-Focus

Vũ Ngọc San: Topology 42 (2003)



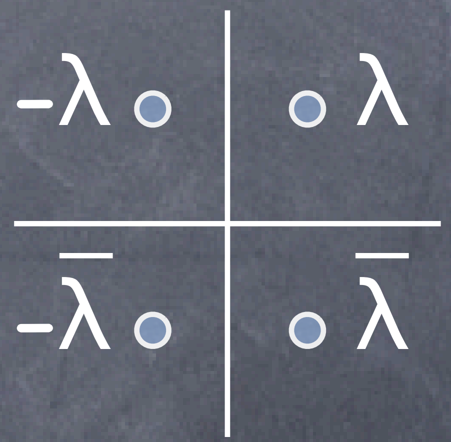
Focus-Focus

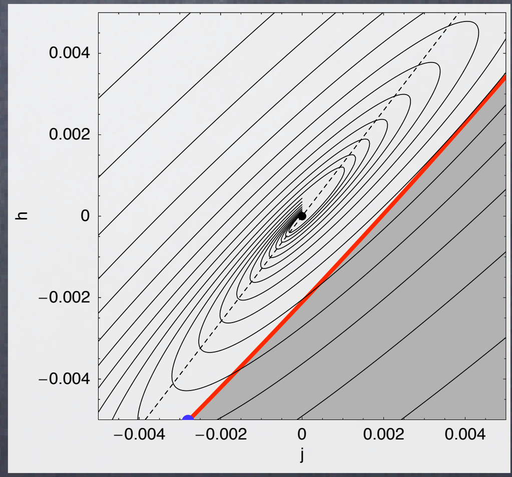
- Momentum map $F=(J_1, J_2)=(x_1y_1+x_2y_2, x_1y_2-x_2y_1)$
- J_2 is an S^1 action
- $A = \mathrm{Re}(\zeta \log \zeta - \zeta)$ + smooth higher order terms, where $\zeta = J_1 + i J_2$

  semiglobal symplectic invariants
- Action of foliation, i.e. $H=J_1$
- $T = dA/dJ_1 = -\log |\zeta|$, $R = dA/dJ_2 = \arg \zeta$
  singular part of $A = J_1 T - J_2 R$ (Monodromy!)
- Using San's results and $H(J_1, J_2) = aJ_1 + bJ_2 + \text{hot}\ldots$:

Focus-Focus Equilibria

Theorem: There is a local diffeomorphism on the image of the Energy-Momentum map in a neighbourhood of a simple focus-focus point which is C^1 at the origin and C^∞ elsewhere, which maps the level lines of the rotation number to the integral curves of the ODE $d\zeta/ds = -\lambda\zeta$, $\zeta \in \mathbb{C}$. These curves are spirals determined by the eigenvalue λ.





HRD und Vũ Ngọc San, Nonlinearity, 17, 1777-86 (2004)

FF continued

- ...now it is straightforward to consider the non-degeneracy of the frequency (ratio) map
- Thm: The frequency ratio map is degenerate on a line near non-degenerate focus-focus values
- Thm: The frequency map is non-degenerate near focus-focus values (everywhere!, improving (a little) the result by Nguyen Tien Zung, Phys. Lett. A 215 (1996))

HRD und Vũ Ngọc San, Nonlinearity, 17, 1777-86 (2004)

Resonant Equilibra: Differences

- Origin is not an isolated critical value
- Thus period lattice is not defined in a neighbourhood of the origin
- Flow of F is not linear, technical difficulties
- No Eliasson normal form result, hence formal
- But: the action is still a continuous function on the image of the energy momentum map, even across the singularity

Main Lemma:

compare FF:

For the momentum map $F=(H_2, H_{1+q})$

$F = (J_1, J_2)$

- Chose $\zeta = h_2 + i\, h_{1+q}{}^{2/(q+1)}$, then:
- Action $A = h_{1+q}\, T - h_2\, R$
- with period $T = f(|\zeta|)\, g(\arg \zeta)$
- $f(r) = 1/r^{(q-1)/2}$, smooth,
  $g(\Phi)$ log-singularity at $\Phi=\pi$
- Rotation number $R = h(\arg \zeta)$, smooth

$\zeta = J_1 + i\, J_2$

$A = J_1\, T - J_2\, R$

$T = \log|\zeta|$

$R = \arg \zeta$

Steps of the Proof:

- symplectic structure on leaves: $\left\{ \pi_2, \cos^{-1} \frac{\pi_3}{\sqrt{\pi_1^2 \pi_2}} \right\} = 2$
- assume compactness of singular fibres
- decompose into singular contribution near critical point and smooth part away from it using Poincare sections transversal to invariant manifolds
- find nice coordinates on image of F

Theorem

- The action A of the foliation $F=(H_2,H_{q+1})$ of the 1:-q resonance has fractional monodromy 1/q
- for $H = H_2 + c\,H_3 + \ldots$:
- The frequency ratio map is degenerate along the line h3=0, h2>0
- The frequency map is non-degenerate (in a sector around h3=0, h2>0)

The Image of F

1:-2

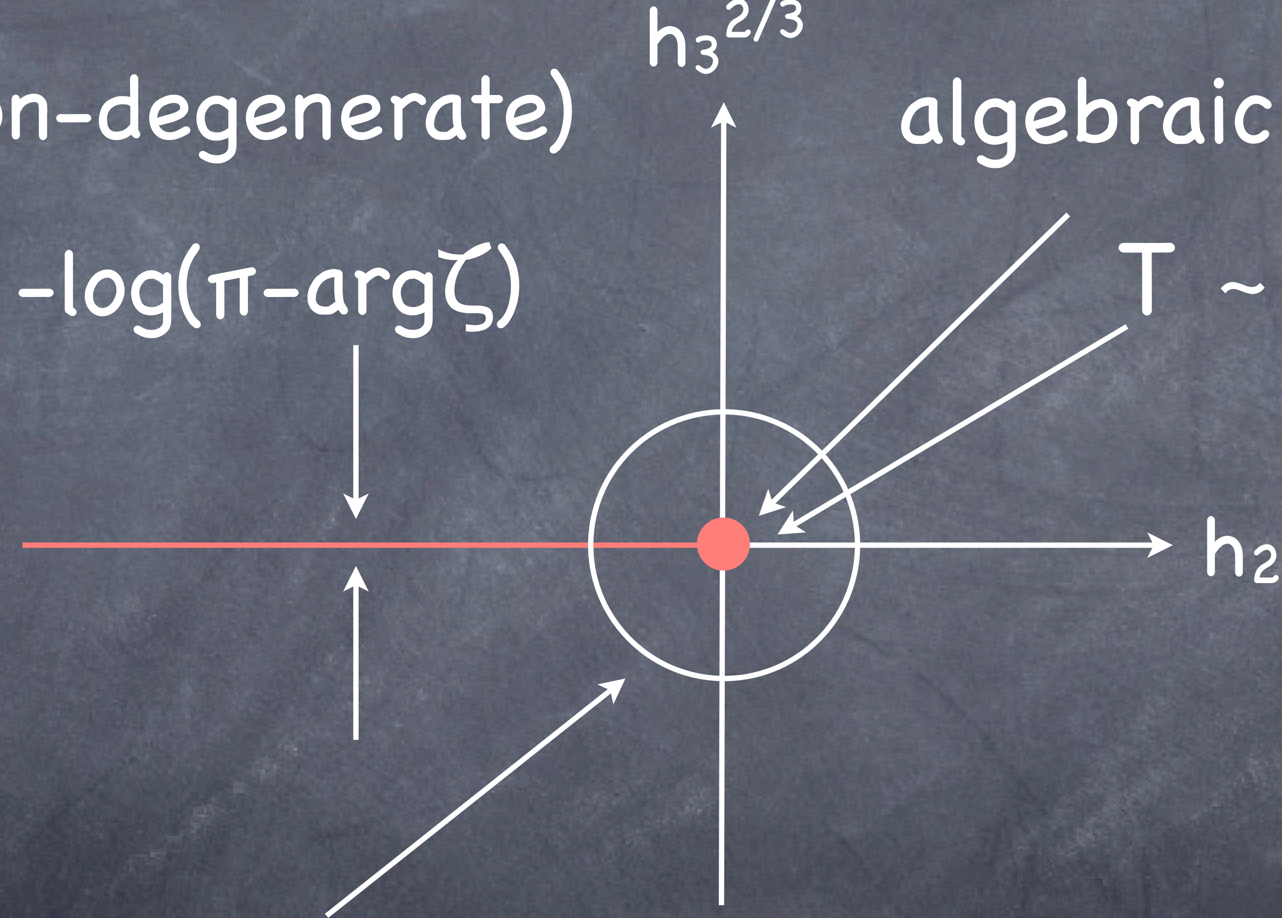
ζ-plane (deformed image of the momentum map)

logarithmic (non-degenerate)

algebraic (degenerate)

$T \sim -\log(\pi - \arg\zeta)$

$T \sim |\zeta|^{-(q-1)/2}$

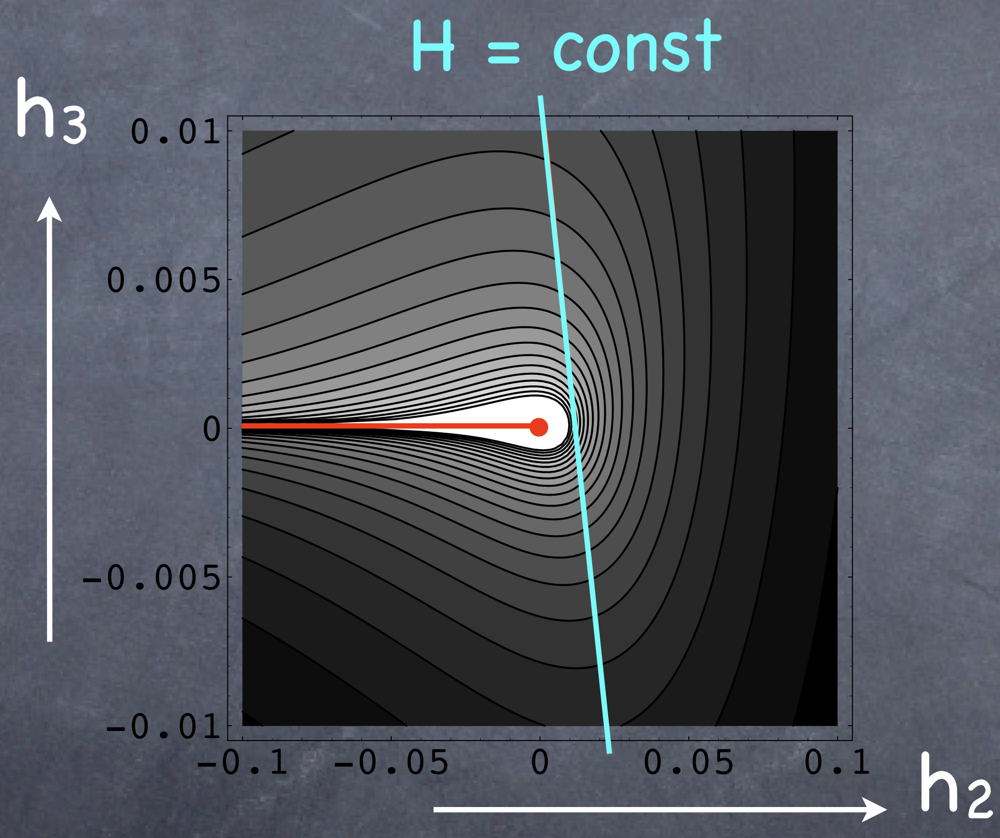


$R \to R + 1/2$ upon encircling the origin

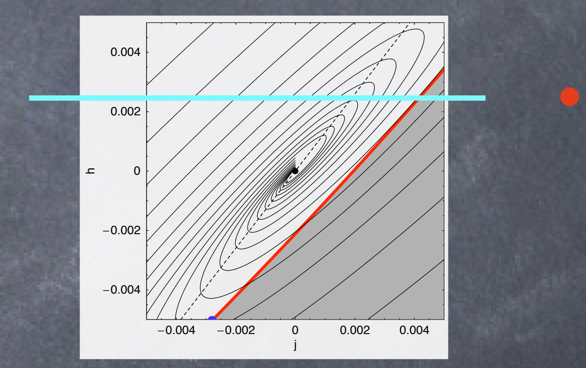
$(H_2, A) \to (H_2, A + H_2/2)$ fractional Monodromy

Contourplot Rotation Number

for $H = H_2 + c\, H_3$ (not foliation with $H = H_3$)



cmp. FF:



Conclusion

- degenerate singularities are interesting (Bifurcations!)
- fill the image of the momentum map
- deform the image of the momentum map
- study actions