Non-commutative integrability and Mischenko-Fomenko conjecture

Alexey Bolsinov
Loughborough University, UK
A.Bolsinov@lboro.ac.uk

Geometry of Integrable Systems
Hanoi National University of Education
Hanoi, Vietnam

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Liouville Integrability

Consider a symplectic manifold \((M^{2n}, \omega)\), a smooth Hamiltonian \(H : M^{2n} \to \mathbb{R}\) and the corresponding Hamiltonian system

\[
\frac{dx}{dt} = X_H(x) = \omega^{-1} dH(x).
\]

**Definition.** This system is called *Liouville integrable* if it admits \(n\) commuting independent first integrals \(f_1, \ldots, f_n\).

Very often: the number of integrals is \(> n\) but they do not commute.
Non-commutative Integrability

**Definition.** A Hamiltonian system is *integrable in non-commutative sense* if it admits an algebra $\mathcal{F}$ of first integrals satisfying the following condition:

at a generic point $x \in M$, the subspace $K$ in $T^*_x M$ generated by $df(x)$, $f \in \mathcal{F}$, is **coisotropic**.

**Coisotropy condition:** $\tilde{K} \subset K$, where

$$
\tilde{K} = \{ l \in T^*_x M \mid \omega^{-1}(l, K) = 0 \}.
$$
Non-commutative Liouville theorem.
(Nekhoroshev, Mischenko, Fomenko, Brailov)
Consider a common level of the integrals

\[ X_f = \{ f_1(x) = c_1, f_2(x) = c_2, \ldots, f_k(x) = c_k \} \].

If \( X_f \) is regular, compact and connected, then \( X_f \) is a torus of dimension \( 2n - k \) with quasi periodic motion on it.

Relationship: commutative and non-commutative

\{\text{Commutative integrability}\} \subset \{\text{Non-commutative integrability}\}

The converse is true at least locally:

\{\text{Non-commutative integrability}\} \rightarrow \{\text{Commutative integrability}\}
Explanation:
Let $f_1, \ldots, f_k$ be local generators of $\mathcal{F}$, then
\[
\{f_i, f_j\} = h_{ij}(f_1, \ldots, f_k)
\]

This formula defines a Poisson structure on the image of the "momentum mapping" $\Phi : M \to \mathbb{R}^k$ given by the first integrals $\Phi(x) = (f_1(x), f_2(x), \ldots, f_k(x))$.

Darboux-Weinstein theorem: there is a canonical coordinate system $h_1, \ldots, h_s, p_1, \ldots, p_r, q_1, \ldots, q_r$ ($k = s + 2r$) such that $h_i$ are Casimirs and
\[
\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}
\]

Then $h_1, \ldots, h_s, p_1, \ldots, p_r$ is a desired set of commuting integrals for the original system.
Mischenko-Fomenko conjecture
Non-commutative integrability always implies classical commutative integrability in the same class of integrals (smooth, analytic, polynomial,...).

The image of the momentum mapping $\Phi : M \to \mathbb{R}^k$ can be considered as a (singular) Poisson manifold $X$. If $g_1, \ldots, g_m$ is a complete commutative set of functions on $X$ with respect to the reduced Poisson structure, then $g_1 \circ \Phi, \ldots, g_m \circ \Phi$ is a complete commutative set of integrals for the original system.

Problem. Given a symplectic (Poisson) manifold $X$, does there exist any integrable Hamiltonian system on it?
"Trivial" examples:
Symplectic vector space $(\mathbb{R}^{2n}, \omega)$, cotangent bundles $T^*N$, etc.

First non-trivial example: coadjoint orbits of Lie groups or, equivalently, dual spaces of Lie algebras $\mathfrak{g}^*$ with Lie-Poisson bracket.

Mischenko-Fomenko conjecture (1981). For any finite-dimensional Lie algebra $\mathfrak{g}$, there is an integrable Hamiltonian system on $\mathfrak{g}^*$ with polynomial integrals.

Algebraic reformulation. The Lie-Poisson algebra $P(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ always admits a complete commutative subalgebra $\mathcal{F} \subset P(\mathfrak{g})$ such that $\text{tr.deg.} \mathcal{F} = \frac{1}{2}(\dim G + \text{ind} G)$. 
Relation to Hamiltonian Reduction
Consider a Hamiltonian system invariant with respect to a Hamiltonian action of a Lie group $G$. Then each element $\xi \in g$ generates a first integral $H_\xi$ in such a way that $\{H_\xi, H_\eta\} = H_{[\xi, \eta]}$. Assume that these integrals are functionally independent and coisotropy condition holds:

$$\dim g + \text{ind } g = \dim M = 2n$$

Mischenko-Fomenko conjecture. Under above conditions, the Hamiltonian system is integrable in classical sense, i.e., there exist independent commuting integrals $h_1, \ldots, h_n$. Moreover, these integrals are polynomials in $H_\xi$'s.
Semisimple Lie algebras (Mischenko-Fomenko, 1978) Nilpotent Lie algebras (Vergne, 1972) and Solvable algebraic Lie algebras Semi-direct sums $\mathfrak{g} +_{\rho} V$, with $\mathfrak{g}$ semisimple and $V$ abelian (many types, many authors)

General case (Sadetov 2003)

Theorem. Mischenko-Fomenko conjecture holds for any finite-dimensional Lie algebra over an arbitrary field of zero characteristic.
Main idea is induction by dimension: we reduce the problem for $g$ to the same problem for a certain Lie algebra $f$ of smaller dimension step by step until we get either trivial, or semisimple Lie algebra.

Linear algebra:
Integrable system = maximal isotropic subspace

How to construct a maximal isotropic subspace $U$ in a vector space $L$ endowed with a skew-symmetric bilinear form $\beta$ (by induction)?

Take an arbitrary subspace $V \subset L$ and consider

$$\tilde{V} = \{l \in L : \beta(l, V) = 0\}.$$  \hspace{1cm} (1)

If $A \subset V$ is maximal isotropic in $V$ and $\tilde{A} \subset \tilde{V}$ is maximal isotropic in $\tilde{V}$, then $A + \tilde{A}$ is maximal isotropic in $L$. 
Non-linear case:
Let $M$ be a manifold endowed with a Poisson bracket. How to construct an integrable Hamiltonian system on it?

Take an arbitrary subalgebra $\mathcal{F}$ in $C^\infty(M)$ and consider $\tilde{\mathcal{F}} = \{ f \in C^\infty(M) : \{ f, \mathcal{F} \} = 0 \}$.

Assume that at a generic point $x \in M$, we have the same condition as in linear case: the subspaces $V, \tilde{V} \subset T^*_x M$ generated by the differentials of functions from $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively satisfy (1).

If $\mathcal{A}$ is a complete commutative subalgebra in $\mathcal{F}$ and $\tilde{\mathcal{A}}$ is a complete commutative subalgebra in $\tilde{\mathcal{F}}$, then $\mathcal{A} + \tilde{\mathcal{A}}$ is a complete commutative subalgebra in $C^\infty(M)$ (i.e., an integrable system on $M$).
In the case of a Lie algebra:
How to construct a complete commutative subalgebra in $P(g)$?

Take an arbitrary subalgebra $\mathfrak{h} \subset g$ and let $\mathcal{F} = P(\mathfrak{h}) \subset P(g)$.

Consider $\tilde{\mathcal{F}} = \text{Ann} (\mathfrak{h}) = \{ f \in P(g) : \{ f, h \} = 0 \ \forall h \in \mathfrak{h} \}$. Equivalently, Ann($\mathfrak{h}$) can be defined as the algebra of invariants of the coadjoint action of $H$ on $g^*$.

If (1) holds (this happens very often), then a complete commutative subalgebra in $P(g)$ can be obtained as $\mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are certain complete commutative subalgebras in $P(\mathfrak{h})$ and Ann($\mathfrak{h}$) respectively.
What is the problem? No problem with $P(\mathfrak{h})$. But Ann($\mathfrak{h}$) may have a very complicated algebraic structure.

Problem. How to find $\mathfrak{h}$ in such a way that Ann($\mathfrak{h}$) has a nice algebraic structure?

**Lemma 1.** Any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ satisfies at least one of the following conditions:

1. $\mathfrak{g}$ is semisimple;
2. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbb{K}$, and $\mathfrak{g}_0$ is semisimple;
3. there is a commutative ideal $\mathfrak{h}$ (which is not one-dimensional center as in (2));
4. there is an ideal $\mathfrak{h}$ isomorphic to the Heisenberg algebra.
Cases (1) and (2) are simple.

Case (3). 
\( \mathfrak{h} \) is a commutative ideal in \( g \); \( G \) naturally acts on \( \mathfrak{h}^* \).

\[ \text{St}(h) \subset g \]
the stationary subalgebra for \( h \in \mathfrak{h}^* \);

\[ \Psi : \mathfrak{h}^* \to g \]
a rational map s.t. \( \Psi(h) \in \text{St}(h) \) for any \( h \);

\[ f_\Psi : g^* \to K \]
the function defined by \( f_\Psi(x) = \langle x, \Psi(\pi(x)) \rangle \).

**Lemma 2.** \( f_\Psi \in \text{Ann}(\mathfrak{h}) \).
\[ L = \{ \text{the set of all } \Psi \}; \]

\[ \mathcal{F}_L = \{ \text{the set of all } f_\Psi \} \subset \text{Ann } \mathfrak{h}; \]

\[ L \longrightarrow \mathcal{F}_L, \ \Psi \rightarrow f_\Psi \quad \text{homomorphism of Lie algebras} \]

**Main observation:** All these objects are defined not only over \( \mathbb{K} \), but also over the field \( \mathbb{K}(\mathfrak{h}^*) \) of rational functions on \( \mathfrak{h}^* \). Moreover, \( L \) and \( \mathcal{F}_L \) become finite-dimensional over \( \mathbb{K}(\mathfrak{h}^*) \).

**Lemma 3.** If MF conjecture holds for \( \mathcal{F}_L \) (over the new field \( \mathbb{K}(\mathfrak{h}^*) \)), then MF conjecture holds for \( g \) (over \( \mathbb{K} \)).
Case (4).
\( \mathfrak{h} \subset \mathfrak{g} \) is an ideal isomorphic to Heisenberg algebra:
\[ \mathfrak{h} = \langle p_1, \ldots, p_k, q_1, \ldots, q_k, e \rangle \text{ and } [p_i, q_i] = e. \]

**Lemma 4.** There is a subalgebra \( \mathfrak{b} \subset \mathfrak{g} \) such that
1. \( \mathfrak{b} \cap \mathfrak{h} = \langle e \rangle \)
2. \( \mathfrak{b} \) acts on \( V = \langle p_1, \ldots, p_k, q_1, \ldots, q_k, \rangle \) by symplectic transformations.

Let \( \beta \in \mathfrak{b} \). Consider \( f_\beta : \mathfrak{g}^* \to \mathbb{K} \) of the form
\[
f_\beta(x) = \beta(x) + \frac{\langle \omega^{-1}(\text{ad}^*_\beta \pi(x)), x \rangle}{2e(x)}
\]

**Lemma 5.** \( f_\beta \in \text{Ann}(\mathfrak{h}) \).
Lemma 6. \( \beta \rightarrow f_\beta \) is a monomorphism from \( \mathfrak{b} \) to \( \text{Ann} (\mathfrak{h}) \). The functions \( f_\beta \) generates \( \text{Ann} (\mathfrak{h}) \).

Lemma 7. If MF conjecture holds for \( \mathfrak{b} \), then MF conjecture holds \( \mathfrak{g} \).

Conclusion:
We can always either reduce the dimension of the given Lie algebra \( \mathfrak{g} \) or (if \( \mathfrak{g} \) is semisimple) construct a complete commutative subalgebra in \( P(\mathfrak{g}) \).

Theorem is proved by induction.
Open questions

Q1. Completeness in the sense of flows.

Q2. Singular orbits.

Q3. Other important polynomial Poisson algebras like Ann $\mathfrak{h}$. 


