Noetherian first integrals of nonholonomic systems

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Joint work with A. Giacobbe and A. Sansonetto

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0. Content

Link symmetries—integrals of motion in nonholonomic mechanics

Hamiltonian systems:

- Momentum map is conserved for all systems with given symmetry group: has the Noetherian property [*Ortega-Ratiu 2004*]

Nonholonomic systems:

- Sometimes symmetries do produce first integrals
- Sometimes they do not
- Sometimes not clear whether first integrals descend from symmetries

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In this talk:

- Idea: Noetherianity is an indication of a link symmetry–first integral
- Question: First integrals of a specific nonholomic system with symmetry are Noetherian?
- Answer: I present a method to obtain a (partial) answer, and some (still a little bit incomplete) aplications.

1. Example: the ball in a cup — an integrable system

Holonomic system: heavy sphere with center of mass on a cup[a]

- $Q = \mathbb{R}^2 \times \mathrm{SO}(3)$, $\dim TQ = 10$
- Symmetry group $G = S^1 \times \mathrm{SO}(3) \times \mathrm{SO}(3)$
- (Energy–)momentum map gives 5 first integrals
- Motions quasi–periodic on $\mathbb{T}^3$

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Nonholonomic system: sphere rolls without sliding in the cup

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Reduction

- Reduced phase space M/G is 4–dimensional
- Reduced system on M/G has 3 first integrals:
  - E = energy
  - J_1, J_2 solutions of a linear ODE
- Reduced system has periodic dynamics

---

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1. Example: the ball in a cup — reconstruction from periodic dynamic

Reconstruction from periodic reduced dynamics Setting:

- Free action of compact connected Lie group G on manifold M
- G–invariant vector field X on M
- Reduced system on M/G has periodic dynamics (with continuous period)

Then, the dynamics (generically) reconstructs to quasi–periodic motions on $\mathbb{T}^{r+1}$, $r = \operatorname{rank} G$.
[*M. Field 1970's, J. Hermans 1995]*

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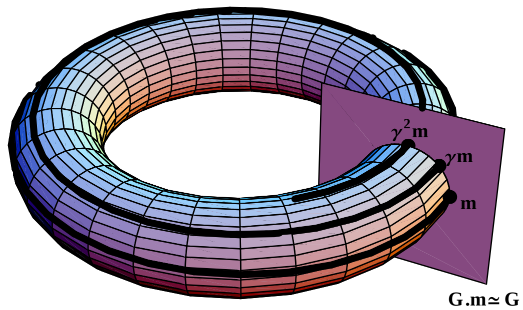

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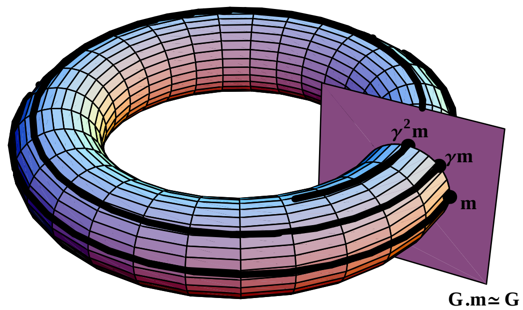
Ball in the cup:

- $r = \operatorname{rank} \mathrm{SO}(3) \times S^1 = 2$
- $\dim M - (r+1) = 5$ first integrals

Two other first integrals $G_1, G_2 \implies$ Superintegrability.
[*J. Hermans 1995, F. and Giacobbe 2006]*



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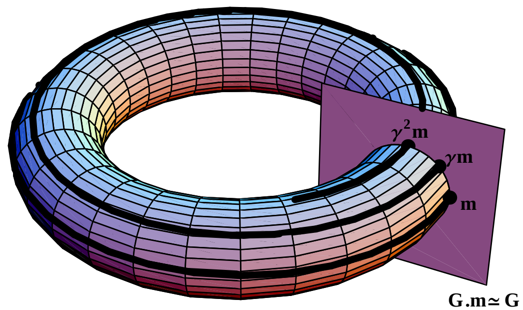

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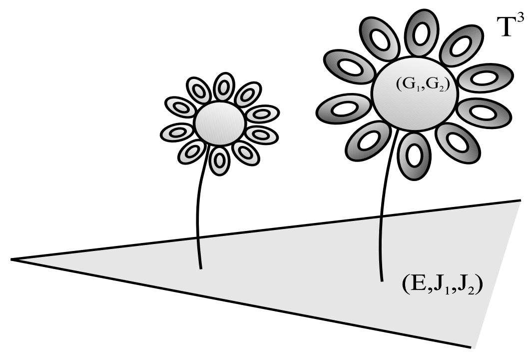

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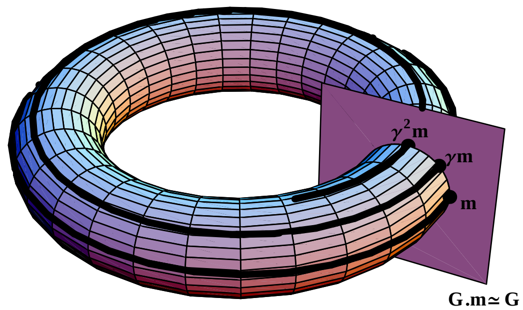
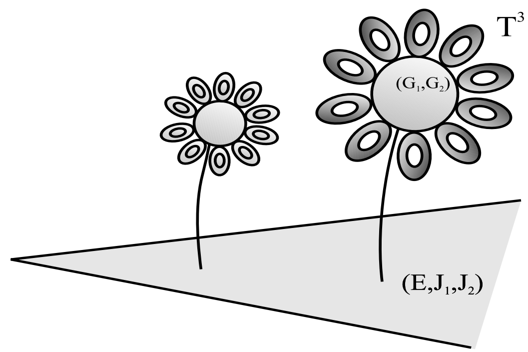


Are J_1, J_2, G_1, G_2 due to G–action?

- J_1, J_2 are gauge–momenta in the sense of Bates, Graumann and MacDonnel [1996] [*Ramos, Sansonetto*]
- G_1, G_2?



2. Nonholonomic Noether theorem — review of basic theory

Review now the nonholonomic version of Noether theorem.

- Need short review of basis of theory of nonholonomic mechanics.
- Only simplest cases: linear constraints, constant rank, free actions, natural Lagrangian, etc.
- Coordinate treatment wherever sufficient.

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Start with a holonomic system:

- Configuration manifold Q, $\dim Q = n$
- Lagrangian $L = T - V$

Add linear nonholonomic constraints: $\forall q \in Q$

- $\dot{q} \in D_q$ subspace of T_qQ
- $\dim D_q = k$

Terminology:

- D = *constraint distribution* (nonintegrable)
- (L, Q, D) = *Nonholonomic (Lagrangian) system*

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D'Alembert principle: reaction forces R at $(q, \dot q)$ annihilate D_q.

Then:

- Eliminate Lagrange multipliers: $R = R(q, \dot q)$
- Get a dynamical system on $D \subset T_Q$ given by $\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = R$

2. Nonholonomic Noether theorem — Agostinelli's (almost) theorem

Versions of Noether theorem for nonholonomic sysems have a long history: *..., Agostinelli 1956, ..., Arnold-Kozlov-Neishtadt 1980's, Bloch et Al. 1990's, Cantrijn, de Leon et al 1990's, Cushman et al 1990's, Sniatycki 1998 , ...*

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Proposition *(Agostinelli)* Let ξ^Q be a section of D. Then

$$F := p \cdot \xi^Q$$

is a first integral of $(Q, L = T - V, D)$ iff

- $L_{\xi^Q} V = 0$ in Q
- $L_{\xi^{TQ}} T = 0$ in D

Here:

- $p = \frac{\partial L}{\partial \dot{q}}$, thus F is the momentum of the $\mathbb{R}$–action on Q generated by ξ^Q
- ξ^{TQ} = tangent lift of ξ^Q.

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Proof Equations of motion give

$$\begin{aligned} \dot{F} &= \dot{p} \cdot \xi^Q + p \cdot \dot{\xi}^Q \\ &= \left(\frac{\partial T}{\partial q} - \frac{\partial V}{\partial q} + R \right) \cdot \xi^Q + \frac{\partial T}{\partial \dot{q}} \cdot \xi^{TQ}_{\dot{q}} \\ &= L_{\xi^{TQ}} T - L_{\xi^Q} V \end{aligned}$$

and the last term is $\dot{q}$–independent. ■

2. Nonholonomic Noether theorem: the simplest case

Agostinelli's theorem directly implies the following, elementary version of Noether theorem:

Corollary (Elementary version of Noether theorem) Given:

- Nonholonomic system $(L = T - V, Q, D)$
- $\mathbb{R}$–action Ψ^Q on Q with infinitesimal generator η^Q and tangent lift η^{TQ} such that
  - $L_{\eta^{TQ}} L_{|D} = 0$
  - η^Q is a section of D

Then $J_\eta := p \cdot \eta^Q$ is a first integral of (L, Q, D)

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Horizontal symmetries: Consider now a group G which acts on Q and preserve the Lagrangian. If $\eta \in \mathfrak{g}$ is such that $\eta^Q \in D$, then the momentum $J_\eta = p \cdot \eta^Q$ is a first integral.

[*Bloch et al, Marle, ...*]

Thus, only certain infinitesimal generators of a symmetry group of the holonomic system produce conserved momenta for the nonholonomic system.

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The notion of horizontal symmetries has

- Generalization to (sub)group actions
- Extensions to non–lifted actions are also possible (see later)

but horizontal symmetries are rare

Note: Ininfluent for first integrals that action preserves the constraints D.

2. Nonholonomic Noether theorem: Gauge momenta

Gauge momenta are an extension of horizontal symmetries.
Idea introduced in [*Bates, Graumann, MacDonnell 1996*], see also [*Marle 2003*]

Proposition

- G acts on Q
- $\eta_1, \dots, \eta_k$ basis of $\mathfrak{g}$

Assume exist functions $f_1, \dots, f_k : Q \longrightarrow \mathbb{R}$ such that

- $\xi^Q := \sum_j f_j \eta_j^Q$ is a section of D
- $L_{\xi^{TQ}}(T - V) = 0$

Then $p \cdot \xi^Q \ = \ \sum_j f_j J_{\eta_j}$ is first integral of $(L = T - V, Q, D)$.

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Variations and generalizations are possible, e.g. to non–lifted actions, but no applications so far.

The notion of gauge momentum makes it possible to link first integrals to symmetry groups in a number of cases, significantly larger than horizontal symmetries.

As of today, it seems to be the most effective nonholonomic version of Noether theorem.

2. Nonholonomic Noether theorem: Examples of HS and GM

Nonholonomic oscillator

$L = \frac{1}{2}(\dot{q}_x^2 + \dot{q}_y^2 + \dot{q}_z^2 - y^2)$

$D = \{\dot{z} = y\dot{x}\}$

$G = \mathbb{R}^2$ translations along x, z

- One GM [a]

Vertical coin

$G = \mathbb{R}^2 \times S^1 \times S^1$, translations in the plane, rotations about vertical, rotations about coin's axis.

- One HS
- One GM

Routh sphere $G = \mathbb{R}^2 \times S^1$

- Two GM
- One not known if GM

Ball in the cup $G = \mathrm{SO}(3) \times S^1$

- J_1, J_2 are GM (HS only in special cases)
- Not known if G_1, G_2 are GM

[a] HS=Horizontal Symmetry. GM=Gauge momentum but not SH

3. Noetherian first integrals

Consider:

- A manifold Q and a (costant rank) distribution D on Q.
- A Lie group G which acts on Q.
- A function $F : D \longrightarrow \mathbb{R}$

Definition

- F is (G, D)–Noetherian if it is a first integral of any nonholonomic system (L, Q, D) with G–invariant Lagrangian L.
- Fix $T : TQ \to \mathbb{R}$. F is weakly (G, D, T)–Noetherian if it is a first integral of any nonholonomic system $(L = T - V, Q, D)$ with G–invariant potential V.

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Proposition/Remark:

- Horizontal symmetries are Noetherian
- Gauge momenta are weakly Noetherian
- Gauge momenta might be non–Noetherian

2. Nonholonomic Noether theorem: Examples of HS and GM revisited

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- Two GM — One Noetherian, one not known
- One not known if GM

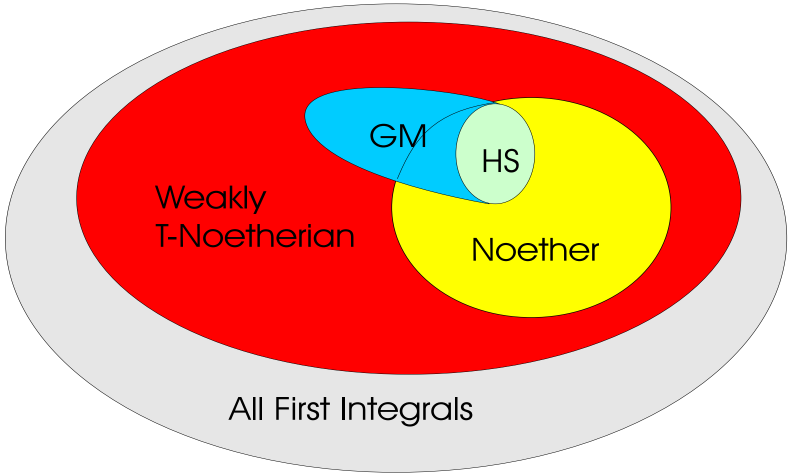
Ball in the cup

$G = \mathrm{SO}(3) \times S^1$

- J_1, J_2 are GM — not (yet) known if Noetherian
- but what about G_1, G_2? Not even known if they are GM.

Note: Noetherianity can be proved with techniques to be seen later

3. Noetherian first integrals — Questions



Questions:

- Are GM ‘typically’ also Noetherian? (They need not be)
- Are there ‘many’ Noetherian first integrals besides HS and possibly GM?
- Are there ‘many’ weakly–Noetherian first integrals besides GM?
- Where do G_1, G_2 of the ball in the cup lie in this diagram?
- Is it possible to apriori compute/bound the number of Noetehrian first integrals?
- And that of the weakly–Noetherian first integrals?

Need geometry.
Mostly for personal preference, pass to the Hamiltonian setting

3. Hamiltonian formulation

Legendre transform

- $\Lambda : TQ \longrightarrow T^*Q$
- $(L = T - V, TQ) \mapsto (H = T + V, T^*Q)$
- Constraint manifold $M := \Lambda(D)$.

3. Hamiltonian formulation

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There is another distribution, $\overline{D}$, along M:

- $\overline{D}_{(q,p)} := \{(\dot{q}, \dot{p}) \ : \ \dot{q} \in D_q\} \neq T_{(q,p)}M$
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Splitting property:

At points $m \in M$:

$$T_m T^*Q = \overline{D}^{\omega}_m \oplus T_m M.$$

[...., *Marle 1995, ...*]

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Equations of motion: $\dot m = X_H^{TM}(m), m \in M.$

(H, Q, M): Nonholonomic (Hamiltonian) system

NB: X_H^{TM} *is section of* $\overline{D} \cap TM$.

3. Hamiltonian formulation — First integrals

Hamiltonian characterization of first integrals.
Sometimes (improperly—given that there is no group action?) called “Nonholonomic Noether theorem”

Proposition A function $F : M \to \mathbb{R}$ is a first integral of (H, Q, M) if and only if any of the following two equivalent conditions is fullfilled:

i. $\ker dF \supset (\ker dH \cap \overline{D})^\omega \cap TM$.

ii. One (and hence any) extension $\tilde{F}$ of F off M satisfies

$$X_{\tilde{F}} \in (\ker dH \cap \overline{D}) \cap TM^\omega.$$

in the points of M.

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Proof (i.) $L_{X_H^{TM}} F = 0 \iff \mathrm{Span} X_H^{TM} \subset \ker dF$. Using splitting $T_m T^*Q = \overline{D}_m^\omega \oplus T_m M$ gives

$$\begin{aligned}
\mathrm{Span} X_H^{TM} &= (\mathrm{Span} X_H + \overline{D}^\omega) \cap TM \\
&= \big((\ker dH)^\omega + \overline{D}^\omega\big) \cap TM \\
&= (\ker dH \cap \overline{D})^\omega \cap TM
\end{aligned}$$

(ii.) Use duality. ■

4. Weakly Noetherian First Integrals (WNFI)

Our aim: to find an upper bound on the number of WNFI

Setting:

- Fix manifold Q, distribution D on Q, kinetic energy $T : TQ \to \mathbb{R}$, and action Ψ^Q of group G on Q.
- I_G := family of all G–invariant functions on Q.

WNFI in Hamiltonian formulation: A weak T–Noetherian first integral is a function $F : D \to \mathbb{R}$ which is first integral of $(T - V, Q, D)$ for any $V \in I_G$.
Since $\Lambda : TQ \to T^*Q$ is fixed, $M = \Lambda(D)$ is fixed.
Hence, F is WNFI iff $F \circ \Lambda^{-1} : M \to \mathbb{R}$ is first integral of $(H = T + V, Q, M)$ for any $V \in I_G$.

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From the characterization of FI: F is WNFI iff

$$\ker dF \supset \big(\ker d(T+V) \cap \overline{D}\big)^{\omega} \cap TM \quad \forall V \in I_G$$

namely iff

$$\ker dF \supset \bigcap_{V \in I_G} \big(\ker d(T+V) \cap \overline{D}\big)^{\omega} \cap TM$$

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WNFI in Hamiltonian formulation: A weak T–Noetherian first integral is a function $F : D \to \mathbb{R}$ which is first integral of $(T - V, Q, D)$ for any $V \in I_G$.
Since $\Lambda : TQ \to T^*Q$ is fixed, $M = \Lambda(D)$ is fixed.
Hence, F is WNFI iff $F \circ \Lambda^{-1} : M \to \mathbb{R}$ is first integral of $(H = T + V, Q, M)$ for any $V \in I_G$.

From the characterization of FI: F is WNFI iff

$$\ker dF \supset \big(\ker d(T+V) \cap \overline{D})^\omega \cap TM \quad \forall V \in I_G$$

namely iff

$$\ker dF \supset \bigcap_{V \in I_G} \big(\ker d(T+V) \cap \overline{D})^\omega \cap TM$$

Proposition Let

$$\mathcal{G} := \bigcap_{V \in I_G} \ker d(T+V)$$

$$\Delta := (\mathcal{G} \cap \overline{D})^\omega \cap TM$$

Then $F : M \to \mathbb{R}$ is WNFI iff

$$\ker dF \supset \Delta$$

4. WNFI — How to bound their number?

Bound on $\#$ of WNFI
A WNFI is a first integral of a distribution Δ on M.
(Existence of WNFI = integrability conditions on Δ)

Number of WNFI can thus be studied:

- Determine $\mathcal{G}$
- Determine $\Delta = (\mathcal{G} \cap \overline{D})^{\omega} \cap TM$
- Determine $\#$ of (local) integrals of Δ.

This gives upper bound on (global) WNFI

Determination of $\mathcal{G}$

Sufficient here to characterize $\mathcal{G} := \bigcap_{V \in I_G} \ker d(T+V)$ in Darboux coordinates (q, p).

Write

$T(q,p) = \frac{1}{2} p \cdot B(q) p$

TO_G = distribution by tangent spaces to G–orbits in Q.

4. WNFI – $\mathcal{G}$

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Lemma Assume G acts freely and properly on Q. Then

$$\mathcal{G}_{(q,p)} = \left\{ (u_q, u_p) \,:\, u_q \in TO_G \,,\; u_p \in \left(\tfrac{\partial T}{\partial p}\right)^{\perp} - \|Bp\|^{-2} \left(u_q \cdot \tfrac{\partial T}{\partial q}\right) Bp \right\} \text{ if } p \neq 0$$

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Proof $u = (u_q, u_p) \in \mathcal{G}_{(q,p)}$ iff $d(T+V)u = 0$ for all $V \in I_G$, namely iff

- $\frac{\partial V}{\partial q} u_q = 0$ for all $V \in I_G$
- $\frac{\partial T}{\partial q} \cdot u_q + \frac{\partial T}{\partial p} \cdot u_p = 0$

namely iff

- $u_q \in TO_G$ (see [*Ortega-Ratiu 2004*])
- $u_p \in \left(\frac{\partial T}{\partial p}\right)^{\perp} - \|\frac{\partial T}{\partial p}\|^{-2} \left(u_q \cdot \frac{\partial T}{\partial q}\right) \frac{\partial T}{\partial p}$ ■

4. WNFI — Estimate of their number

Recall that Δ is a distribution on M. Denote

- Δ^∞ := smallest integrable distribution on M which contains Δ ("involutive closure" of Δ).
- $c_m := \mathrm{corank}_M \Delta^\infty_m$, $m \in M$.
- $M^\infty_{\mathrm{reg}} :=$ set of regular points of Δ^∞.

M^∞_{reg} is open and dense in M.

Proposition For each $m \in M$, there are c_m but not c_{m+1} independent germs of first integrals of Δ.

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Proposition For each $m \in M$, there are c_m but not c_{m+1} independent germs of first integrals of Δ.

Proof: Each point $m \in M^\infty_{\mathrm{reg}}$ has a neighbourhood U_m in which Δ^∞ has constant rank c_m.

- In U_m there are c_m functionally independent first integrals of Δ.
- If in an open set $U \subset U_m$ there were $c_m + 1$ independent first integral of Δ, then Δ^∞ would not be smallest integrable distribution which contains Δ. ■

4. Weakly NFI — Determining Δ^∞

Standard technique for determining Δ^∞, if Δ is *real analytic*.

- Consider (local) generators $X_1, \ldots, X_r$ of Δ, so that $\Delta \; = \; \mathrm{Span}\{X_1, \ldots, X_r\}$.
- Define
  $\Delta^1 = \Delta$
  $\Delta^2 = \mathrm{Span}\{X_1, \ldots, X_r, [X_1, X_2], \ldots, [X_{r-1}, X_r]\}$
  $\Delta^3 = \mathrm{Span}\{X_1, \ldots, [X_1, X_2], \ldots, [X_1, [X_1, X_2]], \ldots\}$
  etc.

Then, $\Delta^\infty = \Delta^s$ where s is the smallest positive integer such that Δ^s is integrable, that is, $\Delta^s = \Delta^{s+1}$.

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Then, $\Delta^\infty = \Delta^s$ where s is the smallest positive integer such that Δ^s is integrable, that is, $\Delta^s = \Delta^{s+1}$.

Can be implemented in two ways:

- Work in M, e.g. by using local coordinates M
- Work in T^*Q using *any* extension $\widehat{X}_1, \ldots, \widehat{X}_r$ of $X_1, \ldots, X_r$ off M. Reason:
  $[\tilde{X}_i, \tilde{X}_j]_{|M} \;=\; [X_i, X_j]$
  Convenient, e.g., to use trivializations.

4. WNFI — The ball in the cup & conclusions

Application:

The two additional integrals G_1 and G_2 of the ball in the cap are WNFI?

4. WNFI — The ball in the cup & conclusions

Application:

The two additional integrals G_1 and G_2 of the ball in the cap are WNFI?

Yes (?) (modulo checking the computations....)

Things to do:

- Check computations for ball in the cup. Try to reproduce G_1, G_2 with some variants of the GM method.
- What happens if other $\mathrm{SO}(3)$–action is taken into consideration?
- Develop a similar method to study the Notherianity of FI
- Systematically check WN and N in known cases.
- Explore variants of the gauge method