Non-commutative integrability and Mischenko-Fomenko conjecture

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April 12, 2007

Liouville Integrability

Consider a symplectic manifold (M^{2n}, ω),
a smooth Hamiltonian $H : M^{2n} \to \mathbb{R}$ and
the corresponding Hamiltonian system

$$\frac{dx}{dt} = X_H(x) = \omega^{-1} dH(x).$$

Definition. This system is called *Liouville integrable* if it admits n commuting independent first integrals $f_1, \dots, f_n$.

Very often: the number of integrals is $> n$ but they do not commute.

Non-commutative Integrability

Definition. A Hamiltonian system is *integrable in non-commutative sense* if it admits an algebra $\mathcal{F}$ of first integrals satisfying the following condition:

at a generic point $x \in M$, the subspace K in T_x^*M generated by $df(x)$, $f \in \mathcal{F}$, is coisotropic.

Coisotropy condition: $\tilde{K} \subset K$,
where $\tilde{K} = \{l \in T_x^*M \mid \omega^{-1}(l, K) = 0\}$.

Non-commutative Liouville theorem.

(Nekhoroshev, Mischenko, Fomenko, Brailov)
Consider a common level of the integrals

$$X_f = \{f_1(x) = c_1, f_2(x) = c_2, \ldots, f_k(x) = c_k\}.$$

If X_f is regular, compact and connected, then X_f is a torus of dimension $2n-k$ with quasi periodic motion on it.

Relationship: commutative and non-commutative

$$\{\text{Commutative integrability}\} \subset \{\text{Non-commutative integrability}\}$$

The converse is true at least locally:

$$\{\text{Non-commutative integrability}\} \longrightarrow \{\text{Commutative integrability}\}$$

Explanation:

Let $f_1, \ldots, f_k$ be local generators of $\mathcal{F}$, then

$$\{f_i, f_j\} = h_{ij}(f_1, \ldots, f_k)$$

This formula defines a Poisson structure on the image of the "momentum mapping" $\Phi : M \to \mathbb{R}^k$ given by the first integrals $\Phi(x) = (f_1(x), f_2(x), \ldots, f_k(x))$.

Darboux-Weinstein theorem: there is a canonical coordinate system $h_1, \ldots, h_s$, $p_1, \ldots, p_r$, $q_1, \ldots, q_r$ $(k = s + 2r)$ such that h_i are Casimirs and

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}$$
$$h, p, q \leftarrow \text{diffeo} \rightarrow f_1, \ldots, f_k$$

Then $h_1, \ldots, h_s$, $p_1, \ldots, p_r$ is a desired set of commuting integrals for the original system.

Mischenko-Fomenko conjecture

Non-commutative integrability always implies classical commutative integrability in the same class of integrals (smooth, analytic, polynomial,...).

The image of the momentum mapping $\Phi : M \to \mathbb{R}^k$ can be considered as a (singular) Poisson manifold X. If $g_1, \dots, g_m$ is a complete commutative set of functions on X with respect to the reduced Poisson structure, then $g_1 \circ \Phi, \dots, g_m \circ \Phi$ is a complete commutative set of integrals for the original system.

Problem. Given a symplectic (Poisson) manifold X, does there exist any integrable Hamiltonian system on it?

"Trivial" examples:
Symplectic vector space $(\mathbb{R}^{2n}, \omega)$, cotangent bundles T^*N, etc.

First non-trivial example: coadjoint orbits of Lie groups or, equivalently, dual spaces of Lie algebras $\mathfrak{g}^*$ with Lie-Poisson bracket.

Mischenko-Fomenko conjecture (1981). For any finite-dimensional Lie algebra $\mathfrak{g}$, there is an integrable Hamiltonian system on $\mathfrak{g}^*$ with polynomial integrals.

Algebraic reformulation. The Lie-Poisson algebra $P(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ always admits a complete commutative subalgebra $\mathcal{F} \subset P(\mathfrak{g})$ such that $\mathrm{tr.deg.}\mathcal{F} = \frac{1}{2}(\dim G + \mathrm{ind}\, G)$.

Relation to Hamiltonian Reduction

Consider a Hamiltonian system invariant with respect to a Hamiltonian action of a Lie group G. Then each element $\xi \in \mathfrak{g}$ generates a first integral H_ξ in such a way that $\{H_\xi, H_\eta\} = H_{[\xi,\eta]}$. Assume that these integrals are functionally independent and coisotropy condition holds:

$$\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g} = \dim M = 2n$$

Mischenko-Fomenko conjecture. Under above conditions, the Hamiltonian system is integrable in classical sense, i.e., there exist independent commuting integrals $h_1, \ldots, h_n$. Moreover, these integrals are polynomials in H_ξ's.

Semisimple Lie algebras (Mischenko-Fomenko, 1978)
Nilpotent Lie algebras (Vergne, 1972) and
Solvable algebraic Lie algebras
Semi-direct sums $\mathfrak{g} +_\rho V$, with $\mathfrak{g}$ semisimple and V abelian (many types, many authors)

General case (Sadetov 2003)

Theorem. Mischenko-Fomenko conjecture holds for any finite-dimensional Lie algebra over an arbitrary field of zero characteristic.

Main idea is induction by dimension: we reduce the problem for $\mathfrak{g}$ to the same problem for a certain Lie algebra $\mathfrak{f}$ of smaller dimension step by step until we get either trivial, or semisimple Lie algebra.

Linear algebra:
Integrable system = maximal isotropic subspace

How to construct a maximal isotropic subspace U in a vector space L endowed with a skew-symmetric bilinear form β (by induction)?

Take an arbitrary subspace $V \subset L$ and consider

$$\tilde{V} = \{l \in L : \quad \beta(l, V) = 0\}. \tag{1}$$

If $A \subset V$ is maximal isotropic in V and $\tilde{A} \subset \tilde{V}$ is maximal isotropic in $\tilde{V}$, then $A + \tilde{A}$ is maximal isotropic in L.

Non-linear case:

Let M be a manifold endowed with a Poisson bracket. How to construct an integrable Hamiltonian system on it?

Take an arbitrary subalgebra $\mathcal{F}$ in $C^\infty(M)$ and consider $\tilde{\mathcal{F}} = \{f \in C^\infty(M) : \quad \{f, \mathcal{F}\} = 0\}$.

Assume that at a generic point $x \in M$, we have the same condition as in linear case:
the subspaces $V, \tilde{V} \subset T_x^* M$ generated by the differentials of functions from $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively satisfy (1).

If $\mathcal{A}$ is a complete commutative subalgebra in $\mathcal{F}$ and $\tilde{\mathcal{A}}$ is a complete commutative subalgebra in $\tilde{\mathcal{F}}$, then $\mathcal{A} + \tilde{\mathcal{A}}$ is a complete commutative subalgebra in $C^\infty(M)$ (i.e., an integrable system on M).

In the case of a Lie algebra:

How to construct a complete commutative subalgebra in $P(\mathfrak{g})$?

Take an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let
$\mathcal{F} = P(\mathfrak{h}) \subset P(\mathfrak{g})$.

Consider $\tilde{\mathcal{F}} = \mathrm{Ann}\,(\mathfrak{h}) = \{f \in P(\mathfrak{g}) \ : \ \{f, h\} = 0 \ \forall h \in \mathfrak{h}\}$. Equivalently, $\mathrm{Ann}\,(\mathfrak{h})$ can be defined as the algebra of invariants of the coadjoint action of H on $\mathfrak{g}^*$.

If (1) holds (this happens very often), then a complete commutative subalgebra in $P(\mathfrak{g})$ can be obtained as $\mathcal{A}+\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are certain complete commutative subalgebras in $P(\mathfrak{h})$ and $\mathrm{Ann}\,(\mathfrak{h})$ respectively.

What is the problem? No problem with $P(\mathfrak{h})$. But Ann $(\mathfrak{h})$ may have a very complicated algebraic structure.

Problem. How to find $\mathfrak{h}$ in such a way that Ann $(\mathfrak{h})$ has a nice algebraic structure?

Lemma 1. Any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ satisfies at least one of the following conditions:

(1) $\mathfrak{g}$ is semisimple;

(2) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{K}$, and $\mathfrak{g}_0$ is semisimple;

(3) there is a commutative ideal $\mathfrak{h}$ (which is not one-dimensional center as in (2));

(4) there is an ideal $\mathfrak{h}$ isomorphic to the Heisenberg algebra.

Cases (1) and (2) are simple.

Case (3).
$\mathfrak{h}$ is a commutative ideal in $\mathfrak{g}$; G naturally acts on $\mathfrak{h}^*$.

$\mathrm{St}\,(h) \subset \mathfrak{g}$
the stationary subalgebra for $h \in \mathfrak{h}^*$;

$\Psi : \mathfrak{h}^* \to \mathfrak{g}$
a rational map s.t. $\Psi(h) \in \mathrm{St}\,(h)$ for any h;

$f_\Psi : \mathfrak{g}^* \to \mathbb{K}$
the function defined by $f_\Psi(x) = \langle x, \Psi(\pi(x))\rangle$.

Lemma 2. $f_\Psi \in \mathrm{Ann}\,(\mathfrak{h})$.

$L = \{$the set of all $\Psi\}$;

$\mathcal{F}_L = \{$the set of all $f_\Psi\} \subset \mathrm{Ann}\,\mathfrak{h}$;

$L \longrightarrow \mathcal{F}_L$, $\Psi \to f_\Psi$ homomorphism of Lie algebras

Main observation: All these objects are defined not only over $\mathbb{K}$, but also over the field $\mathbb{K}(\mathfrak{h}^*)$ of rational functions on $\mathfrak{h}^*$. Moreover, L and $\mathcal{F}_L$ become finite-dimensional over $\mathbb{K}(\mathfrak{h}^*)$.

Lemma 3. If MF conjecture holds for $\mathcal{F}_L$ (over the new field $\mathbb{K}(\mathfrak{h}^*)$), then MF conjecture holds $\mathfrak{g}$ (over $\mathbb{K}$).

Case (4).

$\mathfrak{h} \subset \mathfrak{g}$ is an ideal isomorphic to Heisenberg algebra:
$\mathfrak{h} = \langle p_1, \ldots, p_k, q_1, \ldots, q_k, e \rangle$ and $[p_i, q_i] = e$.

Lemma 4. There is a subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that
(1) $\mathfrak{b} \cap \mathfrak{h} = \langle e \rangle$
(2) $\mathfrak{b}$ acts on $V = \langle p_1, \ldots, p_k, q_1, \ldots, q_k, \rangle$ by symplectic transformations.

Let $\beta \in \mathfrak{b}$. Consider $f_\beta : \mathfrak{g}^* \to \mathbb{K}$ of the form

$$f_\beta(x) = \beta(x) + \frac{\langle \omega^{-1}(\mathrm{ad}^*_\beta \pi(x)), x \rangle}{2\, e(x)}$$

Lemma 5. $f_\beta \in \mathrm{Ann}\,(\mathfrak{h})$.

Lemma 6. $\beta \to f_\beta$ is a monomorphism from $\mathfrak{b}$ to Ann $(\mathfrak{h})$. The functions f_β generates Ann $(\mathfrak{h})$.

Lemma 7. If MF conjecture holds for $\mathfrak{b}$, then MF conjecture holds $\mathfrak{g}$.

Conclusion:

We can always either reduce the dimension of the given Lie algebra $\mathfrak{g}$ or (if $\mathfrak{g}$ is semisimple) construct a complete commutative subalgebra in $P(\mathfrak{g})$.

Theorem is proved by induction.

Open questions

Q1. Completeness in the sense of flows.

Q2. Singular orbits.

Q3. Other important polynomial Poisson algebras like $\mathrm{Ann}\,\mathfrak{h}$.

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