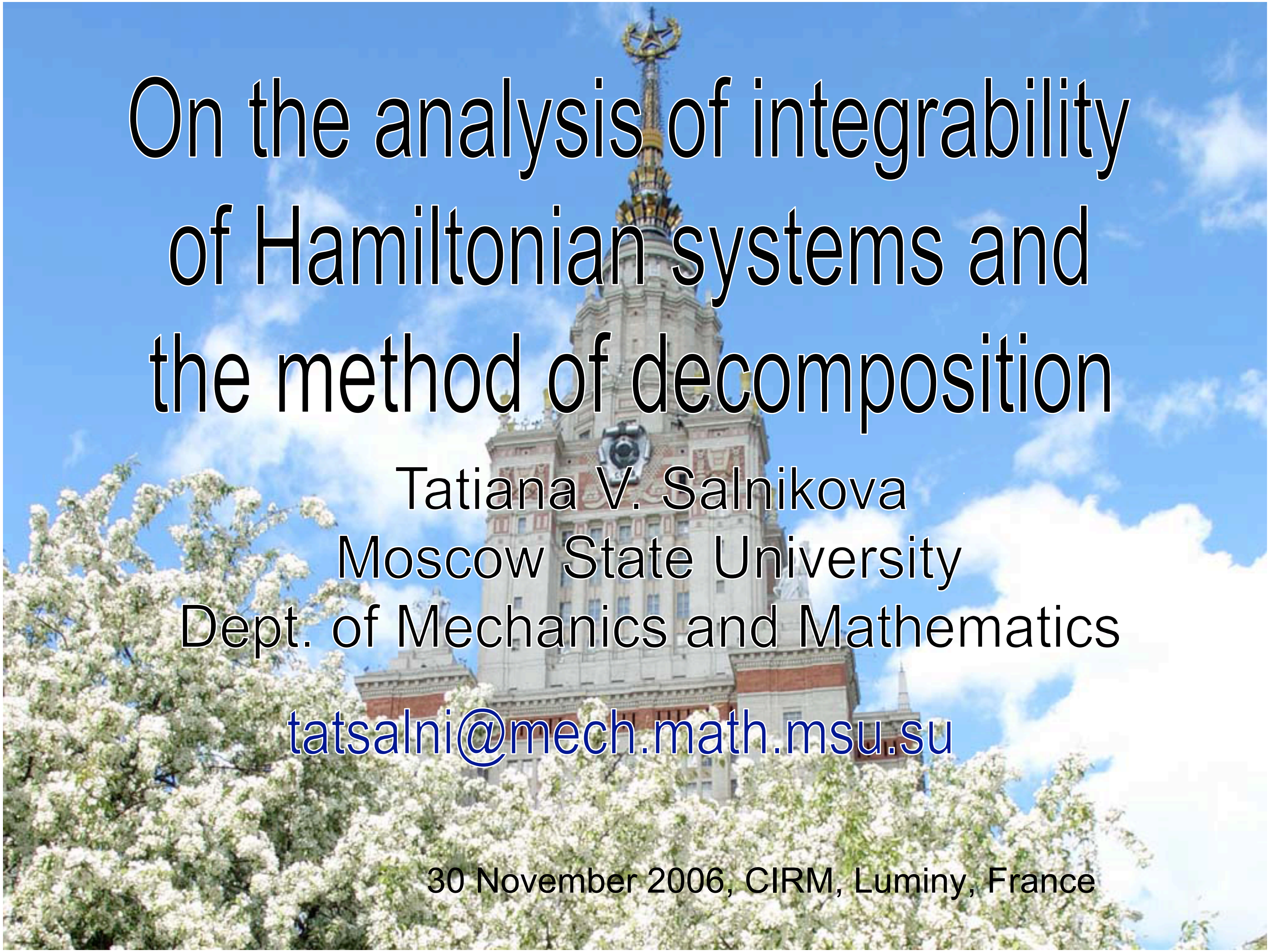

On the analysis of integrability of Hamiltonian systems and the method of decomposition

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- Birkhoff:
  “When, however, one attempts to formulate a precise denition of integrability, many possibilities appear, each with a certain intrinsic theoretical interest”

Theorem (Liouville – Arnold)

$(M, \underline{\omega}, H)$, $\dim M = 2n$, $\underline{\omega} = dp \wedge dq$

$F_1, \ldots, F_n$ – first integrals, $\{F_i, F_j\} = 0$,

$M_f = \{z \in M, F_j(z) = f_j = \text{const}, j = 1 \ldots n\}$,

F_j are independent on M_f. Then

1) M_f is a smooth manifold, invariant with respect to Hamiltonian system $\dot{z} = v_H$
2) Each compact connected component of M_f is diffeomorphe to T^n
3) There exist coordinates $(\varphi_1, \ldots \varphi_n)$ mod 2π, such that Hamilton equations read

$$\dot{\varphi} = \omega(f) = const \in \mathbb{R}^n$$

Action – angle variables

(I,φ): $\underline{\omega} = dI \wedge d\varphi$ $I = \frac{1}{2\pi}\int_{\gamma_n} p\,dq$

$H = H(I)$,

$\varphi = \varphi(\mathrm{mod}\ 2\pi)$ (i.e. φ – *angular coordinates on* M_h)

Poincaré H. Les Méthodes nouvelles de la mecanique céleste. Paris, Gauthier-Villars, 1882.

V.V.Kozlov. Methods of qualitative analysis in dynamics of rigid bodies, Moscow, 1980

Theorem (Poincaré)

If 1) (*) $H(I,\varphi,\varepsilon) = H_0(I_1,I_2)+ \varepsilon H_1(I_1,I_2, \varphi_1, \varphi_2)+\ldots$ - analytical function of I, φ, ε, where

$$H_1(I,\varphi) = \sum_{k\in\mathbb{Z}^2} H_1^{\,k}(I)e^{i(k,\varphi)}, k = (k_1,k_2)$$

2) $\det\left\|\dfrac{\partial^2 H_0}{\partial I^2}\right\| \neq 0$

3) The Poincaré set B of variables

$$I\in D\subset\mathbb{R}^2 : \quad a) k_1\omega_1(I) + k_2\omega_2(I) = 0$$

$$b) H_1^{\,k}(I) \neq 0$$

is that any real analytic function, equal to 0 on B is equal to 0 on the whole *D*.

Then (*) doesn't possess another analytic integral of the form

$$F(I,\varphi,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k F_k(I,\varphi)$$

independent of H for small $\varepsilon\neq 0$

- Now we fix energy h and set I_1=I, φ_1= φ, φ_1= t
  Then on *H=h* we have
  where K=K_0+εK_1+…

$$\dot{\varphi} = \frac{\partial K}{\partial I}$$

$$\dot{I} = -\frac{\partial K}{\partial \varphi}$$

$$K_0 : H_0(I_1, K_0(h, I)) = h$$

$$K_1 = H_1(I_1, K_0(h, I_1), \varphi_1, \varphi_2)$$

- Theorem
  If 1) $\partial^2 K / \partial I^2 \neq 0$
  2)for some I_0, φ_0 the frequency

$$\omega = \frac{\partial K_0}{\partial I} = \frac{m}{n}$$

and

$$\frac{\partial \overline{K}_1}{\partial \varphi_0} = 0, \quad \frac{\partial^2 \overline{K}_1}{\partial {\varphi_0}^2} \neq 0 \qquad \overline{K}_1(l^0, \varphi^0) = \lim_{s\to\infty} \frac{1}{s} \int_0^s K_1(l^0, \omega t + \varphi^0, t)\, dt$$

Then for small ε there exist a periodic solution of the perturbed system , it depends analytically on ε, and for ε=0 it coincides with the solution of the unperturbed system. The two characteristic indexes $\pm\alpha$ - convergent series in powers of $\sqrt{\varepsilon}$

$$\alpha = \alpha_1 \sqrt{\varepsilon} + \alpha_2 \sqrt{\varepsilon}^2 + \alpha_3 \sqrt{\varepsilon}^3 + \ldots, \quad {\alpha_1}^2 = -\frac{\partial^2 \overline{K_1}}{\partial {\varphi_0}^2} \frac{\partial^2 K_0}{\partial {I_0}^2} \neq 0$$

Lagrange problem

$$H_0 = \frac{1}{2A}\left[\frac{(p_\psi - p_\theta \cos\theta)^2}{\sin^2\theta} + p_\theta{}^2\right] + \frac{p_\varphi{}^2}{2C} + mgz_0\cos\theta$$

$$p_\varphi = I_2, \quad p_\psi = I_3, \quad I_1 = \frac{1}{\pi}\int_{\lambda_3}^{\lambda_2} \frac{\sqrt{f(u)}}{1-u^2}du$$

$$f(u) = 2B\left(h - pz_0 - \frac{I_2^2}{2C}\right)(1-u^2) - (I_3 - I_2 u)^2 = 0$$

$$H = H_0(I_1, I_2, I_3) + \varepsilon H_1(I_1, I_2, I_3, \varphi_1, \varphi_2) \ \ (2\pi \text{ per of } \varphi_1, \varphi_2)$$

$$H_1 = mg[\sum b_{n,1}(I)e^{i(n\varphi_1 + \varphi_2)} + \sum b_{n,-1}(I)e^{i(n\varphi_1 - \varphi_2)}]$$

$$f(u) = (1-u)^2(u - \lambda_3)$$

$$h_0 = pz_0 + \frac{I_2^2}{2C}, \quad \lambda_3 = \frac{I_3^2}{2Apz_0} - 1$$

Theorem. If $z_0 \neq 0$, the reduced canonical equations of the perturbed problem do not have a first integral that is independent with the energy integral and that can be represented as a power series $F = \sum_{n\geq 0} \varepsilon^n F_n$, converging for small ε, with coefficients analytic in phase space.

Method of decomposition

$$\mathbf{\hat{O}}^2;\quad \varphi_1, \varphi_2, \quad \dot{\varphi}_i = \lambda_i, \quad (H = \lambda_2\varphi_2 - \lambda_1\varphi_1)$$

$$f_1 = \sin\varphi_1,\ \ f_2 = \cos\varphi_1$$

$$\begin{cases} \dot{f}_1 = f_2 \\ \dot{f}_2 = -f_1 \end{cases}$$

$$M_C = \{f_1 = c_1, f_2 = c_2\} = \mathbf{\hat{O}}^1;$$

$$\varphi_1 = const, \forall x \in M_C \quad \dot{c} = const$$

Quasi-integrals

- Def. $H(q,p,t)$; $q = (q_1,\ldots,q_n)$, $p = (p_1,\ldots,p_n)$
  $f_1(q,p,t), \ldots f_k(q,p,t)$ – quasi-integrals, if

  $$\dot{f}_i = \Phi_i(f_1,\ldots f_k,t) \quad (\dot{f}_i = \{H, f_i\})$$

  I.e. class of Hamiltonian systems with the set of functions closed with respect to Poisson structure.

- Statement:

1) Let $\forall c\ M_c\{(q,p)\,|\,f_i(q,p)=c_i, i=1,\ldots k\}$

$g_{t_0}^{t}$ -shift along the trajectory of the system with $H(q,p,t)$ from t_0 to t

Then $\forall t\ g_{t_0}^{t}(M_C)=M_{C_t}$ is homeomorphe to the level manifold of $M_{C_{t_0}}$

2) Let's denote π - natural application of the initial

phase space to the space of constants $\square^{k}=(x_1,\ldots,x_k)$

$$\pi(q,p)=\{x_1=f_1(q,p,t),\ldots,x_k=f_k(q,p,t)\}$$

In $\square^{k}$ we can consider non-autonomous system

(*) $x_i=\Phi_i(x_1,\ldots,x_k,t)$

$$\begin{array}{ccc} M_C & \xrightarrow{g_{t_0}^{t}} & M_{C_t} \\ \downarrow\pi & & \downarrow\pi \\ x & \xrightarrow{h_{t_0}^{t}} & x_t \end{array}$$

levels of $M_C \leftrightarrow x$

Decomposition

Notice 1. Quasi-integrals are well defined.

Ex. $M^n \subset \mathbf{R}^{2n+1}$, coordinates

Ex. $\mathbf{T}^2 \subset \mathbf{R}^3$ but no reduction

$$\begin{cases} f_1 = x(u,v) \\ f_2 = y(u,v) \\ f_3 = z(u,v) \end{cases} \qquad \dot{f}_i = \Phi_i(f_1, f_2, f_3)$$

Notice 2. Globally defined. Independence is not necessary

Notice 3. The dynamics of the point is replaced by the dynamics of a manifold.

Example 1.

$$H = \frac{1}{2}(M, A(t)M) - (k(t), M)$$

$$\{M_i, M_j\} = -\varepsilon_{ijk} M_k \qquad (SO(3))$$

General case: In 6-dim phase space $SO(3) \times \mathbf{R}^3$ - motion of a torus $\mathbf{T}^2 \equiv M_C$:

$$\begin{cases} \dot{H} = \Phi_1(M, t) \\ \dot{M} = \Phi_2(M, t) \end{cases}$$

Illustration: Euler's top

$\tilde{r}_i = \lambda_0(t) r_i,$ where $\lambda_0 = 1 + \lambda(t)$

$$\dot{\tilde{H}} = \frac{2\dot{\lambda}_0}{\lambda_0}\tilde{H} = f(t)\tilde{H}$$

Quasi-intregrals: $\tilde{H}, M_1, M_2, M_3$

$$f(t)dt = d\boldsymbol{\tau}$$

$$\begin{cases} \dfrac{d}{d\boldsymbol{\tau}} q = \dfrac{1}{f}\dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \\ \dfrac{d}{d\boldsymbol{\tau}} p = \dfrac{1}{f}\dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} \end{cases}$$

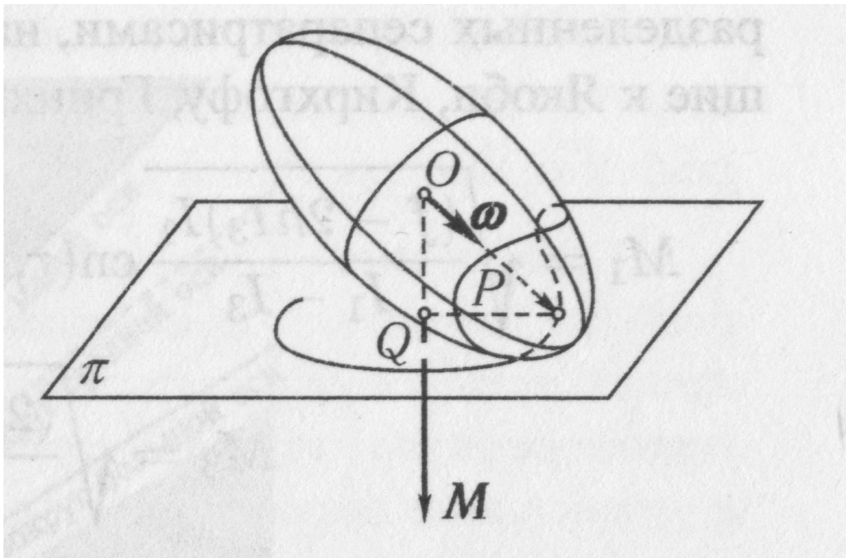


Illustration: Zhukovskiy-Volterra case

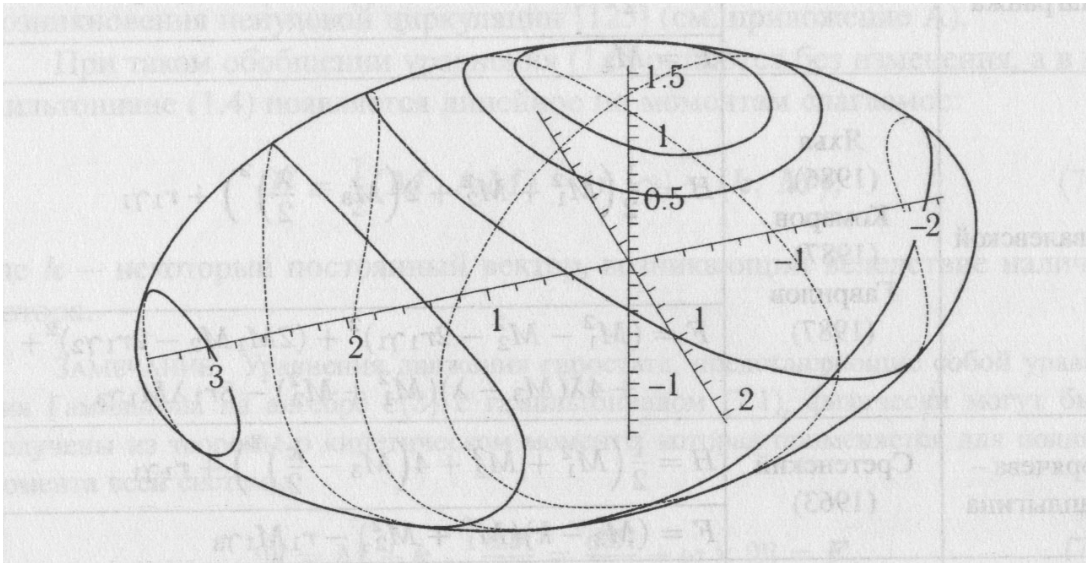
$A = const, \quad k = const$

$$H = \frac{1}{2}(M - k, A(M - k)) = h = const$$

$M^2 = f = const,$

In phase space $SO(3) \times \mathbf{R}^3$ - motion of a torus $\mathbf{T}^2$:

$$\begin{cases} \dot{H} = 0 \\ \dot{M} = \Phi(M) \end{cases}$$



$$\pi^{-1}(h = c_1, M = M_1) \equiv \mathbf{T}^2$$

Example 2

- Lagrange top with $M_z(t)$

$$H_0 = \frac{1}{2}(J^{-1}M, M) - mg(r, \gamma),$$

$$H_0 = h = const,\ p_\psi = M_3 = const,\ p_\varphi = const,\ \gamma^2 = 1$$

$$M_z(t) \Rightarrow \begin{cases} \dfrac{dM_3}{dt} = M_z(t) \\ \dfrac{dH}{dt} = M_3 M_z(t) \end{cases}$$

Quasi-integrals: $H(t),\ \ r(t)$

$M_C = \{H = c_1, r = c_2, \gamma^2 = 1\}$ some 3dim manifold,

moving in the phase space as $H(r)$.

Example 3

$$H = \frac{1}{2}(p_x{}^2 + p_y{}^2 + p_z{}^2) + V(r) + \frac{b(t)}{m}(p_y x - p_x y) = H_0 + b(t) M_z$$

$$(\mathbf{B} = (0,0,b(t)), \quad V_1 = -\frac{b(t)}{2m}(x^2 + y^2))$$

Quasi-integrals: H, M_x, M_y, M_z

$$\begin{cases} \dot{H} = \dot{b} M_z \\ \dot{M}_x = b M_y \\ \dot{M}_y = -b M_x \\ \dot{M}_z = 0 \end{cases}$$

$$M_C = \{H = c, M_i = c_i\} \equiv \mathbf{T}^2$$

Thank you for your attention!