On the dynamics of the triple pendulum: non-integrability, topological properties of the phase space.

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In this paper the planar motion of a multiple pendulum is studied. A multiple pendulum is a system of mass points with constraints given by second degree polynomial equations. Employing the reduction of the problem (Routh transform), some topological properties (dimensions of the intersection of the general manifolds, and foliation of the phase space, generated by the Routh transform) and direct numerical modeling the non-integrability of the problem has been shown and also the chaotisation of the dynamics has been observed. Special attention is given to the connection of dynamics of the system with constraints to the behavior of the geodesics on the three-dimensional torus. It results in the presence of returnability geodesics.

I. INTRODUCTION.

In this paper the planar motion of the multiple pendulum is studied. The triple pendulum is the system of four mass points, connected by weightless inextensible rods in the case of the first mass point being fixed. This conditions can be cast in the form:

$$(\vec{r}_i - \vec{r}_{i-1})^2 = const \qquad i = 1, 2, 3$$

where $\vec{r}_i$ corresponds to i-th mass point, the point $\vec{r}_0$ is fixed. Or in more general case the conditions may be given by a second degree polynomial equations.

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Let us note that the case of double pendulum has been carefully studied. Particularly, the free motion of the planar double pendulum is a classical system described in various textbooks. It is proved to be completely integrable (See for example [8]), as there exist the energy and the third component of angular momentum integrals. But for this system in the gravity field the non-integrability has been shown, and the chaotic motion can be observed. The problem of control has also been studied for this system [2]. It turns out that the control of only one degree of freedom of two is enough to set the system in the upright vertical position. And it is possible to construct an algorithm of control to observe the motion on the given curve with some conditions.

At the same time the dynamics of the triple pendulum is not that well studied, although it is very interesting for different applications. As it will be shown in this paper the system is non-integrable and possesses the irregular behavior. Furthermore the employed methods may be useful for studying the dynamics of the multiple pendulum which is even one of the models of a polymer molecule. The idea of describing the interaction of atoms with the help of Lagrange mechanics can be the continuance of the harmonic model employed in ([3]) for studying thermodynamical properties of the system or may in some sense contrast with it.

Thus, we consider a mechanical system of three mass points, the position of it is described by radius-vectors $\vec{r}_i$ and velocities $\vec{v}_i$ $(i = 1, 2, 3)$ of the points. The motion is restricted by the constraints - the conditions of the form:

$$\begin{aligned} \varphi_1 = (\vec{r}_1)^2 - l_1^2 &= 0 \\ \varphi_i = (\vec{r}_i - \vec{r}_{i-1})^2 - l_i^2 &= 0 \qquad i = 2, 3 \end{aligned} \tag{1}$$

As is know ([1]), there are at least two approaches to the analysis of the systems of the kind: the constraints can be resolved (i.e. we can choose new variables instead of $\vec{r}_i$ and $\vec{v}_i$, that consider the constraints in explicit form) or we can employ Lagrange multipliers (for details see section III). The first method turns out to be more convenient for analytical studies of the properties of the phase space of the system, while the second one permits to obtain the equations of motion in explicit form in the initial coordinates and may be employed for numerical modeling of the dynamics of the system. We should note that in the first case the connection between the problem of dynamics if the system, coming from mechanics, and the problem of behavior of geodesics on the corresponding manifold becomes obvious. At that some non-trivial but not that artificial metric is generated on the manifold. This consideration will be extremely useful for us in studying the connection between the non-integrability and topological properties of the system.

II. TOPOLOGICAL PROPERTIES OF THE PHASE SPACE OF THE SYSTEM.

The considered problem permits the convenient description in terms of generalized coordinates. The angles α_i between the units of the pendulum and some fixed straight line on the plane can be used for this purpose (See fig. 1) For the description of the position of the pendulum it is sufficient to set three angles.

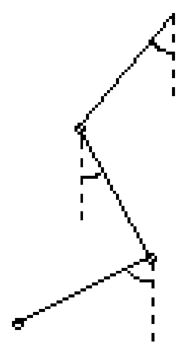

FIG. 1: The triple pendulum. Angle α_i parametrization.

The Lagrange function ([1]) in terms of this variables reads

$$\mathcal{L} = E_{kin} - U = \mathcal{L}(\alpha_1, \dot{\alpha}_1, \alpha_2, \dot{\alpha}_2, \alpha_3, \dot{\alpha}_3) =$$
$$= \frac{m}{2}(3l_1^2\dot{\alpha}_1^2 + 2l_2^2\dot{\alpha}_2^2 + 2cos(\alpha_2 - \alpha_3)l_1 l_2 \dot{\alpha}_2\dot{\alpha}_3 + l_3^2\dot{\alpha}_3^2 +$$
$$+4l_1 l_2 cos(\alpha_1 - \alpha_2)\dot{\alpha}_1\dot{\alpha}_2 + 2l_1 l_3 cos(\alpha_1 - \alpha_3)\dot{\alpha}_1\dot{\alpha}_3)$$

While the potential energy of the particles $U = 0$, as we consider the free motion. The configuration space in our case is a three dimensional torus $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. And the phase space is the tangent bundle of it $T(\mathbb{T}^3)$, which, as is know is the Cartesian product $\mathbb{T}^3 \times \mathbb{R}^3$ - the 6-dimensional manifold [7]. The Lagrange function corresponds to the metric on the torus ([4]):

$$ds^2 = \frac{m}{2}(3l_1^2 d\alpha_1^2 + 2l_2^2 d\alpha_2^2 + 2cos(\alpha_2 - \alpha_3) l_1 l_2 d\alpha_2 d\alpha_3 +$$
$$+ l_3^2 d\alpha_3^2 + 4 l_1 l_2 cos(\alpha_1 - \alpha_2) d\alpha_1 d\alpha_2 + 2 l_1 l_3 cos(\alpha_1 - \alpha_3) d\alpha_1 d\alpha_3)$$

The Lagrange function explicitly doesn't depend on time, it means that our systems possesses the energy integral. Let us note that our system also possesses the cylindrical symmetry: the Lagrange function is invariant with respect to rotation around the point $\vec{r}_0$. That is in the 6-dimensional phase space $T(\mathbb{T}^3)$ the trajectory belongs to some 4-dimensional manifold.

More efficiently and pictorially the analysis of the topology of the systems trajectory can be performed with the use of Routh transform. The idea is that the presence of the angular momentum integral results in the existence of the cyclic variable ([1]). That is, we can choose new generalized coordinate system in which the Lagrange function reads

$$\mathcal{L} = \mathcal{L}(\beta_1, \dot{\beta}_1, \beta_2, \dot{\beta}_2, \dot{\beta}_3) \tag{2}$$

It is sufficient to perform the following coordinate transform

$$\begin{aligned} \beta_i &= \alpha_{i+1} - \alpha_i, \qquad i = 1, 2, \\ \beta_3 &= \alpha_1 \end{aligned} \tag{3}$$

In that case

$$\begin{aligned} \mathcal{L} = \frac{m}{2} \Big[&\left(3l_1^2 + 2l_2^2 + l_3^2 + 2l_1 l_2 (cos\beta_2 + 2\cos\beta_1) + 2l_1 l_3 cos(\beta_1 + \beta_2)\right) \dot{\alpha}_1^2 + \\ &+ \left(2l_2^2 + l_3^2 + 2l_1 l_2 cos\beta_2\right) \dot{\beta}_1^2 + \left(l_3^2\right) \dot{\beta}_2^2 + \left(4l_2^2 + 2l_3^2 + 4l_1 l_2 (cos\beta_1 + cos\beta_2) + \right. \\ &\left. + 2l_1 l_3 (cos(\beta_1 + \beta_2))\right) \dot{\alpha}_1 \dot{\beta}_1 + \left(2l_3^2 + 2l_1 l_2 cos\beta_2 + 2l_1 l_3 (cos(\beta_1 + \beta_2))\right) \dot{\alpha}_1 \dot{\beta}_2 + \\ &+ \left(2l_3^2 + 2l_1 l_2 cos\beta_2\right) \dot{\beta}_1 \dot{\beta}_2 \Big] \end{aligned} \tag{4}$$

From this we can obtain that $\dot\beta_3 = \dot\alpha_1 = const$. Now we can introduce the Routh function:

$$\mathcal{R} = \frac{\partial \mathcal{L}}{\partial \dot\beta_3}\dot\beta_3 - \mathcal{L} =$$
$$= \frac{m}{2}\left[\left(3l_1^2 + 2l_2^2 + l_3^2 + 2l_1l_2(cos\beta_2 + 2\cos\beta_1) + 2l_1l_3cos(\beta_1+\beta_2)\right)\dot\alpha_1^2 - \right.$$
$$\left. - \left(2l_2^2 + l_3^2 + 2l_1l_2cos\beta_2\right)\dot\beta_1^2 - \left(l_3^2\right)\dot\beta_2^2 - \left(2l_3^2 + 2l_1l_2cos\beta_2\right)\dot\beta_1\dot\beta_2\right] \tag{5}$$

When $\dot\beta_3$ is fixed the function depends on only two variables and its derivatives.

$$\mathcal{R} = \mathcal{R}(\beta_1, \dot\beta_1, \beta_2, \dot\beta_2; \dot\beta_3 = const)$$

Then we can consider $\mathcal{R}$ as a new Lagrange function. That is the trajectory in the phase space belongs to the manifold of dimension 4 (certainly depending on the value of the cyclic integral $\dot\beta_3$). Thus we obtain a continuous set of nonintersecting 4-dimensional manifolds - layers, parameterized by the value of the cyclic integral; at that any trajectory belongs to one of them. That is the foliation of the phase space with respect to Routh transform. And as $\mathcal{R}$ still doesn't depend explicitly on time, the system again possesses the "energy" integral. That's why the manifold (let's name it M_β), containing the trajectory is actually three-dimensional.

This foliation is of primary importance in our analysis. let's consider this fact from topological point of view. The motion takes place on the 3-dimensional manifold in the 4-dimensional space. It permits us to study the structure of the manifold M_β by intersecting it with the planes (of dimension 2) which is very convenient for visualization. Really, the dimension of the intersection of the manifolds with dimensions n_1 and n_2 in the N-dimensional space is given by the equation [7]

$$dim = n_1 + n_2 - N$$

In our case it results in $dim = 1$, i.e. the finite set of curves.

The ideas discussed above allow us to prove the non-integrability of the problem. In case of existence of another integral the dimension of M_β would be 2, i.e. the intersection would be the set of points. The concept is the variant of Poincaré application for the case of more complicated phase space than usual. The presence of foliation allows to analyze the geodesics flow and its intersection with the plane.

The presence of curves in the intersection indicates not only the absence of integrability, but also that the trajectory is rather dense on the manifold. It means that the behavior of the system is quite irregular, that is the system can be chaotic.

It is worth noting that the numerical investigation of our problem with the resolved constraints leads to rather complicated equations and to the needs of employing other methods for obtaining constructive results.

But before considering this problem let us note that the case of the triple lattice (the pendulum with the first point being not fixed) from topological point of view is equivalent to the one investigated above. Really, we can choose the same α_i generalized coordinates and also the values of coordinates of the center of mass of the system. In this case the system will be invariant with respect to translation along the X and Y axes. That is the momentum conservation law will be fulfilled. The Lagrange function can be cast in the form

$$\mathcal{L} = \mathcal{L}(\dot{x}, \dot{y}, \alpha_1, \dot{\alpha}_1, \alpha_2, \dot{\alpha}_2, \alpha_3, \dot{\alpha}_3)$$

and the Routh transform can be performed. First with respect to x:

$$\mathcal{R}_1 = \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{x} - \mathcal{L} = \mathcal{R}_1(\dot{x} = const, \dot{y}, \alpha_1, \dot{\alpha}_1, \alpha_2, \dot{\alpha}_2, \alpha_3, \dot{\alpha}_3)$$

then similarly with respect to y

$$\mathcal{R}_2 = \frac{\partial \mathcal{R}_1}{\partial \dot{y}}\dot{y} - \mathcal{R}_1 = \mathcal{R}_2(\dot{x} = const, \dot{y} = const, \alpha_1, \dot{\alpha}_1, \alpha_2, \dot{\alpha}_2, \alpha_3, \dot{\alpha}_3)$$

And as described above we can choose angle variables β_i:

$$\mathcal{R}_2 = \mathcal{R}_2(\dot{x} = const, \dot{y} = const, \beta_1, \dot{\beta}_1, \beta_2, \dot{\beta}_2, \dot{\beta}_3)$$

and perform the Routh transform with respect to β_3

$$\mathcal{R} = \frac{\partial \mathcal{R}_2}{\partial \dot{\beta}_3}\dot{\beta}_3 - \mathcal{R}_2 = \mathcal{R}(\dot{x} = const, \dot{y} = const, \dot{\beta}_3 = const, \beta_1, \dot{\beta}_1, \beta_2, \dot{\beta}_2,)$$

That is, the method worked out above is applicable also for the lattice.

III. DYNAMICS OF THE SYSTEM. VISUALIZATION.

Let us now turn to the second way of treating the systems with constraints, more convenient for numerical modeling. If we investigate the system of N mass points, described by radius-vectors $\vec{r}_i$ and velocities $\vec{v}_i$ $(i = 1, \ldots, N)$, and the motion is restricted by the conditions

$$\varphi_j(\vec{r}_1, \ldots, \vec{r}_N) = 0, \qquad j = 1, \ldots, K$$

In that case, according to [10] the equations read

$$m_i \ddot{\vec{r}}_i = -\frac{\partial}{\partial \vec{r}_i} U(\vec{r}_1, \ldots, \vec{r}_N) + \sum_{j=1}^{K} \lambda_j \frac{\partial \varphi_j}{\partial \vec{r}_i} \tag{6}$$

where λ_j are Lagrange multipliers. As we consider the free motion of the system $U(\vec{r}_1, \ldots, \vec{r}_N) = 0$ For our concrete system the equations of motion read:

$$\begin{aligned} m_1 \ddot{\vec{r}}_1 &= \lambda_1 \vec{r}_1 - \lambda_2(\vec{r}_2 - \vec{r}_1) \\ m_2 \ddot{\vec{r}}_2 &= \lambda_2(\vec{r}_2 - \vec{r}_1) - \lambda_3(\vec{r}_3 - \vec{r}_2) \\ m_3 \ddot{\vec{r}}_3 &= \lambda_3(\vec{r}_3 - \vec{r}_2) \end{aligned} \tag{7}$$

Here we denote the doubled Lagrange multipliers as λ_j just for convenience.

The equations of this form are very convenient for computer simulation using various algorithms of numerical integration, as we can employ the standard way of reducing the order of differential equation and cast the system (7) of N vector differential equations of second order to the system of $2N$ vector equations of the first order, introducing the variables

$$\vec{v}_i = \dot{\vec{r}}_i, \qquad i = 1, \ldots, N$$

Their physical meaning is velocities. It is important to note, that Lagrange multipliers are the functions of coordinates and velocities:

$$\lambda_i = \lambda_i(\vec{r}_1, \ldots \vec{r}_N, \vec{v}_1, \ldots \vec{v}_N) \qquad i = 1, \ldots, N$$

This fact is true for a general system with constraints. For our system it is easy to obtain the system of equations for λ_i in an explicit form. We should just subtract the $i - th$ equation from the $i + 1 - st$ one, and notice that the conditions of conservation of the constraints read:

$$\begin{aligned} \psi_1 = \dot{\varphi}_1/2 = \vec{r}_1 \vec{v}_1 &= 0 \\ \psi_i = \dot{\varphi}_i/2 = (\vec{r}_i - \vec{r}_{i-1})(\vec{v}_i - \vec{v}_{i-1}) &= 0 \qquad i = 2, \ldots, N \end{aligned}$$

From these equations we can obtain another condition for the initial data:

$$\dot{\psi}_1 = \vec{r}_1\dot{\vec{v}}_1 + \vec{v}_1^2 \;=\; 0 \qquad (8)$$
$$\dot{\psi}_i = (\vec{r}_i - \vec{r}_{i-1})(\dot{\vec{v}}_i - \dot{\vec{v}}_{i-1}) + (\vec{v}_i - \vec{v}_{i-1})^2 \;=\; 0 \qquad i = 2,\ldots,N$$

From these equations we can easily obtain the left-hand sides of equations (7) and thus obtain the linear system for the Lagrange multipliers λ_i.

It is also worth noting that this kind of treating the problem also permits to perform some algebraic analysis, connected with existence of polynomial first integrals. Having written the system of equations describing the motion in explicit form, we can notice, that it is invariant under the similarity transform ([12]), i.e. $t \to \alpha^{-1}t, \qquad \vec{r} \to \alpha^{g}\vec{r}, \qquad \dot{\vec{r}} \to \alpha^{g+M}\dot{\vec{r}}$ That is we can apply Yoshida method, but it results in nonlinear algebraic equations, which can't be solved in a general case. So, the only approach to detailed analysis of the system is the numerical one.

Direct numerical simulations allows us to obtain the pictures like Fig. 2, visually demonstrating the dynamics of the system.

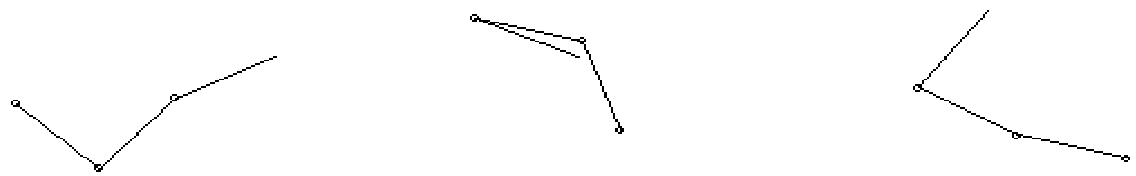

FIG. 2: Triple pendulum. Dynamics.

To apply the method, worked out in section II it is important to choose the efficient way of the results visualization. As it has been shown, as β_1 and β_2 (see eq.2) it is possible to take the difference of corresponding angles α_i

$$\beta_i = \alpha_{i+1} - \alpha_i, \qquad i = 1, 2,$$

that is the angles between the units of the pendulum, calculated by the coordinates $\vec{r}_i$ $\vec{v}_i$. The angles α_i are taken modulo 2π. And the intersecting plane is any coordinate plane of the space $(\beta_1, \dot{\beta}_1, \beta_2, \dot{\beta}_2)$.

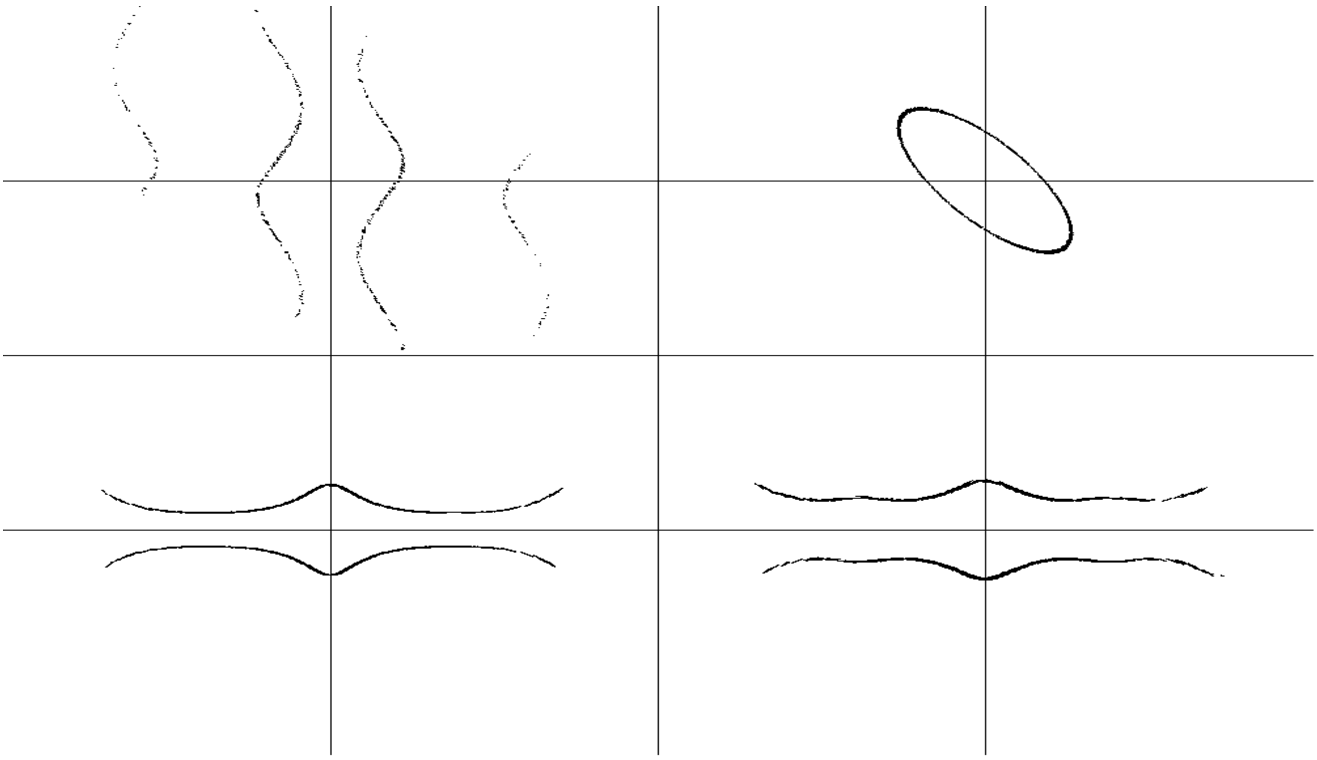

FIG. 3: The intersection of the manifold M_β with the planes $(\dot{\beta}_1 = 1, \dot{\beta}_2 = 1)$, $(\beta_2, \dot{\beta}_2)$, $(\beta_1, \dot{\beta}_2)$, (β_2, β_1)

As it can be seen from Fig. 3, there is some region on the manifold M_β swept by the trajectory. It means not only non-integrability of the problem, but also its strongly irregular behavior.

We should also note, that the system possesses the important property of returnability: the trajectory (as is seen from numerical experiment) passes an infinite number of times close to the initial point. This fact is interesting both from the point of view of pendulum dynamics and also for the behavior of geodesics, as it ir the first step to chaotic motion.

IV. CONCLUSION

Thus, we have investigated the dynamics of the planar triple pendulum, employing two approaches to the analysis of the problem. One is the direct numerical modeling of the system, using Lagrange multipliers in the equations of motion. Another one - is the topological analysis of the phase space and the reduction of the problem using Routh transform. At that each of these methods doesn't give the complete description of properties of the system, due to the difficulties of direct visualization and full analytical investigation.

The analogy between the mechanical problem of motion and the geometric problem of behavior of geodesics on the manifold. The non-integrability, showed explicitly for the mechanical problem, means the possibility of chaotic behavior of geodesics on rather simple objects, like three-dimensional torus with metric, having rather natural origin.

Methods and results, described in this paper may be useful for studying the multiple pendulum or multiple lattice, having more applied interest, and also other multidimensional problems, having analogs in geometry.

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