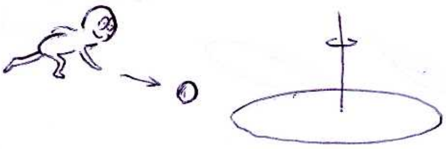
Turning Table, Tippy Tops, Tapped Turtles

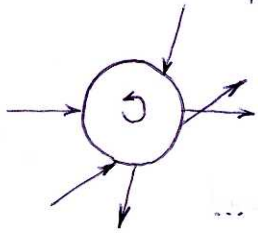
Tadashi TOKIEDA

CIRM Marseille, November 29th, 2006

ball rolling across a turntable



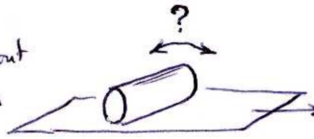
Thm : 'scattering' operator is id



... billiard of ANAIS
[Lévy - Leblond 1986]

Generalize : arbitrary deformation of floor

sheet pulled out from under pen



also beach ball

if tilted, ball drifts horizontally

▶ 15 min

ball rolling on a cylinder

Thm : Ball goes down <u>and up</u> : ↺ unif revol, ↕ harmonic osci

Coupling between these modes :

angle ↺ per period of ↕ = ... could depend on mass, radii, g, initial cond...

= actually universal $\pi\sqrt{14}$ for ball

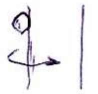
$\pi\sqrt{10}$ for sphere

cf. $\frac{\hat{I}}{1+\hat{I}}$

Same result if

cylinder spins

roll outside



roll outside spinning cylinder

And $\pi\sqrt{14}$ persists

(quasi-) periodicity

Golfer's dilemma [Gualtieri, T^2, ... 2006]

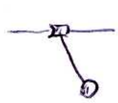
SLIDE
Photos

▶ 20 min

Digression • Do you believe in conservation of lin mom ?

magnetic croquet

• Friction mysterious

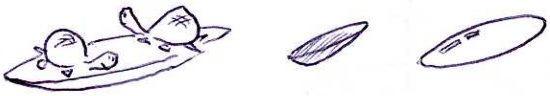


▶ 25 min

celt / rattleback / wobblestone

pron. [selt] not [kelt]

[meteorologist Walker 1896]



~ αω dynamo
in magnetohydrodynamics

Exegesis of the word

SLIDE
graphs

40 min

tippy top

SLIDE
Bohr & Pauli

CHIRAL tippy top

Thank Alain Albouy
Delphine Boucher
Guy Casale
Jacky Cresson
Jean-Pierre Marco
Juan-Pablo Ortega Lahuerta
Maria Przybylska
Nguyen Tien Zung
Alexei Tsygvintsev
Jacques-Arthur Weil

$\Lambda(t,z)$ deformation vel of floor (can be discont)

$z = x + iy$ horiz comp of ball's center

$\omega = \omega_x + i\omega_y$ ... of ang vel

$R = R_x + iR_y$ ... of reaction by floor

$F = F_x + iF_y$ ... of external force

$$\left.\begin{array}{ll} \text{lin mom:} & m\ddot{z} = R + F \\ \text{ang mom:} & \hat{I} m a^2 \dot{\omega} = -iaR \end{array}\right\} \xrightarrow{\text{eliminate } R} \left.\begin{array}{l} \\ \\ \end{array}\right\} \xrightarrow{\text{eliminate } \omega}$$

$$\text{rolling cond:} \quad \dot{z} + ia\omega = \Lambda$$

$$\Rightarrow (1+\hat{I})\ddot{z} = \hat{I}\dot{\Lambda} + \frac{F}{m}$$

$\Rightarrow$ Strange first integral

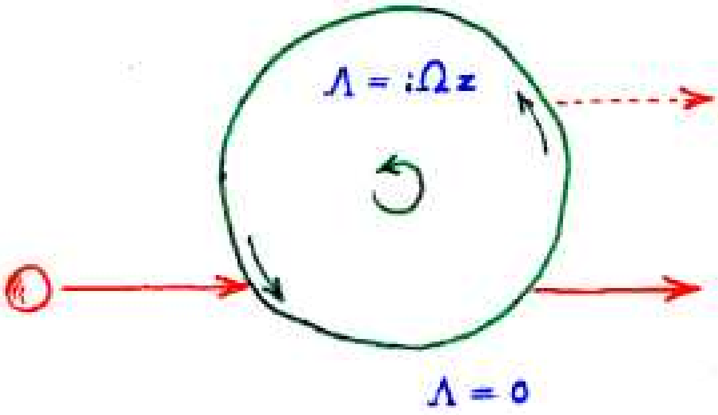
$$\boxed{\dot{z}(t) - \frac{\hat{I}}{1+\hat{I}}\Lambda(t, z(t)) = \dot{z}(0) - \frac{\hat{I}}{1+\hat{I}}\Lambda(0, z(0)) + \frac{1}{1+\hat{I}}\int_0^t \frac{F(\tau)}{m}\,d\tau}$$

Appearence of

$$\frac{\hat{I}}{1+\hat{I}} = \begin{cases} \frac{2}{7} & \text{solid ball} \\ \frac{2}{5} & \text{spherical shell} \end{cases}$$

typical in rolling problems

Across turntable :



// because of first integral

actually = because of symmetry

Within turntable : circular motion $\boxed{\text{periodic}}$

~ charged particle in electromagnetic field
E B

$$\mathbf{v} \longleftrightarrow z$$

$$\frac{qB}{m} \longleftrightarrow \frac{\hat{I}}{1+\hat{I}}\,\Omega$$

$$\frac{qE}{m} \longleftrightarrow \dot{z}(0) - i\,\frac{\hat{I}}{1+\hat{I}}\,\Omega\, z(0)$$

$\boxed{\text{Is rolling (nonholo) cond physically realizable ?}}$

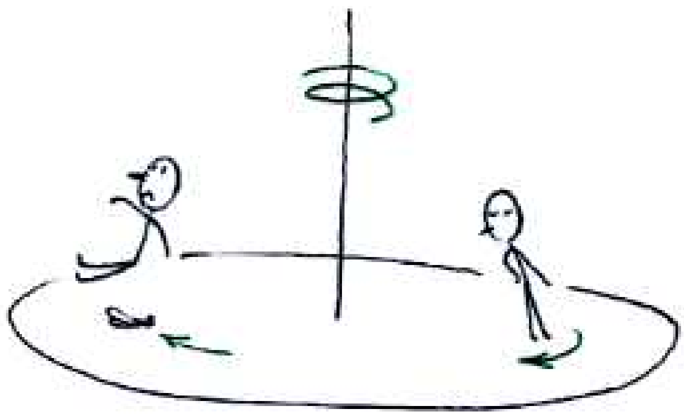
Yes if $|R| \leqslant \mu m g$;

for turntable $|\Omega|\,|\dot{z}(0)| \leqslant \dfrac{1+\hat{I}}{\hat{I}}\,\mu g$

With $\mu \sim \frac{1}{3}$, $\hat{I} \sim \frac{2}{5}$, $|\dot{z}(0)| \sim 0.5\,m/sec$

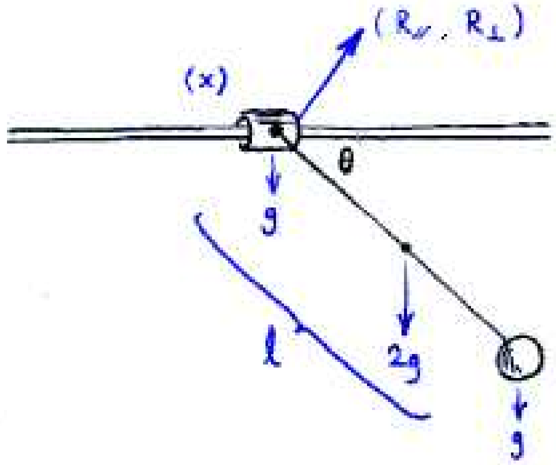
$|\Omega|$ o.k. up to $\sim 3\,Hz$

$\sim \frac{1}{2}$ turn/sec



Usual friction « law » incompatible with classical mechanics

 same mass



friction : $R_\parallel = -\varepsilon\mu R_\perp$

ε chosen s.t.

$$\boxed{\varepsilon \dot{x} R_\perp > 0}$$

$$\left.\begin{aligned}
\text{lin mom :}\quad & \frac{d^2}{dt^2}\{x + (x + \ell\cos\theta)\} = R_\parallel \\
& \frac{d^2}{dt^2}\{0 + \ell\sin\theta\} = R_\perp + 2g \\
\text{ang mom :}\quad & \frac{\ell^2}{2}\frac{d^2}{dt^2}\theta = \frac{\ell}{2}(R_\parallel\sin\theta - R_\perp\cos\theta)
\end{aligned}\right\}$$

Replace $R_\parallel$ by $-\varepsilon\mu R_\perp$ and solve for $R_\perp$, $\ddot{x}$, $\ddot{\theta}$ ...

$$\rightarrow \quad (1 + \cos^2\theta + \varepsilon\mu\sin\theta\cos\theta)\,R_\perp = -\ell\dot{\theta}^2\sin\theta - 2g$$

etc.

Suppose $0 < \theta_0 < \frac{\pi}{2}$ and μ large enough that

$$1 + \cos^2\theta_0 - \mu\sin\theta_0\cos\theta_0 < 0$$

If $\dot{x}_0 > 0$, choice of ε impossible in

If $\dot{x}_0 < 0$, ... ambiguous ...

« Large enough » if $\mu > 2\sqrt{2}$;

physically improbable.

How the word « celt » came to be

Job xix 24

Vulgata :

... stylo ferreo, et plumbi lamina,

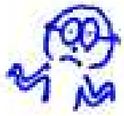
vel celte sculpantur in silice.

looks like ablative of a putative word celtus

in Hebrew original :

לָעַד (for certain, for ever)

so turns out to be a typo for certe !



King James Authorized Version :

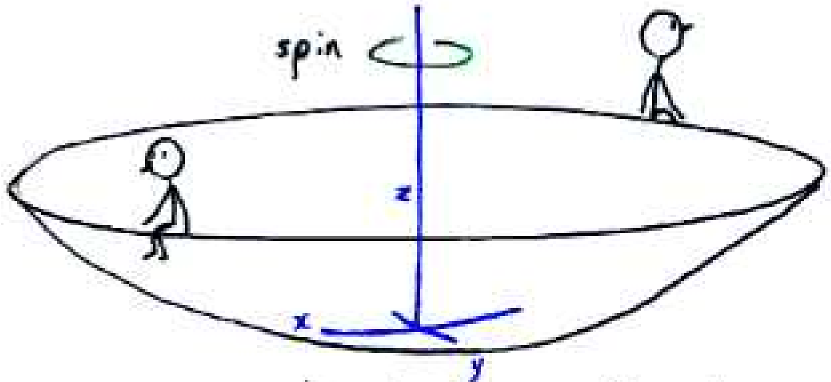
(Oh that my words were now written !

Oh that they were printed in a book !)

That they were graven with an iron pen and lead

in the rock for ever !

Work in dimensionless variables



xyz aligned with axes of inertia

contact surface

$$z = -1 + \frac{x^2}{2a^2} + \chi \frac{xy}{a^2} + \frac{y^2}{2b^2}$$

$\boxed{\text{chirality}}$ (as in χείρ «hand»)

Normal freqs when $\chi = 0$:

$$\omega_x^2 = 5 \frac{a^2 - 1}{a^2 + b^2} \qquad\qquad \omega_y^2 = 5 \frac{b^2 - 1}{a^2 + b^2}$$

たてゆれ	pitching	rolling	よこゆれ
	tangage	roulis	
	килевая качка	бортовая качка	

Let $S = 5\left(1 + \frac{6}{b^2}\right)$ spin and suppose $a \gg b$ (long and thin).

Then

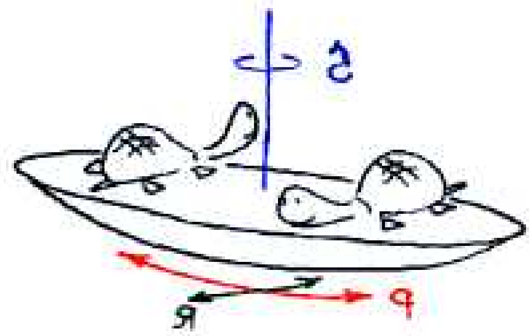
$$\ddot{x} + \omega_x^2 x + \chi \ddot{y} = \ldots$$
$$\ddot{y} + \omega_y^2 y + S\dot{x} = \ldots$$

(these terms slide $\omega_{x,y}$ along Re)

chiral / Coriolis terms push $\omega_{x,y}$ off Re into Im,

so responsible for instabilities.

Averaging / sepa of slow and fast time-scales



Rescale variables

$$P = s\left(1+\frac{c}{b^2}\right)\sqrt{\frac{a^2+c}{a^2+b^2}} \times \text{pitching amplitude}$$

$$R = s\left(1+\frac{c}{b^2}\right)\sqrt{\frac{b^2+c}{a^2+b^2}} \times \text{rolling amplitude}$$

$$S = s\left(1+\frac{c}{b^2}\right) \times \text{spin itself} \quad (\text{not just amplitude})$$

In slow time $\tau = \dfrac{|\chi|}{2(\lambda-1)}\, t$, $\quad \lambda = \left(\dfrac{\omega_x}{\omega_y}\right)^2 > 1$.

$$\boxed{\frac{d}{d\tau}\begin{pmatrix} P \\ R \\ S \end{pmatrix} = \operatorname{sign}\chi \begin{pmatrix} R \\ \lambda P \\ 0 \end{pmatrix} \wedge \begin{pmatrix} P \\ R \\ S \end{pmatrix}}$$

[Moffatt & T², 2007]

« Slow » if $\chi \ll \dfrac{a}{\sqrt{\text{energy}}}$

$$\frac{d}{d\tau}\begin{pmatrix} P \\ R \\ S \end{pmatrix} \perp \begin{pmatrix} P \\ R \\ S \end{pmatrix} \Rightarrow P^2 + R^2 + S^2 = \text{const.}$$

... energy

$$\frac{d}{d\tau}\begin{pmatrix} P \\ R \\ S \end{pmatrix} \perp \begin{pmatrix} R \\ \lambda P \\ 0 \end{pmatrix} \Rightarrow P R^{\lambda} = \text{const.}$$

... $\boxed{\text{strange first integral}}$

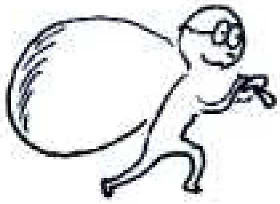
In PRS space every motion is $\boxed{\text{periodic}}$:

infinite succession of reversals

Dissipative version

$$\boxed{\frac{d}{d\tau}\begin{pmatrix} P \\ R \\ S \end{pmatrix} = \operatorname{sign} \chi \begin{pmatrix} R \\ \lambda P \\ 0 \end{pmatrix} \wedge \begin{pmatrix} P \\ R \\ S \end{pmatrix} - \begin{pmatrix} \mu_P P \\ \mu_R R \\ \mu_S S \end{pmatrix}}$$

In conclusion, please take home :



- Important to analyse physical realizability of nonholonomic constraints

- strange first integral
  (quasi-) periodicity
  chirality
  finite-time singularity
  ⋮

  common, even generic (?) features of « nonholo integrability »

merci / thank you