CIRM, Luminy
30/11/2006

Integrable Chaos

A.P. Veselov
Loughborough, UK

$\longleftrightarrow$

Ref.: A.V. "What is an integrable mapping." In "What is Integrability" (Ed. V.E. Zakharov), Springer, 1991

A.V. "Integrable maps" Russian Math. Surveys 46:5 (1991), 1-51

Main theme / leitmotiv

Chaos and Integrability could be compatible

Moreover,

many of standard examples of chaotic systems are actually integrable (in some sense)

and

many of integrable maps exhibit chaotic behaviour

2

Integrability of maps

$$f : M^N \longrightarrow M^N \quad \text{smooth}$$

Iterations: $\boxed{x_{n+1} = f(x_n)}$

3 approaches to Integrability:

(I1) Solvability, or Linearisability

$$\begin{array}{ccc} V & \xrightarrow{A} & V \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array} \quad \text{linear}$$

(I2) Existence of symmetries

$g : M \to M$ is a $\overbrace{\text{symmetry}}^{\text{non-trivial}}$ of f

if $\boxed{f \circ g = g \circ f}$ $\quad f^{(k)} \neq g^{(\ell)}$

(I3) Liouville Integrability (for symplectic maps)

$$f : (M^{2n}, \omega) \longrightarrow (M^{2n}, \omega)$$

n independent integrals in involution $F_1, \dots, F_n$

3

Chaos

$f : X \longrightarrow X$ metric space

(C1) Transitivity

$\forall \underset{\text{open}}{U, V} \subset X \quad \exists k : f^{(k)}(U) \cap V \neq \phi$

(C2) Density of periodic points

$f^{(N)}(x) = x$ period N poi

(C3) Sensitive dependance on initial conditions

$\exists \varepsilon > 0 : \forall x_0 \in X$ and open $U \ni x_0$

$\exists y_0 \in U$ and $k \in \mathbb{N}$ such that

$$d\left(f^{(k)}(x_0), f^{(k)}(y_0)\right) > \varepsilon$$

4

Standard examples of chaotic maps

1 Double angle map

$X = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

$f : z \to z^2$

$z = e^{2\pi i \varphi}, \quad \varphi \in \mathbb{R}/\mathbb{Z}$

$$\boxed{f : \varphi \longrightarrow 2\varphi \pmod{\mathbb{Z}}}$$

$\varphi = 0.a_1 a_2 a_3 \dots, \quad a_i \in \{0,1\}$ binary repres.

$f \downarrow$

$2\varphi = 0.a_2 a_3 a_4 \dots$

(Bernoulli shift)

Chaos C1–3 ✓

Integrability I1–2 ✓

5

② Cat-map (or Anosov map)

$X = T^2 = \mathbb{R}^2/\mathbb{Z}^2 \;(\neq$  $!)$

$f = A : T^2 \to T^2$ $\qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

hyperbolic torus automorphism $\qquad ad - bc = 1$

$a + d > 1$

Example $\quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ (hyperbolic)

$\boxed{\text{Th (Anosov) } A \text{ is structurally stable in } C^1 \text{ topology}}$

Chaos C1–3 ✓ $\quad \boxed{h_{top} = \log \lambda}$

topological entropy > 0

Integrability I1–2 ✓ $\qquad \lambda = \frac{3+\sqrt{5}}{2} >$

I3 ? $\qquad$ eigenvalue of A

⑥ Consider extension of cat-map

$$\hat{A} : T^*T^2 \longrightarrow T^*T^2$$

$\boxed{\text{Th. } \hat{A} \text{ is integrable in Liouville sense.}}$

Proof (Bolsinov, Taimanov, 2000)

In the eigenbasis of A

$$f : \begin{pmatrix} u \\ v \end{pmatrix} \xrightarrow[(\mathrm{mod}\ \mathcal{L})]{} \begin{pmatrix} \lambda u \\ \lambda^{-1} v \end{pmatrix}, \quad \begin{pmatrix} p_u \\ p_v \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda^{-1} p_u \\ \lambda p_v \end{pmatrix}$$

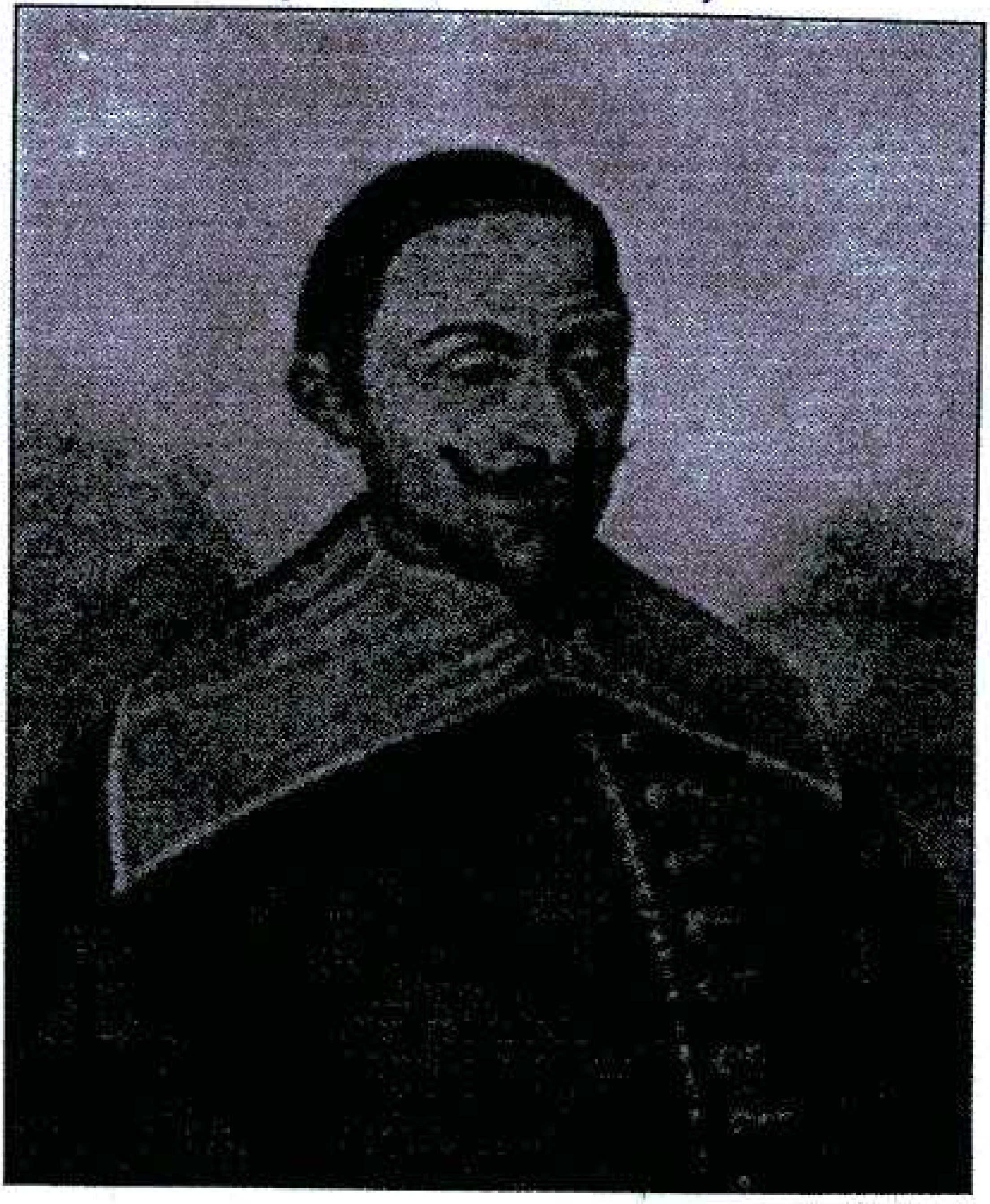
Two integrals (smooth, but not analytic)

$$\boxed{\begin{aligned} F_1 &= p_u p_v \\ F_2 &= e^{-\frac{1}{p_u^2 p_v^2}} \sin\left(2\pi \frac{\log p_u^2}{\log \lambda^2}\right) \end{aligned}}$$

Critical level is chaotic! $p_u = p_v = 0$

Claude Bachet

(1581 – 1638)



Mathematician, poet, linguist,
one of the first members of
Académie Française

DIOPHANTI ALEXANDRINI ARITHMETICORVM LIBRI SEX,

ET DE NVMERIS MVLTANGVLIS LIBER VNVS.

CVM COMMENTARIIS C. G. BACHETI V. C. & obseruationibus D. P. de FERMAT Senatoris Tolosani.

Accessit Doctrinæ Analyticæ inuentum nouum, collectum ex varijs eiusdem D. de FERMAT Epistolis.

TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Iesu.
M. DC. LXX.

Bachet's translation of Diophantus' "Arithmetica", in which Fermat wrote his famous "Last Theorem" marginal note

(7)

Integrable rational maps with chaotic dynamics

Claude Gaspar Bachet (1581–1638)

considered rational solutions to Diophantine equation

$$\boxed{y^2 - x^3 = c}$$

Th. (Bachet, 1621) If (x,y) is a solution, then so is

$$\boxed{(x',y') = \left(\frac{x^4 - 8cx}{4y^2}, \frac{-x^6 - 20cx^3 + 8c^2}{8y^3}\right)}$$

For example, if $\boxed{c = -2}$ then

$$(3,5) \to \left(\frac{129}{100}, -\frac{383}{1000}\right) \to \left(\frac{2340922881}{7660^2}, \frac{113259286337292}{7660^3}\right)$$

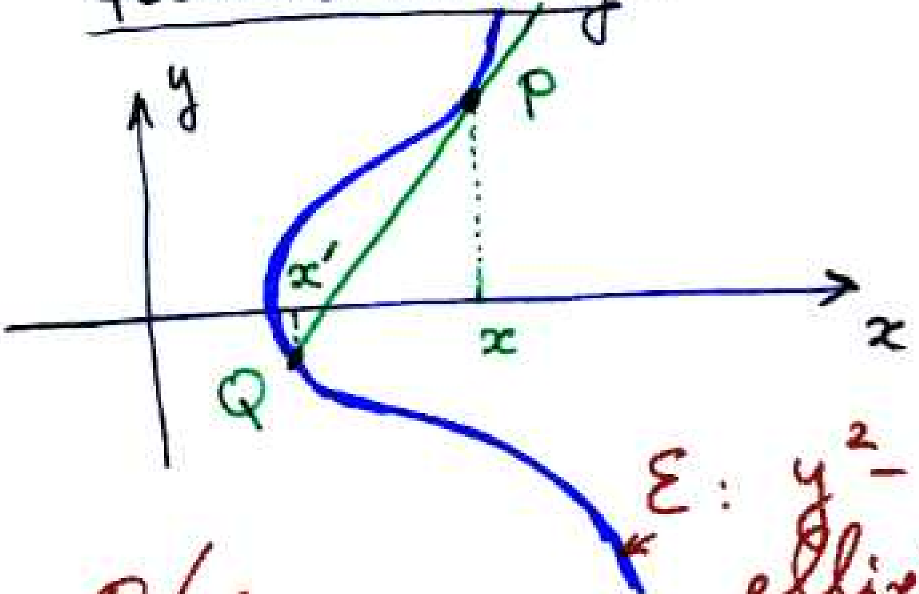
allows to produce infinitely many rational solutions (fastly growing!)

(8) Rational map $B : \mathbb{CP}^1 \to \mathbb{CP}^1$

$$\boxed{x \longmapsto B(x) = \frac{x^4 - 8cx}{4(x^3 + c)}}$$

is called <u>Bachet map</u>

<u>Geometrically</u>:

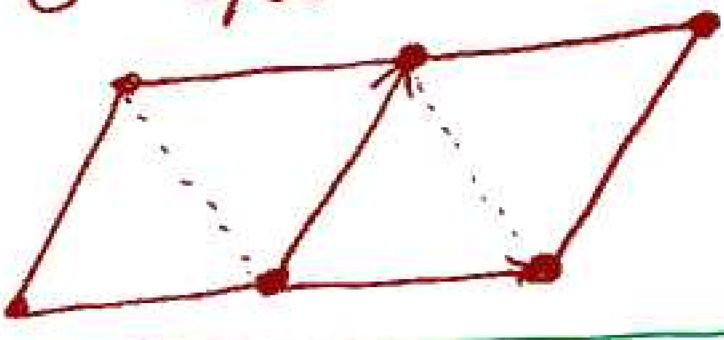


$\mathcal{E} : y^2 - x^3 = C$

<u>elliptic curve</u>

(equi anharmonic)

$\mathcal{E} = \mathbb{C}/\mathcal{L}$



$2P + Q \sim 0$

$Q \sim -2P$

$$\boxed{\begin{tikzcd} \mathcal{E} \arrow[r, "z \to -2z"] \arrow[d, "x = \wp(z)"'] & \mathcal{E} \arrow[d, "\wp(z)"] \\ \mathbb{CP}^1 \arrow[r, "B"] & \mathbb{CP}^1 \end{tikzcd}}$$

$$\boxed{\wp(2z) = B(\wp(z))}$$

<u>duplication formula</u>

⑨ Bachet map is chaotic with Julia set $\boxed{\mathcal{J}(B) = \mathbb{C}P^1}$!

Integrability: I1 ✓ I2 ?

Commuting maps

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{z \to nz} & \mathcal{E} \\ \wp \downarrow & & \downarrow \wp \\ \mathbb{C}P^1 & \xrightarrow{B_n} & \mathbb{C}P^1 \end{array}$$

$$\boxed{\wp(nz) = B_n(\wp(z))}$$

$$\boxed{B_n \circ B_m = B_{nm} = B_m \circ B_n}$$

In particular, $B = B_2$ has infinitely many non-trivial symmetries B_n

$\Rightarrow$ I2 ✓

Commuting rational maps

Th. (J. Ritt (1924))

Let $f, g : \mathbb{C}P^1 \to \mathbb{C}P^1$, degree > 1

$\boxed{f \circ g = g \circ f}$ and $\boxed{f^{(k)} \neq g^{(\ell)}}$

Then $\boxed{\varphi(\alpha x + \beta) = f(\varphi(x))}$, where

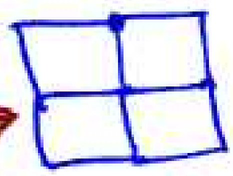
1) $\varphi = e^x$

2) $\varphi = \cos x$

3) $\varphi = \wp(x)$ for any elliptic curve

4) $\varphi = \wp^2(x)$, $\tau = i$

(lemniscatic)



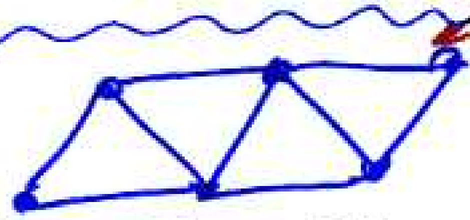
5) $\varphi = \wp'(x)$, $\tau = \varepsilon$, $\varepsilon^3 = 1$

6) $\varphi = \wp^3(x)$, $\tau = \varepsilon$, $\varepsilon^3 = 1$

$\Big\}$ equi-anharmonic

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{z \to \alpha z + \beta} & \mathcal{E} \\ G \downarrow & & \downarrow G \\ \mathbb{C}P^1 & \longrightarrow & \mathbb{C}P^1 \end{array}$$

Symmetry group



$G = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_4$

(11)

Commuting polynomial maps

G. Julia, P. Fatou (1922), J. Ritt (1924)

$f = a_n z^n + \dots + a_0, \quad g = b_m z^m + \dots + b_0$

$$\boxed{f \circ g = g \circ f} \Rightarrow \boxed{f = z^n} \text{ or } \boxed{f = T_n(z)}$$

T_n – Chebyshev (Tchebycheff) polynomials

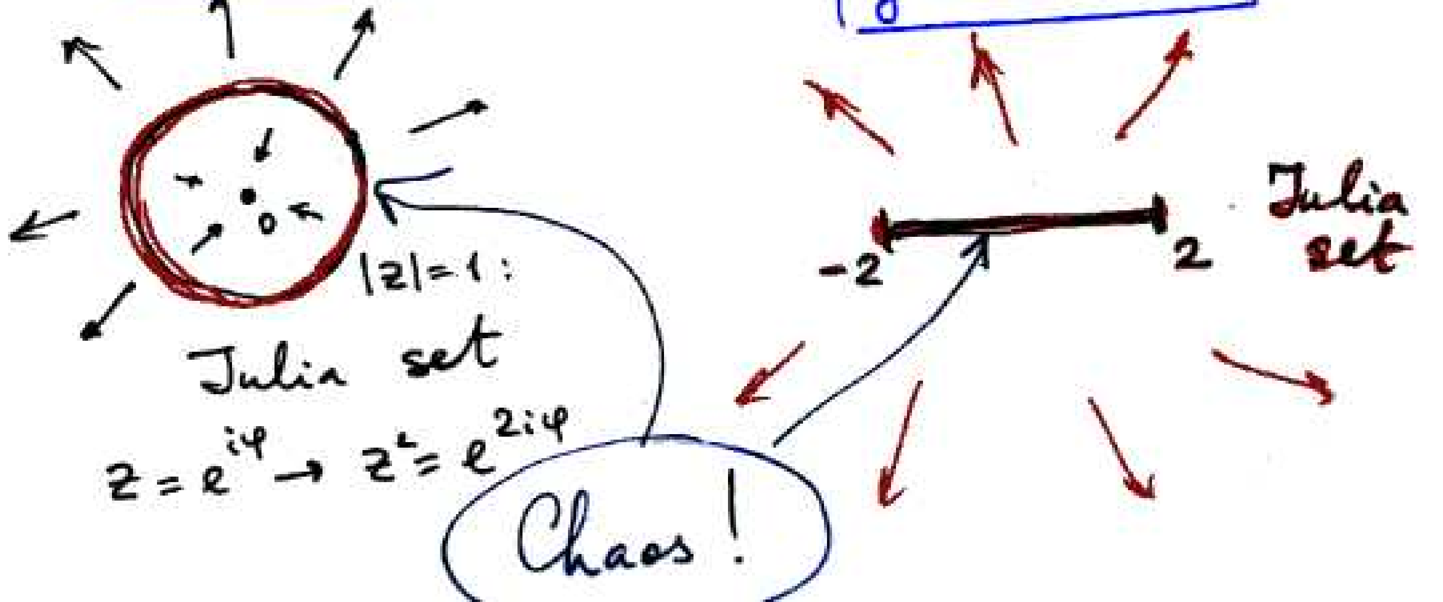
$$\boxed{\cos n\varphi = T_n(\cos\varphi)}$$

$$T_n \circ T_m = T_{nm} = T_m \circ T_n$$

Example
$n = 2$

$f = z^2 + c$ has commuting onl

if $\boxed{f = z^2}\boxed{g = z^3}$ or $\boxed{f = z^2 - 2}$ $\boxed{g = z^3 - 3z}$



(12) Multidimensional Chebyshev maps and Weyl groups

A.V. (1987):

$\mathfrak{g}$ simple complex Lie algebra (of rank n)

H Cartan subalgebra, $H \simeq \mathbb{C}^n$

$\chi_1, \dots, \chi_n$ characters of fund. rep.

$$\begin{array}{ccc} H & \xrightarrow{h \to mh} & H \\ \chi \downarrow & & \downarrow \chi \\ \mathbb{C}^n & \xrightarrow[\text{polynomial}]{P_m^{\mathfrak{g}}} & \mathbb{C}^n \end{array}$$

$$\boxed{P_n^{\mathfrak{g}} \circ P_m^{\mathfrak{g}} = P_m^{\mathfrak{g}} \circ P_n^{\mathfrak{g}}}$$

Th. (Chevalley, 1955) $\chi_1, \dots, \chi_n$ freely generate the algebra of exponential invariants of Weyl group W

$$\Rightarrow \boxed{\chi_k(mh) = P_m^k(\chi_1, \dots, \chi_n)}$$

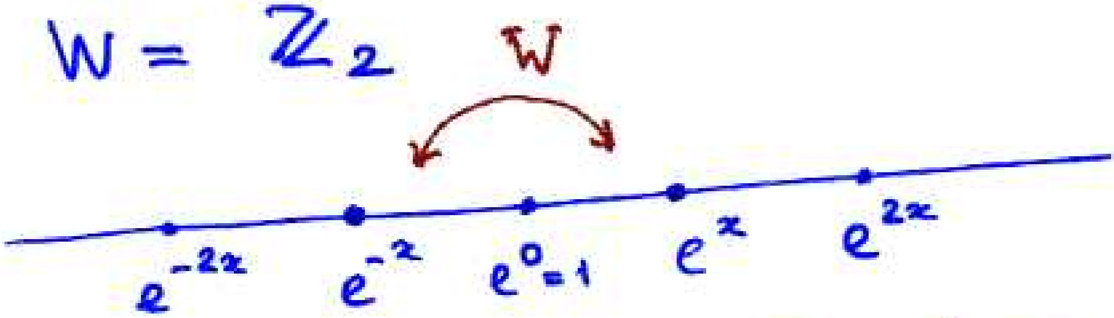
for some polynomial P_m^k

(13)

Examples

A_1-type, $\mathfrak{g} = sl(2, \mathbb{C})$

$W = \mathbb{Z}_2$



$$\chi = e^x + e^{-x} = 2\cosh x$$

$$x \to 2x$$

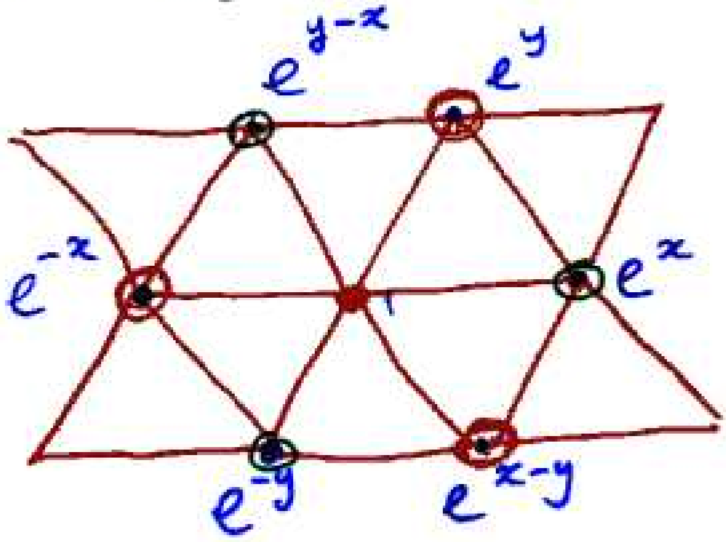
$$\chi \longrightarrow 2\cosh 2x = 2(2\cosh^2 x - 1)$$

$$= 4\cosh^2 x - 2 = \chi^2 - 2$$

so $\boxed{P_2^{A_1} : z \to z^2 - 2}$

(14) A_2-type, $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$

$W = S_3$



$$\begin{cases} \varphi_1 = e^{x} + e^{y-x} + e^{-y} \\ \varphi_2 = e^{-x} + e^{x-y} + e^{y} \end{cases}$$

$$(x, y) \longrightarrow (2x, 2y)$$

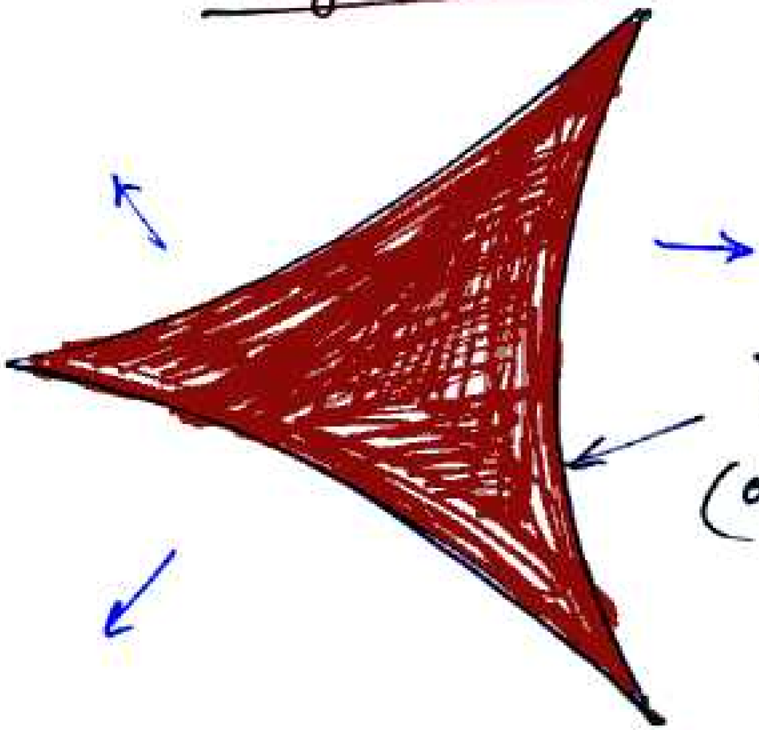
$$\varphi_1' = e^{2x} + e^{2y-2x} + e^{-2y}$$

$$= \varphi_1^2 - 2\varphi_2$$

$$\varphi_2' = \varphi_2^2 - 2\varphi_1$$

so $\boxed{P_2^{A_2} : \begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u^2 - 2v \\ v^2 - 2u \end{pmatrix}}$

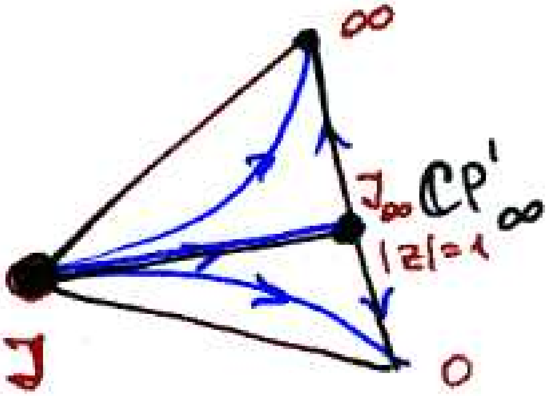
(15) Julia set : image of <u>Weyl alcove</u>



<u>Deltoid</u>
(or <u>Steiner curve</u>)

Dynamics on $\mathbb{CP}^2$:

toric varieties $\xrightarrow[\text{map}]{\text{momentum}}$ polyhedra



$P|_{\mathbb{CP}^1_\infty} : z \to z^2$

$$\hat{P}_2^{A_2} : \begin{pmatrix} u \\ v \\ w \end{pmatrix} \to \begin{pmatrix} u^2 - 2vw \\ v^2 - 2uw \\ w^2 \end{pmatrix}$$

15

Elliptic version of Chevalley theorem (E. Loojenga (1976), I. Bernshtein, O. Schwartsman (1978))

A.V. '87 $\Rightarrow$ Commuting rational maps

$$\mathbb{C}P^n \longrightarrow \mathbb{C}P^n$$

Example

$$\boxed{f(x:y:z) = \left(\frac{1}{2i}x(y-z) : -\frac{1}{4}[(y+z)^2 - x^2] : yz\right)}$$

has a commuting map, but is chaotic on the whole $\mathbb{C}P^2$.

$$\begin{tikzcd} \mathcal{E}^n \arrow[r, "z \to mz"] \arrow[d, "W"'] & \mathcal{E}^n \arrow[d, "W"] \\ \mathbb{C}P^n \arrow[r, "R_m"] & \mathbb{C}P^n \end{tikzcd}$$

$$R_m \circ R_k = R_k \circ R_m$$

Some open problems

1. Classification of commuting polynomial maps in $\mathbb{C}^n$, $n \geq 2$ (cf. O. Chalykh, M. Feigin, A.V.: commutative rings of diff operators of Calogero-Moser type)

2. Classification of commuting correspondences in $\mathbb{CP}^1$

Example: Modular correspondences

$$\boxed{\Phi_n(j(\tau), j(n\tau)) = 0}$$

j is modular invariant of $\mathcal{E}(\tau)$

$n=2$:

$$\boxed{\begin{aligned}\Phi_2(x,y) &= x^3+y^3-x^2y^2+2^4\cdot 3\cdot 31\, xy(x+y)\\ &- 2^4\cdot 3^4\cdot 5^3(x^2+y^2)+3^4\cdot 5^3\cdot 4027\, xy\\ &+ 2^8\cdot 3^7\cdot 5^6(x+y)-2^{12}\, 3^9\, 5^9 = 0\end{aligned}}$$