Singularity confinement for a class of recursion relations

Joint work with Mark Adler and Pierre van Moerbeke

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Unitary integrals & combinatorics

Object of interest:

$$\tau_n(t,s) = \int_{U(n)} \exp\left(\sum_{j=1}^{N} \mathrm{Tr}(t_j M^j - s_j M^{-j})\right) dM$$

- $U(n)$ unitary group
- dM Haar measure
- t_j, s_j "time variables"

Basic examples:

$$\tau_n(t) = \int_{U(n)} e^{t\,\mathrm{Tr}(M+M^{-1})}\, dM$$

$$\tau_n(t) = \int_{U(n)} e^{t\,\mathrm{Tr}(M^2+M^{-2})}\, dM$$

Example 1: increasing subsequences

Gessel's Theorem (1990)

$$\int_{U(n)} e^{t\,\mathrm{Tr}(M+M^{-1})}\,dM = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} P_k(\pi \in S_k \mid \ell(\pi) \leq n)$$

- S_k permutations of $\{1, 2, \ldots, k\}$
- $\ell(\pi)$ length of the longest increasing subsequence of π

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$$P_2(\pi \in S_2 \mid \ell(\pi) \leq 1) = \frac{1}{2}; P_2(\pi \in S_2 \mid \ell(\pi) \leq 2) = 1.$$

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- $\ell(\pi)$ length of the longest increasing subsequence of π
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$$P_3(\pi \in S_3 \mid \ell(\pi) \le 1) = \frac{1}{6}; P_3(\pi \in S_3 \mid \ell(\pi) \le 2) = \frac{5}{6}.$$

Example 2: odd permutations

Rains (1998); Tracy-Widom (1999)

$$\int_{U(n)} e^{s\,\mathrm{Tr}(M^2+M^{-2})}\,dM = \sum_{k=0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P_{2k}(\pi \in S_{2k}^{odd} \mid \ell(\pi) \le n)$$

1. S_{2k}^{odd} permutations σ of $\{-k, \dots, -1, 1, \dots, k\}$ with $\sigma(-i) = -\sigma(i)$
2. P_{2k} uniform probability on S_{2k}^{odd}
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4. In the case of

$$S_{2k+1}^{odd} = \text{odd permutations of } \{-k,\dots,-1,0,1,\dots,k\}$$

the unitary integral involves

$$e^{t\,\mathrm{Tr}(M+M^{-1}) \pm s\,\mathrm{Tr}(M^2+M^{-2})}$$

Unitary integrals & integrable systems

$$\tau_n(t,s) = \int_{U(n)} \exp\left(\sum_{j=1}^{N} \mathrm{Tr}(t_j M^j - s_j M^{-j})\right) dM$$

$\Longrightarrow$ an inner product on functions on the circle

$\Longrightarrow$ (bi-)orthogonal polynomials

$\Longrightarrow$ an integrable system $\subset$ 2-Toda lattice

$\Longrightarrow$ recursion relations

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$$\langle f, g \rangle_{s,t} = \int_{S^1} \frac{dz}{2\pi i z} f(z) g(z^{-1}) \exp\left(\sum_{j=1}^{N} (t_j z^j - s_j z^{-j})\right) dz$$

$$\tau_n(t,s) = \det\left(\left\langle z^k, z^l \right\rangle_{s,t}\right)_{0 \le k,l \le n-1}$$

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$\Longrightarrow$ (bi-)orthogonal polynomials $p_n^{(1)}(t,s;z),\ p_n^{(2)}(t,s;z)$

$$\left\langle p_n^{(1)}, p_m^{(2)}\right\rangle = \delta_{n,m} \frac{\tau_{n+1}(t,s)}{\tau_n(t,s)}$$

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The Toeplitz lattice

The constant coefficients of the bi-orthogonal polynomials

$$x_n(t,s) := p_n^{(1)}(t,s;0) \qquad y_n(t,s) := p_n^{(2)}(t,s;0)$$

satisfy for $i = 1, \ldots, N$ the differential equations

$$\frac{dx_k}{dt_i} = (1 - x_k y_k)\frac{\partial H_i^{(1)}}{\partial y_k}, \qquad \frac{dy_k}{dt_i} = -(1 - x_k y_k)\frac{\partial H_i^{(1)}}{\partial x_k}$$

and

$$\frac{dx_k}{ds_i} = (1 - x_k y_k)\frac{\partial H_i^{(2)}}{\partial y_k}, \qquad \frac{dy_k}{ds_i} = -(1 - x_k y_k)\frac{\partial H_i^{(2)}}{\partial x_k}$$

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- almost standard Poisson structure $\{x_k, y_l\} = (1 - x_k y_l)\delta_{kl}$
- all $H_i^{(k)}$ and $H_j^{(l)}$ are in involution, $\{H_i^{(j)}, H_j^{(l)}\} = 0$
- $H_i^{(1)} = -\frac{1}{i}\mathrm{Tr}\, L^i$ and $H_i^{(2)} = -\frac{1}{i}\mathrm{Tr}\, M^i$

The Toeplitz matrices

$$L = \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & 0 & \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & \\ -x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 & \\ & & & & \ddots \end{pmatrix}$$

$$M = \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 & \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 & \\ 0 & 1-x_2y_2 & -x_2y_3 & -x_2y_4 & \\ 0 & 0 & 1-x_3y_3 & -x_3y_4 & \\ & & & & \ddots \end{pmatrix}.$$

$$\mathrm{Tr}\, L = -\sum_{i=0}^{\infty} x_{i+1}y_i, \qquad \mathrm{Tr}\, M = -\sum_{i=0}^{\infty} x_i y_{i+1}.$$

The first Toeplitz vector field(s)

$$\frac{dx_k}{dt_1} = (1 - x_k y_k)x_{k+1} \qquad \frac{dy_k}{dt_1} = -(1 - x_k y_k)y_{k-1}$$
$$\frac{dx_k}{ds_1} = (1 - x_k y_k)x_{k-1} \qquad \frac{dy_k}{ds_1} = -(1 - x_k y_k)y_{k+1}$$

duality: $x_i \leftrightarrow y_i \quad L \leftrightarrow M^T \quad \{\cdot\,,\cdot\} \leftrightarrow -\{\cdot\,,\cdot\} \quad s_i \leftrightarrow -t_i$

self-dual: $x_i = y_i \quad L = M^T \quad \{\cdot\,,\cdot\} =? \quad s_i = -t_i$

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The first self-dual Toeplitz vector field

$$\frac{dx_k}{dt} = (1 - x_k^2)(x_{k+1} - x_{k-1})$$

Similar to, but different from, the Kac-van Moerbeke lattice

$$\frac{dx_k}{dt} = x_k (x_{k+1} - x_{k-1})$$

The recursion relations

$(x_k)_{k\in\mathbf{N}}$ and $(y_k)_{k\in\mathbf{N}}$ also satisfy a recursion relation $\Gamma_k = \Delta_k = 0$ $(k = 1, 2, \ldots)$. In the self-dual case:

$$\Gamma_k = kx_k - \frac{1 - x_k^2}{x_k} \sum_{i=1}^{N} it_i \left(L^i_{k+1,k+1} + L^i_{k,k} - 2L^{i-1}_{k+1,k} \right)$$

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- Γ_k is a polynomial in $x_{k-N}, x_{k-N+1}, \ldots, x_k, \ldots, x_{k+N}$ (only!)
- Γ_k is of degree 1 in x_{k+N}
- $(2N + 1)$-step recursion relation $x_n = F_n(x_{n-1}, \ldots, x_{n-2N})$

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$$kx_k + t(1-x_k^2)(\underline{x_{k+1}} + x_{k-1}) = 0$$

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Example (Adler-van Moerbeke, $N = 2$, $v_i := 1 - x_i^2$):

$$kx_k + tv_k(x_{k+1} + x_{k-1}) + 2sv_k(\underline{x_{k+2}}v_{k+1} + x_{k-2}v_{k-1} - x_k(x_{k+1} + x_{k-1})^2) = 0$$

Main result

Theorem (Adler, van Moerbeke and V.)

The recursion relations $\Gamma_k = \Delta_k = 0$ *have the singularity confinement property (in its strongest form).*

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Singularity confinement:

- weak form: there exist formal Laurent solutions $(x_k(\epsilon))_{k \in \mathbf{N}}$ that only blow up for a few k
- stronger form: such a solution exists, depending on $(2N+1) - 1 = 2N$ free parameters
- strongest form: for every k there exists such a solution with at least x_k blowing up (and depending on $2N$ free parameters)

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In a sense: discrete version of the Kowalevski-Painlevé property

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- ... but the precise relation is not properly undertood

For $N = 1$

$$kx_k + t(1 - x_k^2)(\underline{x_{k+1}} + x_{k-1}) = 0$$

invariant (first shown by Borodin):

$$\Phi_n(y, z) = (1 - y^2)(1 - z^2) - \frac{n}{t} yz$$

invariance:

$$\Phi_n(x_{n+1}, x_n) = \Phi_n(x_n, x_{n-1})$$

The examples $N = 1$ and $N = 2$

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- ... but the precise relation is not properly undertood

For $N = 2$

$$kx_k + tv_k(x_{k+1} + x_{k-1}) + \\ 2sv_k(\underline{x_{k+2}}v_{k+1} + x_{k-2}v_{k-1} - x_k(x_{k+1} + x_{k-1})^2) = 0$$

invariant:

$$\Phi_n(x, y, z, u) = nyz - (1 - y^2)(1 - z^2)(t + 2s(x(u - y) - z(u + y)))$$

invariance:

$$\Phi_n(x_{n+3}, x_{n+2}, x_{n+1}, x_n) = \Phi_n(x_{n+2}, x_{n+1}, x_n, x_{n-1})$$

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Conjecture: there is for any N an invariant (self-dual case and general case)
The theorem is in support of this conjecture

The theorem: outline of the proof

Theorem

The recursion relations $\Gamma_k = \Delta_k = 0$ satisfy the singularity confinement property (in its strongest form).

Setup:

1. bi-infinite Toeplitz lattice (semi-infinite: $x_i = y_i = \delta_{i0}$ for $i \leq 0$)
2. L and M becomes bi-infinite matrices

general Toeplitz lattice

$$\begin{cases} \dot{x}_k &= (1 - x_k y_k)(x_{k+1} - x_{k-1}) \\ \dot{y}_k &= (1 - x_k y_k)(y_{k+1} - y_{k-1}) \end{cases} \qquad k \in \mathbf{Z}$$

$$\text{recursion relations } \Gamma_k = \Delta_k = 0, \qquad k \in \mathbf{Z}$$

self-dual Toeplitz lattice

$$\dot{x}_k = (1 - x_k^2)(x_{k+1} - x_{k-1}) \qquad k \in \mathbf{Z}$$

$$\text{recursion relations } \Gamma_k = 0, \qquad k \in \mathbf{Z}.$$

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idea: construct singular solutions for the recursion relations from principal balances for the Toeplitz lattice

principal balances: formal Laurent solutions depending on *many* free parameters

Step 1: Invariant manifold

Theorem

Let $\mathcal{M}$ be the submanifold, defined by $\Gamma_k(x,y) = \Delta_k(x,y) = 0$. Then $\mathcal{M}$ is an invariant submanifold for the (first) Toeplitz flow(s) of the Toeplitz lattice.

Proof.

The recursion relations satisfy differential equations.

General case:

$$\begin{aligned}
\dot{\Gamma}_k &= (1 - x_k y_k)(\Gamma_{k+1} - \Gamma_{k-1}) + (x_{k+1} - x_{k-1})(x_k \Delta_k - y_k \Gamma_k), \\
\dot{\Delta}_k &= (1 - x_k y_k)(\Delta_{k+1} - \Delta_{k-1}) - (y_{k+1} - y_{k-1})(x_k \Delta_k - y_k \Gamma_k).
\end{aligned}$$

Self-dual case: $\dot{\Gamma}_k = (1 - x_k^2)(\Gamma_{k+1} - \Gamma_{k-1})$. □

Corollary : every formal power series solution to Toeplitz that starts out on $\mathcal{M}$ stays on $\mathcal{M}$.

Step 2: Painlevé analysis for the Toeplitz lattice

A few extra features wrt standard Painlevé analysis

- not weight-homogeneous
- infinite number of variables (? how many free parameters)
- existence of all terms, rather than convergence

Step 2: Painlevé analysis for the Toeplitz lattice

$$\frac{dx_k}{dt} = (1 - x_k^2)(x_{k+1} - x_{k-1}) \qquad k \in \mathbf{Z}$$

Theorem

For any $n \in \mathbf{Z}$, the self-dual Toeplitz lattice admits a formal Laurent solution $x(t)$, with only $x_n(t)$ having a pole, given by

$$\begin{aligned}
x_k(t) &= 1\left(a_k + (1 - a_k^2)(a_{k+1} - a_{k-1})t + O(t^2)\right), \quad |k - n| \geq 2, \\
x_{n\pm1}(t) &= 1\left(\mp 1 + 4a_\pm t + 4a_\pm(2a_{n\pm2} \mp (a_- + a_+))t^2 + O(t^3)\right), \\
x_n(t) &= -\frac{1}{2t}\left(1 + (a_+ - a_-)t + \frac{1}{3}((a_+ - a_-)^2 \right. \\
&\quad \left. + 4(a_+ a_{n+2} - a_- a_{n-2} + 1 - 2a_+ a_-))t^2 + O(t^3)\right),
\end{aligned}$$

where a_+, a_- and all a_i, with $i \in \mathbf{Z} \setminus \{n-1, n, n+1\}$ are arbitrary free parameters; also, $1^2 = 1$.

Step 2': Painlevé analysis for the Toeplitz lattice

For the general Toeplitz lattice:

$$\begin{aligned} \dot{x}_k &= (1 - x_k y_k)(x_{k+1} - x_{k-1}) \\ \dot{y}_k &= (1 - x_k y_k)(y_{k+1} - y_{k-1}) \end{aligned} \qquad k \in \mathbf{Z}$$

Features

- not weight-homogeneous
- $2\infty - 1 \neq 2(\infty - 1)$ free parameters
- x_n and y_n have a pole at the same time (n arbitrary)

Step 3: Tangency of the Laurent solutions

In the self-dual case, recall $\dot{\Gamma}_k = (1 - x_k^2)(\Gamma_{k+1} - \Gamma_{k-1})$.

Theorem

Let $\Gamma(t) := \Gamma(x(t))$, where $x(t)$ is the above formal Laurent solution. Then, as formal series in t,

$$\begin{aligned} \Gamma_k(t) &= \Gamma_k^{(0)} + O(t), && k \in \mathbf{Z} \setminus \{n\}, \\ \Gamma_n(t) &= \frac{1}{4t}(\Gamma_{n+1}^{(0)} - \Gamma_{n-1}^{(0)}) + \Gamma_n^{(0)} + O(t). \end{aligned}$$

Moreover, $\Gamma_k(t) = 0$ as a formal series in t, for all $k \in \mathbf{Z}$, as soon as $x(t)$ is such that

$$\Gamma_k^{(0)} = 0, \text{ for all } k \in \mathbf{Z}.$$

Step 3': Tangency of the Laurent solutions

Theorem

Let $\Gamma(t) := \Gamma(x(t), y(t))$, where $(x(t), y(t))$ is the above formal Laurent solution. Then, as a formal series in t, $\Gamma_k(t) = \Gamma_k^{(0)} + O(t)$ and $\Delta_k(t) = \Delta_k^{(0)} + O(t)$ for $k \in \mathbf{Z} \setminus \{n\}$. Also

$$\begin{aligned} \Gamma_n(t) &= \frac{a_{n+1}^2}{a_-(a_{n-1} - a_{n+1})^2 t^2} \left(\Gamma_{n-1}^{(0)} - a_{n-1}^2 \Delta_{n-1}^{(0)} \right) + \frac{1}{t} \Gamma_n^{(-1)} + O(1), \\ \Delta_n(t) &= \frac{a_{n+1} a_{n-1}}{a_-(a_{n-1} - a_{n+1})^2 t^2} \left(\Gamma_{n-1}^{(0)} / a_{n-1}^2 - \Delta_{n-1}^{(0)} \right) + \frac{1}{t} \Delta_n^{(-1)} + O(1), \end{aligned}$$

where $\Gamma_n^{(-1)}$ and $\Delta_n^{(-1)}$ are both linear combinations of $\Gamma_{n\pm1}^{(0)}$ and $\Delta_{n\pm1}^{(0)}$.

Step 4: Parameter restriction (self-dual case)

In the self-dual case, recall that $\Gamma_k(x) = \Gamma_k(x_{n-N}, \ldots, x_{n+N})$. Some very tricky fishing leads to:

Γ_{n-N-1}	$\underline{a_{n-2N-1}}, \ldots, a_{n-1} = 1$
Γ_{n-N-2}	$\underline{a_{n-2N-2}}, \ldots, a_{n-2}$
$\vdots$	$\vdots$
Γ_{n-N}	$a_{n-2N}, \ldots, a_{n-2}, \underline{a_-}$
Γ_{n-N+1}	$a_{n-2N+1}, \ldots, a_{n-2}, a_-, \underline{a_+}$
Γ_{n-N+2}	$a_{n-2N+2}, \ldots, a_{n-2}, a_\pm, \underline{a_{n+2}}$
$\vdots$	$\vdots$
Γ_{n-1}	$a_{n-N-1}, \ldots, a_{n-2}, a_\pm, a_{n+2}, \ldots, \underline{a_{n+N-1}}$
Γ_{n+1}	$a_{n-N+1}, \ldots, a_{n-2}, a_\pm, a_{n+2}, \ldots, \underline{a_{n+N}}, \xcancel{a_{n+N+1}}$
Γ_n	$a_{n-N-1}, \ldots, a_{n-2}, a_\pm, a_{n+2}, \ldots, \underline{a_{n+N+1}}$
Γ_{n+2}	$a_{n-N+2}, \ldots, a_{n-2}, a_\pm, a_{n+2}, \ldots, \underline{a_{n+N+2}}$
$\vdots$	$\vdots$

Step 5: Formal inverse function theorem

The equations

$$\begin{cases} x_k(t) &= a_k + O(t) \qquad k = n-2N, \ldots, n-2 \\ x_{n-1}(t) &= 1 + \sum_{i=1}^{\infty} x_{n-1}^{(i)} t^i \end{cases}$$

can be inverted, as formal power series, into

$$\begin{cases} x_k(t) &= \alpha_k \qquad k = n-2N, \ldots, n-2 \\ x_{n-1}(t) &= 1 + \lambda \end{cases}$$

Final result: Singularity confinement

Theorem

In the self-dual case, there exist formal Laurent solutions to the $(2N+1)$-step recursion relations $\Gamma_k = 0$, depending rationally on $2N$ free parameters $\alpha_{n-2N}, \ldots, \alpha_{n-2}$ and λ, namely

$$
\begin{aligned}
x_k(\lambda,\alpha) &= \sum_{i\in\mathbf{N}} x_k^{(i)}(\alpha)\lambda^i && k < n-2N \\
x_k(\lambda,\alpha) &= \alpha_k && n-2N \le k < n-1 \\
x_{n-1}(\lambda,\alpha) &= 1+\lambda && \\
x_n(\lambda,\alpha) &= \lambda^{-1}\sum_{i\in\mathbf{N}} x_n^{(i)}(\alpha)\lambda^i && \\
x_{n+1}(\lambda,\alpha) &= -1+\sum_{i\in\mathbf{N}} x_{n+1}^{(i)}(\alpha)\lambda^i && \\
x_k(\lambda,\alpha) &= -1+\sum_{i\in\mathbf{N}} x_k^{(i)}(\alpha)\lambda^i && n+1 < k
\end{aligned}
$$