INTEGRABLE FLOWS ON THE SYMPLECTIC GROUP

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PLAN OF THE PRESENTATION

- The equations
- The N-dimensional free rigid body
- The Mischenko-Fomenko free rigid bodies
- Lie-Poisson form of the equations
- Relation to the Mischenko-Fomenko systems
- Lax pair with parameter
- Bi-Hamiltonian structure and integrability
- Linearization of the flows

THE EQUATIONS

Bloch and Iserles (*Foundations of Computational Mathematics* **6** (2006), 121–144) have introduced the system

$$\dot{X} = [X^2, N], \quad X \in \mathsf{Sym}(n), \quad N \in \mathfrak{so}(n) \quad \text{constant.}$$

Note that $[X^2, N] \in \mathsf{Sym}(n)$, so if the initial condition is in $\mathsf{Sym}(n)$ then $X(t) \in \mathsf{Sym}(n)$ for all t.

Note that

$$\langle\langle X, Y \rangle\rangle := \operatorname{trace}(XY), \quad \text{for} \quad X, Y \in \mathsf{Sym}(n)$$

is a positive definite inner product on $\mathsf{Sym}(n)$ and that

$$\frac{1}{2}\operatorname{trace}(X^2) = \frac{1}{2}\langle\langle X, X \rangle\rangle$$

is conserved along the flow. Its level sets are compact so the equation $\dot{X} = [X^2, N]$ has solutions for all $t \in \mathbb{R}$.

Reasons given for studying this system:

1.) $\dot{X} = [X, N]X + X[X, N]$ a special case of a congruent flow

$$\dot{X} = A(X)X + XA(X)^T, \quad A : \mathrm{Sym}(n) \to \mathfrak{gl}(n, \mathbb{R}) \quad \text{smooth}$$

Solution is $X(t) = V(t)X(0)V(t)^T$, where $V(t)$ is the solution of

$$\dot{V}(t) = A\left(V(t)X(0)V(t)^T\right)V(t), \quad V(0) = I.$$

So the solution is given by the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathrm{Sym}(n)$ by congruence. In particular, the signature of $X(t)$ is conserved.

2.) $\dot{X} = [X, XN + NX]$, $B(X) := XN + NX$, is isospectral. The solution is given by the $\mathrm{SO}(n)$-action on $\mathrm{Sym}(n)$ by similarity: $X(t) = Q(t)^{-1}X(0)Q(t)$, where $Q(t) \in \mathrm{SO}(n)$ is the solution of

$$\dot{Q}(t) = Q(t)B\left(Q(t)X(0)Q(t)^{-1}\right), \quad Q(0) = I.$$

3.) It is dual to the SO(n) rigid body

$$\dot{M} = [M, \Omega]$$

$M, \Omega \in \mathfrak{so}(N)$, $M = J\Omega + \Omega J$, where J is a diagonal constant matrix satisfying the condition $J_i + J_j > 0$ for all $i \neq j$. This is an integrable system.

More details soon.

4.) They observed numerically extremely regular behavior of the solutions exactly as for an integrable system.

THE N-DIMENSIONAL FREE RIGID BODY

A: The classical free rigid body; $N=3$

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}$$

$\mathbf{\Pi}, \mathbf{\Omega} \in \mathbb{R}^3$, $\mathbf{\Pi}_i = I_i \mathbf{\Pi}_i$, $I_i > 0$, $i = 1, 2, 3$, principal moments of inertia. The body is in a principal axis body frame.

Conserved quantities:

Casimir: $C(\mathbf{\Pi}) := \|\mathbf{\Pi}\|$
Energy: $H(\mathbf{\Pi}) := \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{\Omega}$

The motion takes place on the intersection of the sphere of radius $\|\mathbf{\Pi}\|$ and the kinetic energy ellipsoid. So the problem has been integrated.

From the point of view of symplectic geometry: $S^2_{\|\mathbf{\Pi}\|}$ is a two-dimensional manifold, and the kinetic energy is a conserved quantity, so the system is Liouville integrable.

B: The N-dimensional rigid body

$$\dot{M} = [M, \Omega]$$

$M, \Omega \in \mathfrak{so}(N)$, $M = J\Omega + \Omega J$, where J is a diagonal constant matrix satisfying the condition $J_i + J_j > 0$ for all $i \neq j$.

If $N = 3$, then $I_1 = J_2 + J_3$, $I_2 = J_3 + J_1$, $I_3 = J_1 + J_2$.

Conserved quantities:

Casimirs: $C_k(M) := \text{trace}(M^k)$, $k = 2, 4, \ldots, 2K-2 = N$, and $C_N(M) := \text{Pf}(M) := \sqrt{\det(M)}$, if $N = 2K$, or $k = 2, 4, \ldots, 2K$, if $N = 2K + 1$

Energy: $H(M) := -\frac{1}{4} \operatorname{trace} M\Omega$

These are Lie-Poisson equations for the bracket

$$\{F, G\}(M) = \frac{1}{2} \operatorname{trace}\left(M[\nabla F, \nabla G]\right),$$

where ∇ taken relative to the inner product $\langle\langle A, B \rangle\rangle := -\frac{1}{2}\text{trace}(AB)$ for any $A, B \in \mathfrak{so}(N)$.

a.) The operator $\mathbb{I}(A) := AJ + JA$ is symmetric relative to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$.

b.) Since $H(M) = \frac{1}{2}\langle\langle M, M \rangle\rangle$, it follows that $\nabla H(M) = \Omega$.

c.) Hence $\dot{F} = \{F, H\}$ is equivalent to

$$\begin{aligned}\langle\langle \dot{M}, \nabla F(M) \rangle\rangle &= \frac{d}{dt}\Big|_{t=0} F(M) = \{F, H\}(M) \\ &= -\langle\langle M, [\nabla F(M), \Omega] \rangle\rangle = \langle\langle M, [\Omega, \nabla F(M)] \rangle\rangle \\ &= \langle\langle [M, \Omega], \nabla F(M) \rangle\rangle\end{aligned}$$

for any F and hence to $\dot{M} = [M, \Omega]$.

Question: Is this system integrable? What exactly does this mean?

Simplest answer is to request that it be integrable on every coadjoint orbit of SO(N). This is also the most difficult to show. Slightly easier answer: request that it be integrable on all *generic* coadjoint orbits of SO(N). This is what will be shown now.

• **Manakov trick**

$$\frac{d}{dt}\left(M + \lambda J^2\right) = [M + \lambda J^2, \Omega + \lambda J]$$

• So

$$\mathrm{trace}(M + \lambda J^2)^k = \sum_{i=0}^{k} p_{ik}(M)\lambda^i$$

are conserved, and hence all $p_{ik}(M)$ are conserved. If N is even then one uses also the Pfaffian and does the same.

• Mishschenko and Fomenko proved that these integrals are in involution and independent (of course, take $k < N$, because of Cayley-Hamilton). Manakov already counted the relevant integrals and showed that their number equals $(\dim \mathfrak{so}(N) - \mathrm{rank}\, \mathfrak{so}(N))/2$.

Thus the N-dimensional rigid body is Liouville integrable.

FREE RIGID BODY ASSOCIATED TO A SEMISIMPLE LIE ALGEBRA

MISCHENKO-FOMENKO CONSTRUCTION

$\mathfrak{g}$ a semisimple complex or real split Lie algebra with Killing form $\langle\cdot,\cdot\rangle$, $\mathfrak{h}$ a Cartan subalgebra, $a, b \in \mathfrak{h}$ and a be regular (i.e. its value on every root is non-zero). A **sectional operator** $C_{a,b,D} : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$C_{a,b,D}(\xi) := \mathrm{ad}_a^{-1}\,\mathrm{ad}_b(\xi_1) + D(\xi_2),$$

where $\xi = \xi_1 + \xi_2$, $\xi_2 \in \mathfrak{h}$, $\xi_1 \in \mathfrak{h}^\perp$ (the perpendicular is taken relative to the Killing form and thus $\mathfrak{h}^\perp$ is the direct sum of all the root spaces), and $D : \mathfrak{h} \to \mathfrak{h}$ is an arbitrary invertible symmetric operator on $\mathfrak{h}$. Then $C_{a,b,D} : \mathfrak{g} \to \mathfrak{g}$ is an invertible symmetric operator (relative to the Killing form) satisfying the condition

$$[C_{a,b,D}(\xi), a] = [\xi, b], \quad \forall \xi \in \mathfrak{g}.$$

The Lie-Poisson bracket on $\mathfrak{g}^* \cong \mathfrak{g}$ (the isomorphism being given by the Killing form) has the expression

$$\{f, g\}(\xi) = -\langle \xi, [\nabla f(\xi), \nabla g(\xi)] \rangle$$

for any $f, g \in C^\infty(\mathfrak{g})$, where ∇ is taken relative to $\langle \cdot, \cdot \rangle$. Hamilton's equations for $h \in C^\infty(\mathfrak{g})$ have thus the form

$$\dot{\xi} = [\xi, \nabla h(\xi)].$$

In particular, if

$$h(\xi) := \frac{1}{2} \left\langle C_{a,b,D}(\xi), \xi \right\rangle$$

then $\nabla h(\xi) = C_{a,b}(\xi)$ since $C_{a,b}$ is $\langle \cdot, \cdot \rangle$-symmetric. Thus the equations of motion are

$$\dot{\xi} = [\xi, C_{a,b,D}(\xi)].$$

They are the body representation of the geodesic flow on a Lie group G underlying $\mathfrak{g}$ for the left invariant metric whose quadratic form at the identity is given by h.

These equations can be written as

$$\frac{d}{dt}(\xi + \lambda a) = [\xi + \lambda a, C_{a,b,D}(\xi) + \lambda b].$$

So $\xi \mapsto f_k(\xi + \lambda a)$, $k = 1, \ldots, \ell :=$ rank$(\mathfrak{g}) = \dim \mathfrak{h}$, are conserved on the flow of of this equation, for any element of the basis of the polynomial Casimir functions $f_1, \ldots, f_\ell$ and any parameter λ. Since the f_k are polynomial, it follows that the coefficients of λ^i in the expansion of $f_k(\xi + \lambda a)$ in powers of λ are conserved along the flow of $\dot{\xi} = [\xi, C_{a,b,D}(\xi)]$. There are redundancies: some coefficients of λ^i vanish and other coefficients are Casimir functions.

Mischenko and Fomenko [1978] proved the following result:

The Lie-Poisson system $\dot{\xi} = [\xi, C_{a,b,D}(\xi)]$ on $\mathfrak{g}$ defined by the Hamiltonian $H(\xi) = \langle C_{a,b,D}(\xi), \xi \rangle / 2$ is completely integrable on the maximal dimensional adjoint orbits of the Lie algebra $\mathfrak{g}$ and its commuting generically independent first integrals are the non-trivial coefficients of λ^i in the polynomial λ-expansion of

$$f_{i,\lambda}(\xi) = f_i(\xi + \lambda a)$$

which are not Casimir functions; here $f_1, \ldots, f_\ell$ is the basis of the ring of polynomial invariants of $\mathfrak{g}$. In addition, all functions $f_{i,\lambda}$ commute with H.

LIE-POISSON FORM OF THE EQUATIONS

The equations to be studied are

$$\dot{X} = [X^2, N] = [X, XN + NX],$$

where $X \in \mathrm{Sym}(n)$, the linear space of $n \times n$ symmetric matrices, $N \in \mathfrak{so}(n)$ is given, and $X(0) = X_0 \in \mathrm{Sym}(n)$ are also given.

N can be thought of as a Poisson tensor on $\mathbb{R}^n$:

$$\{f, g\}_N = (\nabla f)^T N \nabla g \quad \text{or} \quad X_h = N \nabla h$$

Each $X \in \mathrm{Sym}(n)$ defines the quadratic Hamiltonian Q_X by

$$Q_X(z) := \frac{1}{2} z^T X z, \quad z \in \mathbb{R}^n.$$

Define $\mathcal{Q} := \{Q_X \mid X \in \mathrm{Sym}(n)\}$.

Then $Q : X \in \mathrm{Sym}(n) \mapsto Q_X \in \mathcal{Q}$ is a linear isomorphism.

The Hamiltonian vector field of Q_X has the form

$$X_{Q_X} = NX$$

and the Poisson bracket of two such quadratic functions is

$$\{Q_X, Q_Y\}_N = Q_{[X,Y]_N}, \quad \forall X, Y \in \mathrm{Sym}(n)$$

where $[X, Y]_N := XNY - YNX \in \mathrm{Sym}(n)$. In addition, $\mathrm{Sym}(n)$ is a Lie algebra relative to the Lie bracket $[\cdot,\cdot]_N$. Therefore, $Q : X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto Q_X \in (\mathcal{Q}, \{\cdot,\cdot\}_N)$ is a Lie algebra isomorphism.

$\mathcal{LH}$ = the Lie algebra of linear Hamiltonian vector fields on $\mathbb{R}^n$ relative to the commutator bracket of matrices

$$X \in (\mathrm{Sym}(n), [\cdot,\cdot]_N) \mapsto NX \in (\mathcal{LH}, [\cdot,\cdot])$$

is a Lie algebra homomorphism and if N is invertible (so n is even) it induces an isomorphism of $(\mathrm{Sym}(n), [\cdot,\cdot]_N)$ with $\mathfrak{sp}(n, N^{-1}) := \{Z \in \mathfrak{gl}(n) \mid Z^T N^{-1} + N^{-1} Z = 0\}$; $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot N^{-1}\mathbf{v}$ symplectic form.

What if N is not invertible?

If $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then $\mathbb{R}^n$ decomposes orthogonally as $\mathbb{R}^n = \operatorname{im} L^T \oplus \ker L$. Taking $L = N$ and using that $N^T = -N$, we get the orthogonal decomposition $\mathbb{R}^n = \operatorname{im} N \oplus \ker N$. Let $2p = \operatorname{rank} N$ and $d := n - 2p$. Then $\bar{N} := N|_{\operatorname{im} N} : \operatorname{im} N \to \operatorname{im} N$ defines a nondegenerate skew symmetric bilinear form and, by the previous proposition, $(\mathrm{Sym}(2p), [\cdot,\cdot]_{\bar{N}})$ is isomorphic as a Lie algebra to $(\mathfrak{sp}(2p, \bar{N}^{-1}), [\cdot,\cdot])$. In this direct sum decomposition of $\mathbb{R}^n$, the skew- symmetric matrix N takes the form

$$N = \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{N} \in \mathfrak{so}(2p) \quad \text{invertible.}$$

The Lie algebra $(\mathrm{Sym}(2p), [\cdot,\cdot]_{\bar{N}})$ acts on the vector space $\mathcal{M}_{(2p)\times d}$ of $(2p)\times d$ matrices (which we can think of as linear maps of $\ker N$ to $\operatorname{im} N$) by $S\cdot A := S\bar{N}A$, where $S \in (\mathrm{Sym}(2p), [\cdot,\cdot]_{\bar{N}})$ and $A \in \mathcal{M}_{(2p)\times d}$. Form the semidirect product $\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}$:

$$
\begin{aligned}
[(S,A),(S',A')] &= ([S,S']_{\bar{N}}, S\cdot A' - S'\cdot A) \\
&= (S\bar{N}S' - S'\bar{N}S, S\bar{N}A' - S'\bar{N}A)
\end{aligned}
$$

for any $S, S' \in \mathrm{Sym}(2p)$ and $A, A' \in \mathcal{M}_{(2p)\times d}$.

Next, define the $\mathrm{Sym}(d)$-valued Lie algebra two-cocycle

$$
C : \mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d} \times \mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d} \to \mathrm{Sym}(d)
$$

by

$$
C((S,A),(S',A')) := A^T\bar{N}A' - (A')^T\bar{N}A.
$$

Now extend $\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}$ by this cocycle. That is, form the vector space $(\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}) \oplus \mathrm{Sym}(d)$ and endow it with the bracket

$$[(S, A, B), (S', A', B')]^C := \Big(S\bar{N}S' - S'\bar{N}S, S\bar{N}A' - S'\bar{N}A, \\ A^T\bar{N}A' - (A')^T\bar{N}A\Big)$$

for any $S, S' \in \mathrm{Sym}(2p)$, $A, A' \in \mathcal{M}_{(2p)\times d}$, and $B, B' \in \mathrm{Sym}(d)$.

The map

$$\Psi : ((\mathrm{Sym}(2p) \circledS \mathcal{M}_{(2p)\times d}) \oplus \mathrm{Sym}(d), [\cdot,\cdot]^C) \to (\mathrm{Sym}(n, N), [\cdot,\cdot]_N)$$

given by

$$\Psi(S, A, B) := \begin{bmatrix} S & A \\ A^T & B \end{bmatrix}$$

is a Lie-algebra isomorphism.

Hamiltonian. Positive definite inner product on Sym(n)

$$\langle\!\langle X, Y \rangle\!\rangle := \mathrm{trace}\,(XY)\,, \quad \text{for} \quad X, Y \in \mathrm{Sym}(n)$$

identifies Sym(n) with its dual. $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is not ad-invariant relative to the N-bracket but the indefinite symmetric bilinear form

$$\kappa_N(X, Y) := \mathrm{trace}(NXNY)$$

is invariant. If N is non-degenerate so is κ_N.

Define the Hamiltonian $h : (\mathrm{Sym}(n), [\cdot, \cdot]_N) \to \mathbb{R}$ by

$$h(X) = \frac{1}{2}\mathrm{trace}\left(X^2\right) = \frac{1}{2}\mathrm{trace}\left(XX^T\right) =: \frac{1}{2}\langle\!\langle X, X \rangle\!\rangle\,.$$

So the associated Lie-Poisson equations are the body representation of the geodesic flow of the Lie group underlying $(\mathrm{Sym}(n), [\cdot, \cdot]_N)$ relative to the left invariant Riemannian metric which at the identity has the quadratic form h.

Lie-Poisson bracket of $f, g \in C^\infty(\mathrm{Sym}(n))$

$$\{f, g\}_N(X) = -\operatorname{trace}\left[X\Big(\nabla f(X)N\nabla g(X) - \nabla g(X)N\nabla f(X)\Big)\right],$$

where ∇f is the gradient of f relative to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

The equations $\dot{X} = [X^2, N] = [X, XN + NX]$ are the Lie-Poisson equations on $(\mathrm{Sym}(n), [\cdot, \cdot]_N)$ for the Hamiltonian h.

Proof Left Lie-Poisson equations on the dual of a Lie algebra $\mathfrak{g}$ associated with a Hamiltonian $h : \mathfrak{g}^* \to \mathbb{R}$ are

$$\frac{d}{dt}\mu(\eta) = \mu\left(\left[\frac{\delta h}{\delta \mu}, \eta\right]\right), \qquad \forall \eta \in \mathfrak{g},$$

where $\mu \in \mathfrak{g}^*$. In our case, $\mathfrak{g}^* \cong \mathfrak{g}$ via $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, $\mu = \langle\!\langle X, \cdot \rangle\!\rangle$, $\delta h / \delta \mu = X$, so this becomes for any $\eta = Y \in \mathrm{Sym}(n)$

$$\begin{aligned}
\operatorname{trace}\left(\dot{X}Y\right) &= \frac{d}{dt}\langle\!\langle X, Y \rangle\!\rangle = \langle\!\langle X, [X, Y]_N \rangle\!\rangle \\
&= \langle\!\langle X, XNY - YNX \rangle\!\rangle = \operatorname{trace}(X(XNY - YNX)) \\
&= \operatorname{trace}\Big((X^2N - NX^2)Y\Big). \qquad \blacksquare
\end{aligned}$$

Frozen Poisson bracket

$$\{f,g\}_{FN}(X) = -\operatorname{trace}\Big(\nabla f(X) N \nabla g(X) - \nabla g(X) N \nabla f(X)\Big).$$

The Lie-Poisson and frozen Poisson brackets on the dual of any Lie algebra are compatible, which means that any linear combination of these brackets is again a Poisson bracket.

$B, C : T^*(\mathrm{Sym}(n)) \to T(\mathrm{Sym}(n))$ Poisson tensors of the Lie-Poisson and frozen brackets, that is, $B(h) = \{\cdot, h\}_N$ and $C(h) = \{\cdot, h\}_{FN}$ for any locally defined smooth function h.

Their value at $X \in \mathrm{Sym}(n)$ are the linear maps $B_X, C_X : \mathrm{Sym}(n) \to \mathrm{Sym}(n)$ given by

$$B_X(Y) = XYN - NYX \qquad \text{and} \qquad C_X(Y) = YN - NY.$$

Let $n = 2p + k$, where $2p := \operatorname{rank} N$. The generic leaves of the Lie-Poisson bracket $\{\cdot, \cdot\}_N$ are $2p(p+k)$-dimensional.

All the leaves of the frozen Poisson bracket $\{\cdot,\cdot\}_{FN}$ are

- $2p(p+k)$-dimensional if all non-zero eigenvalues of N are distinct

- $p(p+1+2k)$-dimensional if all non-zero eigenvalue pairs of N are equal.

Choose an orthonormal basis of $\mathbb{R}^{2p+k}$ in which N is written as

$$N = \begin{bmatrix} 0 & V & 0 \\ -V & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where V is a real diagonal matrix whose entries are $v_1, \ldots, v_p$.

Notation: S_{kl} is the $p \times p$ (or $k \times k$) symmetric matrix having all entries equal to zero except for the (k,l) and (l,k) entries that are equal to one and A_{kl} is the $p \times p$ skew symmteric matrix with all entries equal to zero except for the (k,l) entry which is 1 and the (l,k) entry which is -1.

The Casimir functions of $\{\cdot,\cdot\}_{FN}$:

(i) If $v_i \neq v_j$ for all $i \neq j$, the $p + k(k+1)/2$ Casimir functions are

$$C_F^i(X) = \mathrm{trace}(E_i X), \quad i = 1, \ldots, p + \frac{1}{2}k(k+1),$$

where E_i is any of the matrices

$$\begin{bmatrix} S_{kk} & 0 & 0 \\ 0 & S_{kk} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{ab} \end{bmatrix}.$$

(ii) If $v_i = v_j$ for all $i, j = 1, \ldots, p$, the $p^2 + k(k+1)/2$ Casimir functions are

$$C_F^i(X) = \mathrm{trace}(E_i X), \quad i = 1, \ldots, p^2 + \frac{1}{2}k(k+1),$$

where E_i is any of the matrices

$$\begin{bmatrix} S_{kl} & 0 & 0 \\ 0 & S_{kl} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & A_{kl} & 0 \\ -A_{kl} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{ab} \end{bmatrix}.$$

The Casimir functions of $\{\cdot,\cdot\}_N$:

Denote

$$\bar{N} = \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix} \quad \text{and} \quad \hat{N} = \begin{bmatrix} \bar{N}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

The $p + k(k+1)/2$ Casimir functions for the Lie-Poisson bracket $\{\cdot,\cdot\}_N$ are given by

$$C^k(X) = \frac{1}{2k} \operatorname{trace}\left[\left(X\hat{N}\right)^{2k}\right], \quad \text{for} \quad k = 1, \ldots, p$$

and

$$C^k(X) = \operatorname{trace}(XE_k), \quad \text{for} \quad k = p+1, \ldots, p + \frac{1}{2}k(k+1)\,,$$

where E_k is any matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{ab} \end{bmatrix}.$$

In the special case when N is full rank the Casimirs are just

$$C^k(X) = \frac{1}{2k} \operatorname{trace}\left[\left(XN^{-1}\right)^{2k}\right], \quad \text{for} \quad k = 1, \ldots, p.$$

RELATION TO THE MISCHENKO-FOMENKO RIGID BODIES

There is a non-trivial relation with the MF systems.

1. The system is not on the MF list

The system $\dot{X} = [X^2, N]$ is not in the Mischenko-Fomenko family of integrable systems, even if N is invertible and already for the case of 2×2 matrices. This is a legitimate question since, if N is invertible, Sym(n, N) with the N-bracket is isomorphic to $\mathfrak{sp}(n, N^{-1})$ and there one has the MF systems. Let

$$N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and denote elements of Sym$(2, N)$ by

$$X = \begin{bmatrix} a & b \\ b & d \end{bmatrix}, \quad a, b, d \in \mathbb{R}.$$

A Cartan subalgebra of $(\mathrm{Sym}(2,N),[\cdot,\cdot]_N)$:

$$A = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}, \qquad \alpha \in \mathbb{R}.$$

Orthogonal complement with respect to the nondegenerate invariant bilinear form κ_N on $(\mathrm{Sym}(2,N),[\cdot,\cdot]_N)$ is $\mathrm{Sym}_d(2)$, the space of diagonal 2×2 matrices.

Now we construct the general sectional operator on $\mathrm{Sym}(2,N)$. We begin with its off-Cartan part. Notice that for any $X \in \mathrm{Sym}(2,N)$ we have

$$[A,X]_N = \begin{bmatrix} -2\alpha a & 0 \\ 0 & 2\alpha d \end{bmatrix}$$

and hence, also in accordance with general theory, if $\alpha \neq 0$, then $\mathrm{ad}_A : \mathrm{Sym}_d(2) \to \mathrm{Sym}_d(2)$ is an isomorphism. Thus the inverse $\mathrm{ad}_A^{-1} : \mathrm{Sym}_d(2) \to \mathrm{Sym}_d(2)$ is defined and hence

$$\mathrm{ad}_A^{-1}(\mathrm{ad}_B X) = \frac{\beta}{\alpha}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \quad \text{for} \quad A = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}, \quad \alpha \neq 0.$$

The Cartan part of the sectional operator is an arbitrary invertible symmetric operator D on the Cartan subalgebra, so it is of the form

$$D(A) = \mu A,$$

for any off diagonal A and $\mu \in \mathbb{R}$, $\mu \neq 0$. So if A is the Cartan part of X, we have

$$[X, D(A)] = \mu \begin{bmatrix} 2ab & 0 \\ 0 & -2bd \end{bmatrix}.$$

Hence the general Mischenko-Fomenko system on $\mathrm{Sym}(2, N)$ has the form

$$\dot{X} = \left[X, \mathrm{ad}_A^{-1}(\mathrm{ad}_B X) + D(X)\right]_N = \left(\frac{\beta}{\alpha} - \mu\right) \begin{bmatrix} -2ab & 0 \\ 0 & 2bd \end{bmatrix}.$$

We shall now prove that $\dot{X} = [X^2, N]$ is not in this family. Indeed,

$$XN + NX = \begin{bmatrix} 0 & a+d \\ -a-d & 0 \end{bmatrix} = (a+d)N$$

the equation becomes

$$\dot{X} = [X, XN + NX] = (a + d) \begin{bmatrix} -2b & a - d \\ a - d & 2b \end{bmatrix}.$$

The only way equations this and the MF system can be identical is if one requires that $a = d$, which is not allowed since X is arbitrary in Sym$(2, N)$.

2. The system is related to MF via a Poisson isomorphism

Firstly, if N is invertible, then there is a Poisson isomorphism from $\mathfrak{sp}(n, N^{-1})$ (identified with its dual) endowed with its Lie-Poisson bracket, to Sym(n, N), likewise with its Lie-Poisson bracket. Secondly, the inverse of this map takes our system to a Mischenko-Fomenko system.

Of course Poisson diffeomorphisms take integrable Hamiltonian systems to integrable Hamiltonian systems, so what the preceding example shows is that linear Poisson diffeomorphisms need not preserve the Mischenko-Fomenko structure.

A Poisson isomorphism for N invertible.

Compare the LP brackets on $\mathrm{Sym}(n, N)$ and on $\mathfrak{sp}(n, N^{-1})^*$.

To get the LP bracket on $\mathfrak{sp}(n, N^{-1})^*$ we identify $\mathfrak{sp}(n, N^{-1})^*$ with $\mathfrak{sp}(n, N^{-1})$ via the invariant nondegenerate symmetric bilinear form

$$\langle\!\langle Z_1, Z_2 \rangle\!\rangle := \mathrm{tr}\,(Z_1 Z_2)\,.$$

So, the LP bracket on $\mathfrak{sp}(n, N^{-1})^* \cong \mathfrak{sp}(n, N^{-1})$ is given by

$$\{\phi, \psi\}_{\mathfrak{sp}}(Z) := -\,\langle\!\langle Z, [\nabla\phi(Z), \nabla\psi(Z)] \rangle\!\rangle\,,$$

where ∇ is taken relative to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\phi, \psi : \mathfrak{sp}(n, N^{-1}) \to \mathbb{R}$ are smooth functions.

The map $Z \in \left(\mathfrak{sp}(n, N^{-1}), \{\cdot, \cdot\}_{\mathfrak{sp}}\right) \mapsto ZN \in (\mathrm{Sym}(n, N), \{\cdot, \cdot\}_N)$ is an isomorphism of Lie-Poisson spaces.

The Mischenko-Fomenko System on $\left(\mathfrak{sp}(n, N^{-1}), \{\cdot, \cdot\}_{\mathfrak{sp}}\right)$.

We now show that if N has distinct eigenvalues, then Φ^* maps the system $\dot{X} = [X^2, N]$ to a MF system on $\left(\mathfrak{sp}(n, N^{-1}), \{\cdot, \cdot\}_{\mathfrak{sp}}\right)$. First, denoting $X := \Phi^*(Z) = ZN$, we get

$$\begin{aligned} \dot{Z} = \dot{X}N^{-1} &= [X^2, N]N^{-1} = X^2 - NX^2N^{-1} \\ &= ZNZN - NZNZNN^{-1} = [Z, NZN]. \end{aligned}$$

Is the invertible operator $C : \mathfrak{sp}(n, N^{-1}) \to \mathfrak{sp}(n, N^{-1})$ defined by $C(Z) = NZN$ is a sectional operator?

Yes: $C = C_{a,b,D}$, for $a = N^{-1}$, $b = -N$, and $D(Y) = NYN$, for Y in the Cartan algebra.

So the Mischenko-Fomenko Theorem applies:

Let N be invertible with distinct eigenvalues. The system

$$\dot{Z} = [Z, NZN]$$

is integrable on the maximal dimensional orbits of $\mathfrak{sp}(n, N^{-1})$ and its generically independent integrals in involution are the non-trivial coefficients of λ^i in the polynomial expansion of $\frac{1}{k}\mathrm{tr}(Z + \lambda N^{-1})^k$ that are not Casimir functions, $k = 2, \ldots, n$. The Hamiltonian is $H(Z) := \mathrm{trace}((ZN)^2)/2$.

Push forward Z by the map Φ^* to get:

Let N be invertible with distinct eigenvalues. The equation $\dot{X} = [X^2, N]$ is an integrable Hamiltonian system on the maximal dimensional symplectic leaf of $\mathrm{Sym}(n, N)$. The independent integrals in involution are the non-trivial coefficients of λ^i in the polynomial expansion of $\frac{1}{k}\mathrm{tr}(XN^{-1} + \lambda N^{-1})^k$ that are not Casimir functions, $k = 2, \ldots, n$.

While this statement proves integrability it does not really explain the role of the other Poisson structures tied to the system $\dot{X} = [X^2, N]$.

There is a direct proof of integrability on Sym(n, N) for N with distinct eigenvalues but not necessarily invertible, that is, N has at most one zero eigenvalue. In the invertible case, the method presented below provides a *different sequence of integrals* and, in addition, derives a second Hamiltonian structure for the Mischenko-Fomenko system on $\mathfrak{sp}(n, N^{-1})$. As far as I know, among all the MF systems, only the higher dimensional rigid bodies on $\mathfrak{so}(n)$ are known to have a second Hamiltonian structure (Morosi and Pizzocchero [1996]) although it is believed that all of them have such a second structure. The MF proof of independence of the integrals is very suggestive, but it is not known how their recursion relation really come from a second Hamiltonian structure. The method also leads directly to the linearization of the flow by standard methods.

LAX PAIRS WITH PARAMETER

The system $\dot{X} = [X^2, N]$ is equivalent to the following Lax pair system

$$\frac{d}{dt}(X + \lambda N) = \left[X + \lambda N, NX + XN + \lambda N^2\right]$$

Now start counting very carefully the non-trivial coefficients of λ^i:

$$\text{trace} \sum_{|i|=k-2r} \sum_{|j|=2r} X^{i_1} N^{j_1} X^{i_2} \dots X^{i_s} N^{j_s}$$

for $i_q, j_q = 0, \dots, k-1$, $r = 1, \dots, \left[\frac{k-1}{2}\right]$ are all the invariants. So the total number of invariants is

$$\left[\frac{n}{2}\right] \left[\frac{n+1}{2}\right]$$

- If N is invertible, then $n = 2p$ and hence

$$\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] = \left[\frac{2p}{2}\right]\left[\frac{2p+1}{2}\right] = p^2 = \frac{1}{2}\Big(2p^2 + p - p\Big)$$
$$= \frac{1}{2}\left(\dim \mathfrak{sp}(2p,\mathbb{R}) - \operatorname{rank} \mathfrak{sp}(2p,\mathbb{R})\right)$$

which is half the dimension of the generic adjoint orbit in $\mathfrak{sp}(2p,\mathbb{R})$.

Therefore, we have the right number of conserved quantities. These functions are the right candidates to prove that the system is integrable on the generic coadjoint orbit of Sym(n).

• If N is non-invertible (which is equivalent to $k \neq 0$), then $n = 2p + k$ and then

$$\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] = p^2 + pk + \left[\frac{k}{2}\right]\left[\frac{k+1}{2}\right].$$

The right number of integrals is $p(p+k)$, so this seems to indicate that there are additional integrals. The situation is not so simple since there are redundancies due to the degeneracy of N. Note, however, that if $d = 1$, then we do get the right number of integrals.

Remark. Recall that in the special case when N is invertible, we already found a sequence of integrals. Note that these are *different*! This is an indication that the system may be super-integrable which will be shown later.

INTEGRABILITY

BIHAMILTONIAN STRUCTURE

The system $\dot{X} = X^2N - NX^2$ is Hamiltonian with respect to the Lie-Poisson bracket $\{\cdot,\cdot\}_N$ for the Hamiltonian $h_2(X) := \frac{1}{2}\text{trace}(X^2)$ and is also Hamiltonian with respect to the compatible frozen bracket $\{\cdot,\cdot\}_{FN}$ for the Hamiltonian $h_3(X) := \frac{1}{3}\text{trace}(X^3)$.

INVOLUTION

Proof is "standard". Show that the $\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$ integrals

$$h_{k,2r}(X) := \text{trace} \sum_{|i|=k-2r} \sum_{|j|=2r} X^{i_1}N^{j_1}X^{i_2}\cdots X^{i_s}N^{j_s},$$

where $k = 1, \ldots, n-1$, $i_q = 1, \ldots, k$, $j_q = 0, \ldots, k-1$, $r = 0, \ldots, \left[\frac{k-1}{2}\right]$, are in involution.

It is convenient to expand as

$$h_k^\lambda(X) := \frac{1}{k}\,\mathrm{trace}\,(X + \lambda N)^k = \sum_{r=0}^{k} \lambda^{k-r} h_{k,k-r}(X)\,.$$

As explained before, not all of these coefficients should be counted: roughly half of them vanish and the last one, namely, $h_{k,k}$, is the constant N^k. Consistently with our notation for the Hamiltonians, we set $h_k = h_{k,0}$.

One shows that:

$$\nabla h_k^\lambda(X) = \frac{1}{2}(X + \lambda N)^{k-1} + \frac{1}{2}(X - \lambda N)^{k-1}.$$

$$B_X(\nabla h_k^\lambda(X)) = C_X(\nabla h_{k+1}^\lambda(X))$$

which implies the recursion relation

$$B_X(\nabla h_{k,k-r}(X)) = C_X(\nabla h_{k+1,k-r}(X)).$$

This has several consequences:

- The functions $h_{k,k-1}(X)$ are Casimirs for $\{\cdot,\cdot\}_{FN}$.

- If $r=0$ the flows are related by

$$B_X\left(\nabla h_k(X)\right) = C_X\left(\nabla h_{k+1}(X)\right).$$

- $h_{k,k-r}$ are in involution with respect to $\{f,g\}_N$ and $\{f,g\}_{FN}$.

- If N is invertible, the Lie-Poisson isomorphism

$$Z \in \left(\mathfrak{sp}(n,N^{-1}),\{\cdot,\cdot\}_{\mathfrak{sp}}\right) \mapsto ZN \in (\mathrm{Sym}(n,N),\{\cdot,\cdot\}_N)$$

induces the second Poisson bracket

$$\{f,g\}_{N^{-1}}(Z) = -\operatorname{trace}\left(N^{-1}[\nabla f(Z),\nabla g(Z)]\right),\ \forall f,g \in C^\infty(\mathfrak{sp}(n,N^{-1}))$$

for the MF system on $\mathfrak{sp}(n,N^{-1})$. The Hamiltonian corresponding to this Poisson structure is $h(Z) = \operatorname{trace}\left((ZN)^3\right)/3$.

INDEPENDENCE

This is done directly "by hand" through an inductive argument and by recursion. The result is:

If N has distinct eigenvalues and is either invertible or has nullity one then the integrals $h_{k,2r}$, $k = 1, \ldots, n-1$, $r = 0, \ldots, [(k-1)/2]$ are independent. Therefore, in these two cases, the system $\dot{X} = [X^2, N]$ is completely integrable.

Remark. Independently Li and Tomei [2006] have shown the integrability of the same system in precisely these two cases employing different techniques; they use the loop group approach suggested by the Lax equation with parameter and give the solution in terms of factorization and the Riemann-Hilbert problem.

LINEARIZATION OF THE FLOWS

Algebraic isospectral curve $Q(\lambda, z) := \det(zI - \lambda N - X) = 0$.

N INVERTIBLE AND GENERIC

Denote $X(\lambda) := X + \lambda N$ and $Y(\lambda) := NX + XN + \lambda N^2$. For N invertible with distinct eigenvalues ($n := 2p$), choose an orthonormal basis of $\mathbb{R}^{2p}$ in which N is written as

$$N = \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix},$$

where V is a real diagonal matrix whose entries are $v_1, \ldots, v_p$.

The *spectral curve* associate to each $X(\lambda)$,

$$\Gamma_{X(\lambda)} := \{(\lambda, z) \in \mathbb{C} \times \mathbb{C} \mid \det(zI - X(\lambda)) = 0\},$$

is preserved by the flow. The functions given by the coefficients of $Q(\lambda, z)$ are constants of the motion.

Similarly, for each $X(\lambda)$ the *isospectral variety of matrices*

$A_{X(\lambda)} := \{X'(\lambda) \mid X(\lambda), X'(\lambda)$ have the same charac. polynomial$\}$ is preserved by the flow.

$\Gamma_{X(\lambda)}$ and $A_{X(\lambda)}$ depend only on the values of the constants of motion, i.e. on the vector $\mathbf{c} = (q_{kl})$, where q_{kl} is the coefficient of $\lambda^k z^l$ in $Q(\lambda, z)$. So write instead $\Gamma_{\mathbf{c}}$ and $A_{\mathbf{c}}$. $\Gamma_{\mathbf{c}}$ is non-singular for generic $\mathbf{c}$. Let $\overline{\Gamma}_{\mathbf{c}}$ be the compactification in the projective plane $\mathbb{P}^2_{\mathbb{C}}$ of $\Gamma_{\mathbf{c}}$. For generic $\mathbf{c}$ the projective curve $\overline{\Gamma}_{\mathbf{c}}$ is also non-singular. Genus of $\overline{\Gamma}_{\mathbf{c}}$ is $g := (p-1)(2p-1)$.

The points at infinity of the spectral curve are

$$\{P_1, ..., P_{2p}\} := \overline{\Gamma}_{\mathbf{c}} \setminus \Gamma_{\mathbf{c}},$$

with $P_{k+1} = (1, \beta_{k+1}, 0)$, $k = 0, 1, ..., 2p-1$, where

$$\beta_{k+1} := v^{1/p} \exp\left(i\frac{(2k+1)\pi}{2p}\right) \qquad \text{and} \qquad v := |v_1 v_2 ... v_p|.$$

At each P_{k+1} the meromorphic functions λ and z on $\overline{\Gamma}_{\mathbf{c}}$ have a pole of order 1.

Take now a generic value of $\mathbf{c}$ such that $\Gamma_{\mathbf{c}}$ is non-singular and note that for generic $(\lambda, z) \in \Gamma_{\mathbf{c}}$, the eigenspace of $X(\lambda)$ with eigenvalue z is one-dimensional. Let $\Delta_{kl}(z, X(\lambda))$ be the cofactor of the matrix $zI_{2p} - X(\lambda)$ corresponding to the (k,l)-th entry. Then the unique eigenvector of $X(\lambda)$ with eigenvalue z, normalized by $\xi_1 = 1$, is $\xi(z, X(\lambda)) := (\xi_1, ..., \xi_{2p})^T$, where

$$\xi_k = \Delta_{1k}(z, X(\lambda))/\Delta_{11}(z, X(\lambda)).$$

Adler, van Moerbeke, Vanhaecke [2004], p.187: When $X(\lambda, t)$ flows

$$\frac{d}{dt}(X + \lambda N) = \left[X + \lambda N, NX + XN + \lambda N^2\right],$$

the corresponding eigenvector $\xi(t) := \xi(z, X(\lambda, t))$ satisfies the autonomous equation

$$\dot{\xi} + Y\xi = \rho\xi,$$

where $Y := Y(\lambda, X(\lambda, t))$ and ρ is the scalar function

$$\rho := \rho(\lambda, z, X(\lambda, t)) = \sum_{l=1}^{2p} Y(\lambda, X(\lambda, t))_{1l} \Delta_{1l}(z, X(\lambda, t))/\Delta_{11}(z, X(\lambda, t)).$$

The role of the eigenvector ξ is to define the *divisor map*

$$i_{\mathbf{c}} : A_{\mathbf{c}} \to \mathrm{Div}^d(\overline{\Gamma}_{\mathbf{c}}), \quad X(\lambda) \mapsto \mathcal{D}_{X(\lambda)},$$

where $\mathcal{D}_{X(\lambda)}$ is the minimal effective divisor on $\Gamma_{\mathbf{c}}$ such that

$$(\xi_k)_{\Gamma_{\mathbf{c}}} \geq -\mathcal{D}_{X(\lambda)},\ k = 1, ..., 2p.$$

Here, $d := \deg(\mathcal{D}_{X(\lambda)})$ is independent of $X(\lambda) \in A_{\mathbf{c}}$ (for generic $\mathbf{c}$ we can assume $A_{\mathbf{c}}$ connected) and so, $\mathcal{D}_{X(\lambda)}$ defines an effective divisor of degree d in $\overline{\Gamma}_{\mathbf{c}}$.

Now choose and fix a divisor $\mathcal{D}_0 \in \mathrm{Div}^d(\overline{\Gamma}_{\mathbf{c}})$, a basis $(\omega_1, ..., \omega_g)$ of holomorphic differentials on $\overline{\Gamma}_{\mathbf{c}}$, and consider the vector $\overline{\omega} := (\omega_1, ..., \omega_g)^T$. One defines the *linearizing map* by

$$j_{\mathbf{c}} : A_{\mathbf{c}} \to \mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}}), \quad X \mapsto \int_{\mathcal{D}_0}^{\mathcal{D}_X} \overline{\omega},$$

where $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$ denotes the Jacobian of the curve $\overline{\Gamma}_{\mathbf{c}}$.

The role of the function ρ is to *linearize* the isospectral flow on $A_{\mathbf{c}}$, that is, to be able to write

$$\int_{\mathcal{D}_{X(0)}}^{\mathcal{D}_{X(t)}} \overline{\omega} = t \sum_{k=1}^{2p} \mathrm{Res}_{P_k}(\rho(\lambda, z, X(\lambda, 0))\overline{\omega}), \quad \mathcal{D}_{X(0)} = \mathcal{D}_0,$$

if it is possible.

The *Linearization Criterion* in Adler, van Moerbeke, Vanhaecke [2004], p.195 (Griffiths [1985]) says that this happens if and only if for each $X \in A_{\mathbf{c}}$ there exists a meromorphic function Φ_X on $\overline{\Gamma}_{\mathbf{c}}$ with

$$(\Phi_X)_{\overline{\Gamma}_{\mathbf{c}}} \geq -\sum_{k=1}^{2p} P_k,$$

such that for all P_k, and

(Laurent tail of $d\rho(\lambda, z, X)/dt$ at P_k) = (Laurent tail of Φ_X at P_k).

Apply this criterion and compute for a while. It applies with $\Phi_X = 0$.

For N invertible with distinct eigenvalues the map $j_{\mathbf{c}}$ linearizes the isospectral flow of $\dot{X} = [X^2, N]$ on the Jacobian $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$.

One can do better. $(X + \lambda N)^T = X - \lambda N \Longrightarrow Q(-\lambda, z) = Q(\lambda, z)$. Thus $\tau : (\lambda, z) \in \overline{\Gamma}_{\mathbf{c}} \mapsto (-\lambda, z) \in \overline{\Gamma}_{\mathbf{c}}$ is an involution. In homogeneous coordinates $\lambda = \nu/z_0$, $z = \zeta/z_0$ it is

$$\tau(\nu, \zeta, z_0) = (-\nu, \zeta, z_0).$$

τ has no fixed points at infinity ($z_0 = 0$ and $\nu = 0$ would imply $\zeta = 0$ from the homogeneous equation of the curve). Thus, the fixed points are obtained from the equation

$$Q(0, z) = 0,$$

which is the characteristic polynomial of the symmetric matrix X. Generically, we obtain $2p$ distinct points $Z_1, ..., Z_{2p}$ as its fixed (ramification) points, where $Z_k = (0, z_k, 1)$, $k = 1, ..., 2p$, with z_k the (real) eigenvalues of the symmetric matrix X. By the Riemann-Hurwitz formula, the quotient (smooth) curve $C_1 := \overline{\Gamma}_{\mathbf{c}}/\tau$ has genus $g_1 := (p-1)^2$.

Associated to the double covering $\overline{\Gamma}_{\mathbf{c}} \to C_1$ is the Prym variety Prym$(\overline{\Gamma}_{\mathbf{c}}/C_1)$, with the property that

$$\text{Jac}(\overline{\Gamma}_{\mathbf{c}}) \quad \text{is isogenous to} \quad \text{Jac}(C_1) \times \text{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1).$$

It follows that $\dim\left(\text{Prym}(\overline{\Gamma}_{\mathbf{c}}/C_1)\right) = g - g_1 = p^2 - p$.

$\Omega_{\overline{\Gamma}_{\mathbf{c}}}$ the sheaf of holomorphic 1-forms on $\overline{\Gamma}_{\mathbf{c}}$. Recall that

$$\text{Jac}(\overline{\Gamma}_{\mathbf{c}}) \cong H^0(\overline{\Gamma}_{\mathbf{c}}, \Omega_{\overline{\Gamma}_{\mathbf{c}}})^*/H_1(\overline{\Gamma}_{\mathbf{c}}, \mathbb{Z}).$$

The involution τ acts on the vector space $H^0(\overline{\Gamma}_{\mathbf{c}}, \Omega_{\overline{\Gamma}_{\mathbf{c}}})$ and on the free group $H_1(\overline{\Gamma}_{\mathbf{c}}, \mathbb{Z})$ having eigenvalues ± 1. The Prym variety Prym$(\overline{\Gamma}_{\mathbf{c}}/C_1)$ can be equivalently described as the quotient

$$H^0(\overline{\Gamma}_{\mathbf{c}}, \Omega_{\overline{\Gamma}_{\mathbf{c}}})^{-*}/H_1(\overline{\Gamma}_{\mathbf{c}}, \mathbb{Z})^-,$$

where the upper $\pm$ index on a vector space denotes the ± 1 eigenspaces. By Griffiths [1985], or by direct computation, we have

$$\tau(\text{Res}_{P_k}(\rho(\lambda, z, X(\lambda, 0)))) = -\text{Res}_{P_k}(\rho(\lambda, z, X(\lambda, 0))).$$

It follows that the flow is actually linearized on Prym$(\overline{\Gamma}_{\mathbf{c}}/C_1)$.

For N invertible with distinct eigenvalues the map $j_{\mathbf{c}}$ linearizes the isospectral flow of the system $\dot{X} = [X^2, N]$ on the Prym variety Prym$(\overline{\Gamma}_{\mathbf{c}}/C_1)$.

Remark. Note that the real part of Prym$(\overline{\Gamma}_{\mathbf{c}}/C_1)$, on which the flow linearizes, is a real torus of dimension less than half of that of the phase space (which is $2p^2$ for a generic coadjoint orbit). Therefore, the system $\dot{X} = [X^2, N]$ is super-integrable since it has invariant tori of dimension strictly less than what the Liouville-Arnold theorem predicts.

To our knowledge, this is the first system that is super-integrable and for which this is not due to the existence of a momentum map for a symplectic nonabelian Lie group action (as is the case, for example, in the Kepler problem, with the Laplace-Runge-Lenz vector, or in the free rigid body motion, with the spatial momentum, as conserved vector quantities).

N MAXIMAL RANK AND NULLITY ONE

Apply again the linearization criterion.

For $N \in \mathfrak{so}(3)$ or $\mathfrak{so}(5)$ having distinct eigenvalues and nullity one, generically the map $j_{\mathbf{c}}$ does not linearize the isospectral flow of the system $\dot{X} = [X^2, N]$ on the Jacobian $\mathrm{Jac}(\overline{\Gamma}_{\mathbf{c}})$.

This is an example of an integrable system all of whose integrals are polynomials but whose flow does not linearize on the Jacobian of the spectral curve. Such examples were known.

Mumford [1984]: The Hamiltonian system on the standard symplectic vector space $(\mathbb{R}^2, \mathbf{d}x \wedge \mathbf{d}y)$ given by $H(x, y) := x^4 + y^4$ is completely integrable but not algebraically completely integrable.

Vanhaecke [1998] has constructed a large class of examples of completely integrable Hamiltonian systems admitting a Lax pair with parameter, having only polynomial integrals, which do not linearize on the Jacobian of the spectral curve. Many of these examples are Lie-Poisson with or without a cocycle.