Foundation of general differential Galois theory

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§1 Why general diff. Galois theory?

1. Galois theory of algebraic equations

$$ax^2+bx+c=0 \qquad a\neq 0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0 \qquad a_0\neq 0$$

Cardano, Ferrari $\qquad n=3,4$

$\vdots$

Impossible to generalize this kind of formula to $n\geqslant 5$.

Galois Theory.

This is not the end of the story but this is the starting point of number theory.

Proof of Fermat's last Theorem without Galois Theory

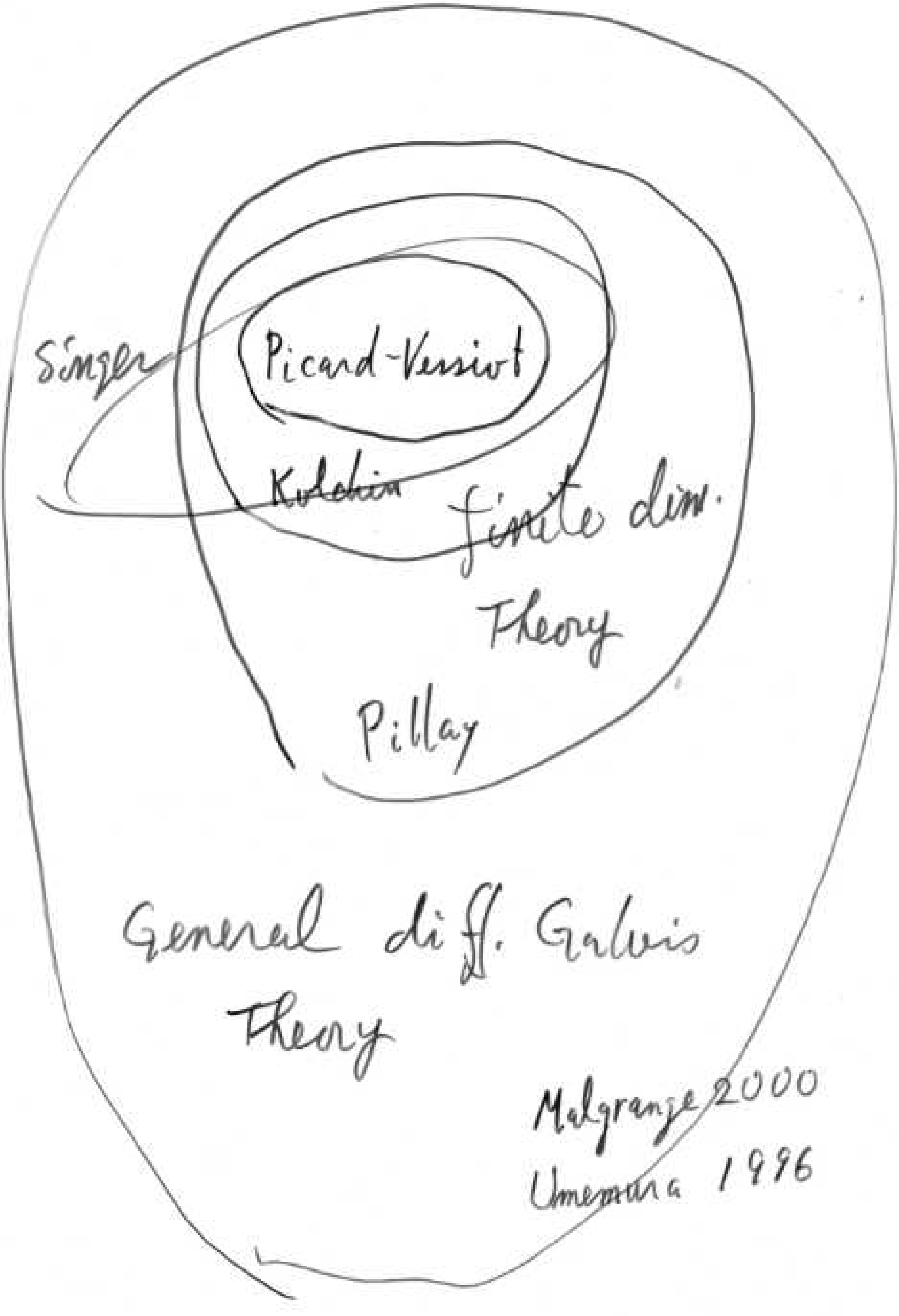
Impossible to imagine

2. Differential Galois Theory

Long history since S. Lie.

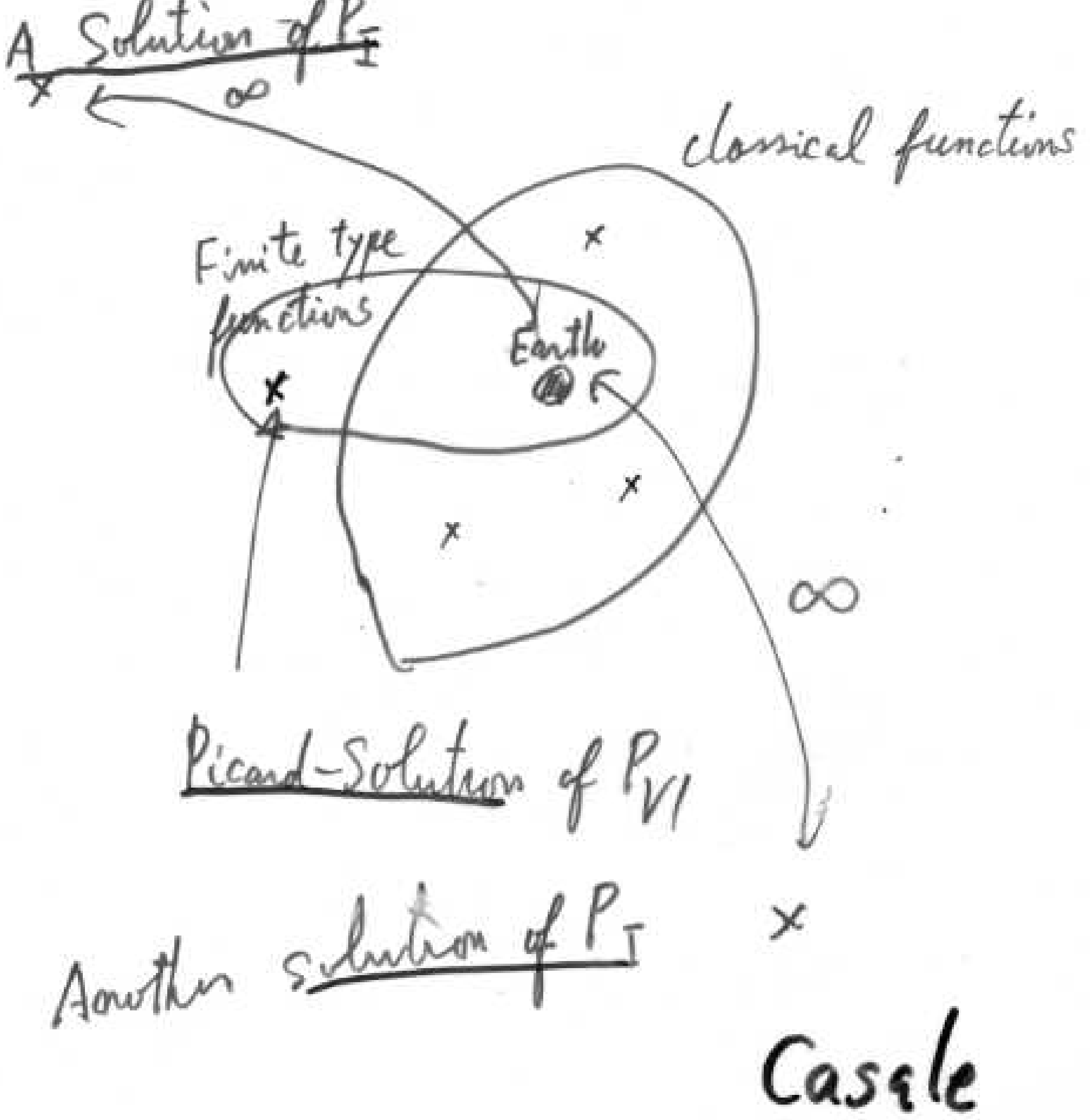
Irreducibility of the Painlevé equations was proved without D.G.T.

Kolchin's Theory, Picard-Vessiot Theory are too restrictive for app. in analysis.


Singer
Picard-Vessiot
Kolchin
finite dim.
Theory
Pillay
General diff. Galois
Theory
Malgrange 2000
Umemura 1996

Remarkable applications to Painlevé equations

Functions = Stars



on a scheme

Dynamical System on a manifold

continuous, discrete

$\Downarrow$

Galois groupoid on the manifold

<u>Observation</u> by <u>algebraic</u>
<u>partial</u> <u>diff.</u> <u>eqs.</u>

given a diff. field $\Longleftrightarrow$ Continuous Dynamical Sys. on a variety

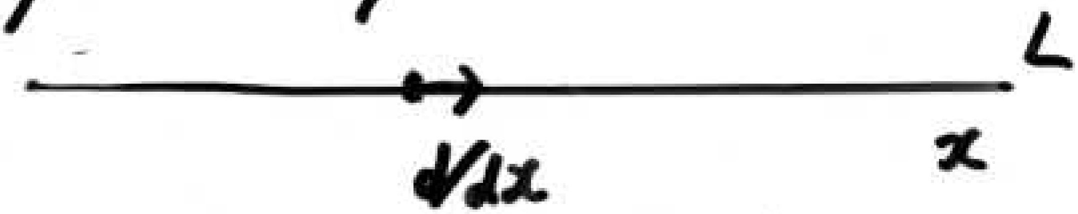
very naturally

We can hear the music of Galois Groupoid.

§2 Diff. field and Dynamical sys on a variety

Example 1. $\mathbb{C}(x)$, d/dx

Dynamical system on the line



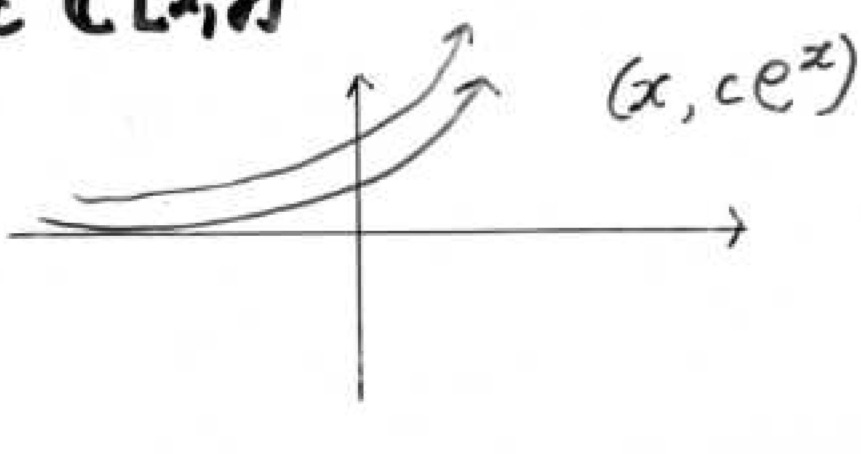
$L = \operatorname{Spec} \mathbb{C}[x]$ Model of $\mathbb{C}(x)$

2. $\mathbb{C}(x, y)$ $y' = y$, $x' = 1$

diff. field

$\left(\mathbb{C}[x, y], \ \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)$ Model of $\mathbb{C}(x, y)$

$\operatorname{Spec} \mathbb{C}[x, y]$



Generally $(L/\mathbb{C}, \delta)$ diff. field

$\delta : L \to L$ derivation

$\delta(ab) = (\delta a)b + a\delta(b)$

$\delta(a+b) = \delta a + \delta b$

$\delta(\mathbb{C}) = 0$

L is finitely generated $/\mathbb{C}$

Proposition $\exists R \subset L$

(i) R is finitely gen. $/\mathbb{C}$ as a ring.

(ii) $\mathbb{C}(R) = L$.

(iii) $\delta(R) \subset R$

δ defines a vector field $X(\delta)$ on Spec R algebraic variety. scheme

Dynamical system $(\mathrm{Spec}\, R, X(\delta))$.

A Geometrization of $(L/\mathbb{C}, \delta)$

§3 Groupoid

Definition Groupoid = Category where $\forall$ morphism is an isomorphism

An object is called a vertex.
a morphism is called an element.
$x, y \in \mathrm{ob}\, G$ vertices

$$\overset{x}{\bullet} \xrightarrow{\varphi} \overset{y}{\bullet} \qquad \varphi \in \mathrm{Hom}(x, y)$$

Brandt 1926, Ehresmann 1950's
Grothendieck 1950's, 60's

$Y := \{\text{elements of } G\} = \{\text{morphisms in } G\}$

$X := \mathrm{ob}\, G = \{\text{vertices}\}$

$Y \underset{t}{\overset{s}{\rightrightarrows}} X$, $\varphi \in Y$ $\varphi = \mathrm{Hom}(A, B)$

$s(\varphi) := A$, $t(\varphi) := B$.

Example 1. Group G

ob $C = \{P\}$ only one object.

$\mathrm{Hom}(P,P) = G$

$$\begin{tikzcd} P \arrow[loop right, "g"] \arrow[loop below, "h"] \end{tikzcd}$$

$g \cdot h := gh$ product in G

Group G operates on $X = \{P\}$ trivially.

2. Equivalence relation on a set X

ob $C := X = \{\text{points of } X\}$

$x \bullet \longrightarrow \bullet\, y$ if $x \sim y$

$$\mathrm{Hom}(x,y) = \begin{cases} \phi & \text{if } x \nsim y \\ 1 \text{ morphism} & \text{if } x \sim y \end{cases}$$

3 Group operation (G, X)

Category $\mathcal{C}$ ob $\mathcal{C} = X$

For $x, y \in X$, $\mathrm{Hom}(x, y) = \{ g \in G \mid x^g = y \}$

$$\underset{x}{\bullet} \xrightarrow{g} \underset{y}{\bullet} \qquad x^g = y$$

$$x \bullet \xrightarrow{g} \bullet y \qquad y \bullet \xrightarrow{h} \bullet z$$

$$x^g = y \qquad\qquad y^h = z$$

so that

$$x^{gh} = (x^g)^h = y^h = z$$

$$x \xrightarrow{gh} z$$

So we can compose morphisms.

Morphism g is an isomorphism.

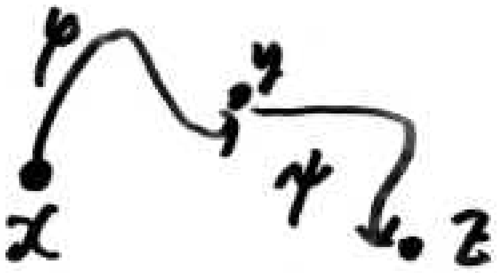
4 Poincaré groupoid

X = topological space

ob $C = \{$ points of $X\}$

For $x, y \in X$,

$\mathrm{Hom}(x, y) = \{ \varphi : [0,1] \to M \mid \varphi$ is continuous, $\varphi(0) = x$, $\varphi(1) = y \} /$ homotopy



Composition of paths

G = groupoid

$Y := \{\text{morphisms in } G\}$

$X :=$ ob G

$$\boxed{Y \overset{s}{\underset{t}{\rightrightarrows}} X}$$

For $\varphi \in \mathrm{Hom}(A, B)$, $A, B \in X$

$s(\varphi) := A$, $t(\varphi) := B$.

$$\boxed{Y {}_{t}{\times}_{s} Y \xrightarrow{\Phi} Y, \quad (\varphi, \psi) \mapsto \psi \circ \varphi}$$

$$\Vert$$

$$\{(\varphi, \psi) \in Y {}_{t}{\times}_{s} Y \mid t(\varphi) = s(\psi)\}$$

$$\underset{A}{\cdot} \longrightarrow \underset{B}{\cdot} \qquad \underset{B}{\cdot} \longrightarrow \underset{C}{\cdot}$$

Associativity

$$\begin{tikzcd}
Y {}_{t}{\times}_{s} Y {}_{t}{\times}_{s} Y \arrow[r, "1 \times \Phi"] \arrow[d, "\Phi \times 1"'] \arrow[dr, phantom, "\circlearrowright"] & Y {}_{t}{\times}_{s} Y \arrow[d, "\Phi"] \\
Y {}_{t}{\times}_{s} Y \arrow[r, "\Phi"] & Y
\end{tikzcd}$$

commutative

a groupoid is defined by arrows and diagrams.

Two sets Y, X

$$Y \underset{t}{\overset{s}{\rightrightarrows}} X, \quad Y \underset{t\ s}{\times} Y \xrightarrow{\Phi} Y \quad \ldots$$

+ commutative diagrams.

$$\Downarrow$$

We can define a groupoid in a category $\mathcal{C}$ where fibre product exists.

Objects Y, X of $\mathcal{C}$

$$Y \underset{t}{\overset{s}{\rightrightarrows}} X, \quad Y \underset{t\ s}{\times} Y \xrightarrow{\Phi} Y, \quad \ldots$$

morphisms in $\mathcal{C}$ + commutative diagrams

Example 5. Lie groupoid of all the local isomorphisms of a manifold M

Let M be a complex manifold.

$$\boxed{J^* \to M \times M \quad x \mapsto (s(x), t(x))}$$

($\mathcal{C}$ = category of complex manifolds)

$$\boxed{J^* {}_t\!\times_s J^* \to J^*,}$$

where J^* is a complex manifold of infinite dim.

<u>$M = \mathbb{C}$, construction of J^*</u>

$$J^* = \mathbb{C}^{\mathbb{N}} = \{(a, b_0, b_1, b_2, \cdots) \in \mathbb{C}^{\mathbb{N}} \mid b_1 \neq 0\}$$

$$\mathbb{N} = \{0, 1, 2, \cdots\}$$

$\mathbb{C} = M$

$$\boxed{\begin{aligned} s &: J^* \to \mathbb{C},\ (a, b_0, b_1, \cdots) \mapsto a \\ t &: J^* \to \mathbb{C},\ (a, b_0, b_1, \cdots) \mapsto b_0 \end{aligned}}$$

Composition

$$\boxed{J^* {}_t\times_s J^* \xrightarrow{\Phi} J^*}$$

$$((a, b_0, b_1, \cdots), (c, d_0, d_1, \cdots)) \in J^*{}_t \times_s J^*$$

$$(\Leftrightarrow b_0 = c)$$

$$\downarrow \Phi$$

$$(e, f_0, f_1, \cdots)$$

$b_0 = b_0(a)$

$d_0 = d_0(c)$

$b_0 = c$

So that

$d_0 = d_0(c) = d_0(b_0(a))$

where $e = a$, $f_0 = d_0$, $f_1 = b_1 d_1$

$\Uparrow$

$f_2 = \cdots$

$d\,\psi(\varphi(x))/dx$

$= \psi'(\varphi(x))\varphi'(x)$

Formula of calculating

$d^m \psi(\varphi(x))/dx^m$ in terms of $\psi^{(i)}$ and $\varphi^{(i)}$

In $(a, b_0, b_1, \cdots)$,

imagine b_0 is a general function of a, $b_0 = b_0(a)$

$b_1 = d\, b_0(a)/da$, $b_2 = d^2\, b_0(a)/da^2$, $\cdots$

Bell polynomial

$$f_1 = b_1 d_1$$

$$f_2 = b_2 d_1 + b_1^2 d_2$$

$$\vdots$$

$$\psi(\varphi)' = \psi'(\varphi)\varphi'$$

$$\psi(\varphi)^{(2)} = \psi'\varphi'' + \psi''(\varphi')^2$$

$$\vdots$$

$J_{*\,t}\times_s J_* \xrightarrow{\Phi} J_*$ is associative

$\vdots$

$J_* \xrightarrow{s\times t} \mathbb{C}\times\mathbb{C}$, $\quad J_{*\,t}\times_s J_* \xrightarrow{\Phi} J_*$

is a groupoid

Groupoid of all the local isomorphisms of $\mathbb{C}$.

We are interested in a very special kind of subgroupoids of $J_* \overset{s}{\underset{t}{\rightrightarrows}} M$.

Subgroupoids of J_* defined by a system of differential equations

Example 6. Schwarzian = 0

Schwarzian derivative

$$\{y : x\} := \left(\frac{d^3y}{dx^3}\right) \Big/ \left(\frac{dy}{dx}\right) - \frac{3}{2}\left[\left(\frac{d^2y}{dx^2}\right) \Big/ \left(\frac{dy}{dx}\right)\right]^2$$

or

$$\{y : x\} = y^{(3)}/y^{(1)} - \frac{3}{2}\left(y^{(2)}/y^{(1)}\right)^2$$

Formula

$$\{z ; x\} = \left(\frac{dy}{dx}\right)^2 \{z ; y\} + \{y ; x\} \qquad (*)$$

cocycle condition

$$J^* \supset Y = \{ (a, b_0, b_1, \cdots) \in J^* \mid$$

$$b_3/b_1 - \frac{3}{2}\left(b_2/b_1\right)^2 = 0 \quad \cdots \qquad \}$$

diff. ideal gen. by $\{b ; a\}$

$$\delta = \sum_{i=0}^{\infty} b_{i+1} \, \partial/\partial b_i$$

$$\begin{array}{c} Y \subset J^* \\ s \downarrow\downarrow t \\ \mathbb{C} \end{array} \qquad \dim Y = 4$$

$Y \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{C}$ $Y \subset J^*$ is closed under the composition by the cocycle condition (*).

$Y {}_t\!\times_s Y \to Y$ is an algebraic D-groupoid.

$\{y; x\}$ is an alg. diff. eq. in y.

<u>Résumé</u> (1) On a complex manifold M, $\exists$ groupoid $J^* \underset{t}{\overset{s}{\rightrightarrows}} M$ of local isom.

(2) An algebraic D-groupoid is a subgroupoid of J^* diffined by diff. eqs.

§4 Algebraic construction of the Jet space

algebra
$R/\mathbb{C}$, of finite type, smooth.

poly. ring
assume R is étale $/\mathbb{C}[t_1, t_2, \cdots, t_n]$

For $\forall$ R-algebra A, consider

$$\{\varphi : R \to A[[X_1, X_2, \cdots, X_n]]\}.$$

Among these, $\exists_1$ the universal one.

$$\varphi : R \to J[[X]]$$

$\mathcal{J} = R \otimes_{\mathbb{C}} J$, $\quad R \otimes_{\mathbb{C}} R$-alg.

$\operatorname{Spec} \mathcal{J} \to \operatorname{Spec} R \otimes_{\mathbb{C}} R = \operatorname{Spec} R \times_{\mathbb{C}} \operatorname{Spec} R$
is the Jet space.

Equivalent definition

$\exists$ derivation $\partial_1, \partial_2, \cdots, \partial_n$ $/\mathbb{C}$

$$\partial_i : J \to J$$

$$\tilde{\partial}_i := \partial_{t_i} \otimes 1 + 1 \otimes \partial_i \quad 1 \le i \le n$$

$$: R \otimes_{\mathbb{C}} J \to R \otimes_{\mathbb{C}} J.$$

$$\begin{tikzcd}
\mathcal{J} = R \otimes_{\mathbb{C}} J \arrow[r, "\tilde{\partial}_i"] \arrow[dr, phantom, "\circlearrowright"] & R \otimes_{\mathbb{C}} J = \mathcal{J} \\
R \arrow[u, dash] \arrow[r, "\partial_{t_i}"'] & R \arrow[u, dash]
\end{tikzcd}$$

<u>universal</u> among such ext. of derivations

$R\otimes_{\mathbb{C}} R$ - alg. S with derivations

$\tilde{\partial}_i \quad 1\le i\le m$

$$
\begin{tikzcd}
a\otimes 1 & S \arrow[r, "\tilde{\partial}_i"] \arrow[dr, phantom, "\circlearrowleft"] & S \\
a \arrow[u, mapsto] & R \arrow[u] \arrow[r, "\partial_i"] & R \arrow[u]
\end{tikzcd}
\qquad 1\le i\le m
$$

$(\mathcal{T}, \{\tilde{\partial}_i\}_{1\le i\le m})$ is the universal among such $(S, \{\tilde{\partial}_i\}_{1\le i\le m})$

$$
\begin{tikzcd}
 & S \arrow[r, "\tilde{\partial}_i"] & S & \\
 & R\otimes_{\mathbb{C}} R \arrow[u, dash] \arrow[r, phantom, "\curvearrowright"] & R\otimes_{\mathbb{C}} R \arrow[u, dash] & \\
a\otimes 1 & & & a\otimes 1 \\
a \arrow[u, mapsto] & R \arrow[uu] \arrow[r, "\partial_i"] & R \arrow[uu] & a \arrow[u, mapsto]
\end{tikzcd}
$$

§5 Construction of Galois groupoid

Algebraically construction of the Galois closure.

$(L/\mathbb{C}, \delta)$ $R/\mathbb{C}$, $\delta(R) \subset R$

R is étale $/\mathbb{C}[t_1, t_2, \cdots, t_n]$

Derivations $\partial_i : R \to R$, $1 \le i \le n$.

$C_R = \{x \in R \mid \delta(x) = 0\} \supset \mathbb{C}$

$$\iota : R \to R^q[\![X]\!], \quad a \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(a) X^n$$

universal Taylor morphism

(1) Ring hom.

(2) compatible with δ and d/dX

$$\mathcal{R} = \langle R^q, \iota(R) \rangle \subset R^q[\![X]\!]$$

$\partial_1, \partial_2, \cdots, \partial_n$ operate on coefficients

Lemma $R^q\iota(R) \simeq R \otimes_{C_R} R$

Cor. $R^q\iota(R)$ is an $R \otimes_{\mathbb{C}} R$-alg.

and we have surj. $R\otimes_{\mathbb{C}} R$-alg. hom

$$\mathcal{J} \longrightarrow R.$$

In fact derivations ∂_i on $R \otimes 1$ extend to R in $R^q[[x]]$.

Universality of $\mathcal{J}$

$$\mathcal{J} \longrightarrow R$$

that is surjective.

$$\begin{array}{c} (\operatorname{Spec} \mathcal{J}) \supset \operatorname{Spec} R \\ \downarrow \\ \operatorname{Spec} R \otimes_{\mathbb{C}} \operatorname{Spec} R \end{array}$$

closed subscheme.

§6 Example

1. M. Singer   Parameterized Picard-Vessiot

$\Delta = \{\partial/\partial x, \partial/\partial y\} = \{\partial_x, \partial_y\}$

$k := \mathbb{C}(x,t), \quad K := \mathbb{C}(x, t, \underset{y}{\underset{\|}{x^t}}, \log x)$

$$\partial_x y = \frac{t}{x} y$$

$$\partial_t y = (\log x) y$$

$$\partial_x \log x = \frac{1}{x}$$

$$\partial_t \log x = 0$$

$$\begin{array}{c} \mathbb{C}(x, t, x^t, \log x) \\ | \\ \mathbb{C}(x,t) \end{array}$$

$Gal_\Delta(K/k) =$ Aut. gr of K/k.

$\sigma \in Gal_\Delta(K/k)$

$$\sigma(y) = \exp(ct + c')y, \quad c, c' \in \mathbb{C}$$

$Gal_\Delta(K/k) = \{\exp(ct + c') \mid c, c' \in \mathbb{C}\}$

$= \{ t \mapsto a(t) \mid 0 = (\log a)'' = \left(\frac{a'}{a}\right)' \}$ D-group(oid)

<u>Our interpretation</u> Ordinary diff. field

$\mathbb{C}(x,t)$

$\mathbb{C}(x,t,y)$ x, t, y are variables

$\partial_x + (t/x)\partial_y \longleftrightarrow \frac{d}{dx}$

on $\mathbb{C}^3$

$$\frac{dt}{dx} = 0, \quad \frac{dx}{dx} = 1, \quad \frac{dy}{dx} = \frac{t}{x} y$$

Ordinary diff. field ext. $\mathbb{C}(x,t,y)/\mathbb{C}(x,t)$

What is the Galois groupoid of the extension?

$$\mathrm{Gal}\left(\mathbb{C}(x,t,y)/\mathbb{C}(x,t)\right) \quad \frac{d}{dx}$$

$S := \mathbb{C}[x,t,y], \quad R := \mathbb{C}[x,t]$

$\iota : S \longrightarrow S[\![X]\!]$ $\quad a \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n a}{dx^n} X^n$

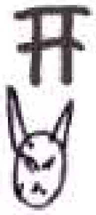
$t \longmapsto t$

$x \longmapsto x + X$

$$y \mapsto \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n y}{dx^n} X^n = y\left(1+\frac{X}{x}\right)^t$$

where

$$\left(1+\frac{X}{x}\right)^t = \sum_{m=0}^{\infty} \frac{t(t-1)\cdots(t-m+1)}{m!}\left(\frac{X}{x}\right)^m$$



$$y = x^t \mapsto (x+X)^t = x^t\left(1+\frac{X}{x}\right)^t = y\left(1+\frac{X}{x}\right)^t$$

$$(t, x, y) \overset{L}{\mapsto} (\tilde{T}, \tilde{X}, \tilde{Y})$$

$$L(t)\ \tilde{T} = t, \quad \overset{L(x)}{\tilde{X} = x + X}, \quad \tilde{Y} = \overset{L(y)}{y\left(1+\frac{X}{x}\right)^t}$$

Diff. eq. satisfied by $\tilde{T}, \tilde{X}, \tilde{Y}$

they are functions of t, x, y,

X being a parameter transcendental

$$\tilde{T} = t, \quad \partial_t \tilde{X} = 0 \qquad \partial_t \tilde{Y} = \log\left(1+\frac{X}{x}\right)\tilde{Y}$$

$$\partial_x \tilde{X} = 1 \qquad \partial_x \tilde{Y} = t\left(\frac{1}{\tilde{X}} - \frac{1}{x}\right)\tilde{Y}$$

$$\partial_y \tilde{X} = 0 \qquad y\,\partial_y \tilde{Y} = \tilde{Y}$$

$$
(*)\begin{cases}
\tilde{T} = t, \quad \partial_t \hat{X} = 0, & \left(\partial_t(\partial_x \tilde{Y}/\tilde{Y}) = \frac{1}{\hat{X}} - \frac{1}{x}\right), \quad (1) \\
\partial_x \hat{X} = 1, & \partial_t(\partial_t \tilde{Y}/\tilde{Y}) = 0, \quad (2) \\
\partial_y \hat{X} = 0, & \partial_y(\partial_t \tilde{Y}/\tilde{Y}) = 0, \quad (3) \\
 & \partial_x \tilde{Y} = t\left(\frac{1}{\hat{X}} - \frac{1}{x}\right)\tilde{Y}, \quad (4) \\
\hat{X} = x & y\partial_y \tilde{Y} = \tilde{Y}. \quad (5)
\end{cases}
$$

Dynamical system

$$(t, x, y) \longmapsto (\tilde{T}, \hat{X}, \tilde{Y})$$

is algebraically described by the system (*) if you observe it by alg. diff. eq.

Galois groupoid $\mathrm{Gal}(\mathbb{C}(t,x,y)/\mathbb{C}, d/dx)$

Solutions of (*)?

$$\hat{T} = t, \quad \tilde{X} = x + c_1, \quad \tilde{Y} = y\left(\frac{x+c_1}{x}\right)^t e^{c_2 t + c_3}$$

$c_1, c_2, c_3 \in \mathbb{C}.$

Why? $\hat{Y} = f(x,t)\, y$ by (5)

$f_x / f = t\,(1/\hat{X} - 1/x) = t\left(\frac{1}{x+c_1} - \frac{1}{x}\right)$ by (4)

$$f = \left(\frac{x+c_1}{x}\right)^t g(t)$$

$\partial_t\left(\partial_t g(t) / g(t)\right) = 0$ by (2)

$$g(t) = e^{c_2 t + c_3}$$

$$\mathrm{Gal}(\mathbb{C}(t,x)/\mathbb{C})$$

$$(t, x) \mapsto (\hat{T}, \hat{X})$$

$$\hat{T} = t, \quad \frac{\partial \hat{X}}{\partial x} = 1, \quad \frac{\partial \hat{X}}{\partial t} = 0$$

Natural groupoid morphism

$$\mathrm{Gal}(\mathbb{C}(t,x,y)/\mathbb{C}(x,t))$$

$$\downarrow$$

$$\mathrm{Ker} \overset{\mathrm{Def}}{=} \mathrm{Gal}(\mathbb{C}(x,t,y)/\mathbb{C}(x,t))$$

$$\downarrow$$

$$\mathrm{Gal}(\mathbb{C}(x,t,y)/\mathbb{C})$$

$$\downarrow$$

$$\mathrm{Gal}(\mathbb{C}(x,t)/\mathbb{C})$$

In $(*)$ to add $\hat{X}=x$ $(\tilde{T}=t)$

$(*)$ reduces to

$$(**)\begin{cases} \tilde{T}=t, \quad \hat{X}=x, \quad \partial_x \tilde{Y}=0 \\ \partial_t(\partial_t \tilde{Y}/\tilde{Y})=0 \\ y\partial_y \tilde{Y}=\tilde{Y} \end{cases}$$

$$(t,x,\tilde{Y}) \mapsto (t,x,\hat{Y})$$

solution $\hat{Y}=ye^{c_1t+c_2}$

as Singer's calculation.

In this particular case, the groupoid $\mathrm{Gal}(\mathbb{C}(x,t,y)/\mathbb{C}(x,t))$ is <u>a group</u>. !