# AVERAGING AND RECONSTRUCTION IN HAMILTONIAN SYSTEMS

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# Goal and Method

**Goal:**

Investigating the existence, stability and bifurcation of periodic solutions and invariant tori to a Hamiltonian vector field which is a small perturbation of a Hamiltonian vector field **whose orbits are all periodic.**

**Method:**

By averaging the perturbation over the fibers of the circle bundle one obtains a Hamiltonian system on the reduced (orbit) space of the circle bundle.

We state and prove results which have hypotheses on the reduced system and **have conclusions about the full system.**

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# The Original Reduction Theorem

**Theorem 1** (Reeb, 1952)

- $(M, \Omega)$ symplectic manifold of dimension $2n$;
- $\mathcal{H}_0 : M \to \mathbb{R}$ a smooth Hamiltonian, which defines a Hamiltonian vector field $Y_0 = (d\mathcal{H}_0)^{\#}$ with symplectic flow $\phi_0^t$;
- $\mathbb{I} \subset \mathbb{R}$ an interval such that each $h \in \mathbb{I}$ is a regular value of $\mathcal{H}_0$;
- $\mathcal{N}_0(h) = \mathcal{H}_0^{-1}(h)$ is a compact connected circle bundle over a base space $B(h)$ with projection $\pi : \mathcal{N}_0(h) \to B(h)$;
- the vector field $Y_0$ is everywhere tangent to the fibers of $\mathcal{N}_0(h)$, i.e. all the solutions of $Y_0$ in $\mathcal{N}_0(h)$ are periodic.

The base space $B$ inherits a symplectic structure $\omega$ from $(M, \Omega)$, i.e. $(B, \omega)$ is a symplectic manifold.

# Perturbation Theorem

**Theorem 2** (Reeb, 1952)

- $\varepsilon$ a small parameter and $\mathcal{H}_1 : M \to \mathbb{R}$ is smooth;
- $\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$; $Y_\varepsilon = Y_0 + \varepsilon Y_1 = d\mathcal{H}_\varepsilon^\#$;
- $\mathcal{N}_\varepsilon(h) = \mathcal{H}_\varepsilon^{-1}(h)$;
- $\phi_\varepsilon^t$ the flow defined by $Y_\varepsilon$;
- the average of $\mathcal{H}_1$ is a smooth function on $B(h)$

$$\bar{\mathcal{H}} = \frac{1}{T}\int_0^T \mathcal{H}_1(\phi_0^t)dt,$$

# Perturbation Theorem #2

- If $\bar{\mathcal{H}}$ has a nondegenerate critical point at $\pi(p) = \bar{p} \in B$ with $p \in \mathcal{N}_0$, then there are smooth functions $p(\varepsilon)$ and $T(\varepsilon)$ for $\varepsilon$ small and **the solution of $Y_\varepsilon$ through $p(\varepsilon)$ is $T(\varepsilon)$-periodic.**
- If $\bar{\mathcal{H}}$ is a Morse function then $Y_\varepsilon$ **has at least $\chi(B)$ periodic solutions**, where $\chi(B)$ is the Euler-Poincaré characteristic of $B$.

# Sketch of the Proofs

**Idea**: Construct symplectic coordinates $(I, \theta, y)$ valid in a tubular neighbourhood of the periodic solution $\phi_0^t(p)$ of $Y_0(h)$.

- $(I, \theta)$ are action-angle variables and $y \in \mathbb{N}$ where $\mathbb{N}$ is an open neighbourhood of the origin in $\mathbb{R}^{2n-2}$.
- The point $p$ corresponds to $(I, \theta, y) = (0, 0, 0)$.
- The Hamiltonian is

$$\mathcal{H}_\varepsilon(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon \bar{\mathcal{H}}(I, y) + O(\varepsilon^2).$$

- We make use of the Hamiltonian Flow Box Theorem.

# Sketch of the Proofs #2

- Up to terms of order $O(\varepsilon^2)$ the equations are

$$\dot{I} = O(\varepsilon^2), \quad \dot{\theta} = 2\pi/T(I) + O(\varepsilon^2), \quad \dot{y} = \varepsilon J \nabla_y \bar{\mathcal{H}}(I, y) + O(\varepsilon^2).$$

- Construct **a section map** in an energy level ($I = 0$):

$$P(y) = y + \varepsilon T J \nabla_y \bar{\mathcal{H}}(0, y) + O(\varepsilon^2).$$

- A fixed point of $P$ leads to a periodic solution: we solve $P(y) = y$.
- As $y = 0$ is a nondegenerate critical point and $\partial^2 \bar{\mathcal{H}}/\partial y^2(0,0)$ is nonsingular, we apply **the implicit function theorem.**

There is a function $\bar{y}(\varepsilon) = O(\varepsilon)$ such that $P(\bar{y}(\varepsilon)) = \bar{y}(\varepsilon)$.

This fixed point of $P$ is the initial condition for the periodic solution asserted in the Perturbation Theorem.

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# Corollary: Estimate on the Number of Critical Points

(Milnor, 1963)

- $\bar{\mathcal{H}}$ a Morse function;
- $\beta_j$ the $j^{th}$ Betti number of $B$;
- $I_j$ the number of critical points of index $j$.
  (The *index* of a nondegenerate critical point $\bar{p}$ of $\bar{\mathcal{H}}$ is the dimension of the maximal linear subspace where the Hessian of $\bar{\mathcal{H}}$ at $p$ is negative definite.)

# Corollary: Estimate on the Number of Critical Points #2

Then $I_j \geq \beta_i$ or better yet

$$
\begin{aligned}
I_0 &\geq \beta_0 \\
I_1 - I_0 &\geq \beta_1 - \beta_0 \\
I_2 - I_1 + I_0 &\geq \beta_2 - \beta_1 + \beta_0 \\
&\cdots \\
I_k - I_{k-1} + I_{k+2} - \cdots \pm I_0 &\geq \beta_k - \beta_{k-1} + \cdots \pm \beta_0 \quad (k < 2n-2) \\
I_0 - I_1 + I_2 - \cdots - I_{2n-3} + I_{2n-2} &= \beta_0 - \beta_1 + \cdots - \beta_{2n-3} + \beta_{2n-2} = \chi(B).
\end{aligned}
$$

# Corollary: Stability

The eigenvalues of

$$A = J\frac{\partial^2 \bar{\mathcal{H}}}{\partial y^2}(0,0)$$

are the characteristic exponents of the critical point of $\bar{Y}$ at $\bar{p}$ on $B$.

If the characteristic exponents of $\bar{Y}(\bar{p})$ are $\lambda_1, \lambda_2, \ldots, \lambda_{2n-2}$, the characteristic multipliers of the periodic solution through $p(\varepsilon)$ are:

$$1,\ 1,\ 1+\varepsilon\lambda_1 T + O(\varepsilon^2),\ 1+\varepsilon\lambda_2 T + O(\varepsilon^2),\ \ldots,\ 1+\varepsilon\lambda_{2n-2} T + O(\varepsilon^2).$$

# Corollary: Stability #2

Given an autonomous linear Hamiltonian vector field, it is said to be **parametrically stable** or **strongly stable** if it is stable and all sufficiently small linear constant coefficient Hamiltonian perturbations of it are stable.

If the matrix $A$ corresponding to the linearisation around an equilibrium point $\bar{p}$ is parametrically stable then the periodic solution through $p(\varepsilon)$ is elliptic.

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# KAM Tori

Let $p$ be as before and let there be symplectic action-angle variables $(I_1, \ldots, I_{n-1}, \theta_1, \ldots \theta_{n-1})$ at $\bar{p}$ in $B$ such that

$$\bar{\mathcal{H}} = \sum_{k=1}^{n-1} \omega_k I_k + \frac{1}{2} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} C_{kj} I_k I_j + \mathcal{H}^{\#},$$

where $\omega_k \neq 0$ and $\mathcal{H}^{\#}(I_1, \ldots, I_{n-1}, \theta_1, \ldots \theta_{n-1})$ is at least cubic in $I_1, ..., I_{n-1}$.

Assume that $\det C_{kj} \neq 0$ and that $dT/dh \neq 0$.

Near the periodic solutions given before there are invariant KAM tori of dimension $n$.

# Weinstein's Theorem

(Weinstein, 1977, 1978)

Let $X$ be a topological space, then the category of $X$ in the sense of Ljusternik-Schnirelmann, $\mathrm{cat}\,(X)$, is the least number of closed sets that are contractible in $X$ and that cover $X$.

Assume $B$ is simply connected and let $\ell = \mathrm{cat}\,(B)$ be the Ljusternik–Schnirelmann category of $B$.

Then, for small $\varepsilon$ the flow of $Y_\varepsilon$ **has at least $\ell$ periodic solutions** with periods near $T$.

There is no **nondegeneracy** assumption.

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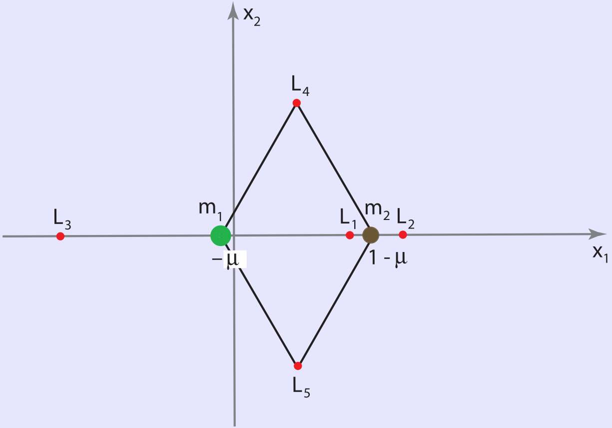
# Rotating Frame




The Hamiltonian has five equilibria:

- $L_1, L_2, L_3$ unstable (Euler),
- $L_4, L_5$ linearly stable (Lagrange).

# The Hamiltonian in the Rotating Frame

$$\mathcal{H} = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - (x_1 y_2 - x_2 y_1) - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2 + x_3^2}} - \frac{1-\mu}{\sqrt{(x_1 + \mu)^2 + x_2^2 + x_3^2}}.$$

- $\mu \in (0, 1/2]$ is the quotient between the mass of one of the primaries

and the sum of the masses of both primaries.

**Lunar case:** In the restricted three-body problem the infinitesimal particle is close to one of the primaries.

# The Hamiltonian in the Rotating Frame #2

Change $y_2$ and $x_1$ to $y_2 - \mu$ and $x_1 - \mu$ and introduce a small parameter, $\varepsilon$, by replacing $y = (y_1, y_2, y_3)$ by $(1-\mu)^{1/3} y/\varepsilon$ and $x = (x_1, x_2, x_3)$ by $(1-\mu)^{1/3}\varepsilon^2 x$:

$$\mathcal{H}_\varepsilon = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} - \varepsilon^3 (x_1 y_2 - x_2 y_1) + \frac{1}{2}\varepsilon^6 \mu (-2x_1^2 + x_2^2 + x_3^2) + \cdots .$$

# Applications

- Detection of **extrasolar planets** with negligible mass around a binary system but rotating around of the stars.
- **Scientific missions** of artificial satellites around the Galilean moon Europa need a precise knowledge of the moon's position.

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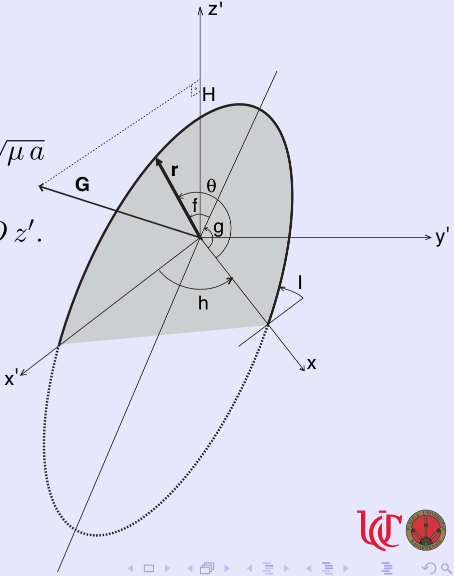
# Delaunay coordinates

- $\ell$: mean anomaly
- $g$: argument of the pericentre
- $\nu \equiv h$: argument of the node
- $L$: action related with $\ell$: $L = \sqrt{\mu\, a}$
- $G$: magnitude of **G**
- $N \equiv H$: projection of **G** onto $O\, z'$.



# Normalised Hamiltonian

In Delaunay elements $(\ell, g, \nu, L, G, N)$, we perform the Delaunay normalisation:

$$
\begin{aligned}
\mathcal{H}_\varepsilon \;=\; & -\tfrac{1}{2\,L^2} - \varepsilon^3 N \\
& + \tfrac{1}{16}\varepsilon^6 \mu\, L^4 \Big( (2 + 3\,e^2)\,\big(1 - 3\,c^2 - 3\,(1 - c^2)\,\cos(2\,\nu)\big) \\
& \qquad - 15\,e^2\,\cos(2\,g)\big(1 - c^2 + (1 + c^2)\,\cos(2\,\nu)\big) \\
& \qquad + 30\,c\,e^2 \sin(2\,g)\sin(2\,\nu)\Big) + \cdots ,
\end{aligned}
$$

where $e = \sqrt{1 - G^2/L^2}$ and $c = N/G$.

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# Reduction

**Base space (orbit space): $S^2 \times S^2$**

Reduction process:

- $\mathbf{G}$ is the angular momentum vector and $A$ is the Runge-Lenz vector: $A = y \times \mathbf{G} - \frac{x}{|x|}$,
- define $\mathbf{a} = \mathbf{G} + LA$, $\mathbf{b} = \mathbf{G} - LA$,
- invariants: $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$,
- constraints:

$$a_1^2 + a_2^2 + a_3^2 = L^2 \quad \text{and} \quad b_1^2 + b_2^2 + b_3^2 = L^2 \quad \text{where} \quad a_i, b_i \in [-L, L].$$

# Reduction #2

The Poisson structure on $S^2 \times S^2$:

$$\{a_1, a_2\} = 2a_3, \qquad \{a_2, a_3\} = 2a_1, \qquad \{a_3, a_1\} = 2a_2,$$
$$\{b_1, b_2\} = 2b_3, \qquad \{b_2, b_3\} = 2b_1, \qquad \{b_3, b_1\} = 2b_2, \qquad \{a_i, b_j\} = 0.$$

- Rectilinear trajectories:

$$\mathcal{R} = \{(\mathbf{a}, -\mathbf{a}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2\}.$$

- Equatorial trajectories

$$\mathcal{E} = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2,\ b_1 = -a_1,\ b_2 = -a_2,\ b_3 = a_3\}.$$

- Circular trajectories:

$$\mathcal{C} = \{(\mathbf{a}, \mathbf{a}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2\}.$$

# The Reduced System

$$\bar{\mathcal{H}} = -\frac{1}{2}\,(a_3 + b_3) - \frac{1}{8}\varepsilon^3\mu L^2\Big(3a_1^2 - 3a_2^2 - 3a_3^2 - 12a_1b_1 + 3b_1^2 + 6a_2b_2 - 3b_2^2 + 6a_3b_3 - 3b_3^2\Big) + \cdots .$$

# Minimum Number of Equilibria

$$B = S^2 \times S^2 = \{a_1^2 + a_2^2 + a_3^2 = L^2, \quad b_1^2 + b_2^2 + b_3^2 = L^2\}$$

Ljusternik-Schnirelmann category of $S^n \times S^n$ is 3.

↓

Weinstein's Theorem

↓

There are **at least three periodic solutions** of the corresponding flow defined by $Y_\varepsilon$ with period near $T = 2\pi L^3$.

This holds for **any perturbation of the spatial Kepler problem.**

# Relative Equilibria

$$\bar{\mathcal{H}} = -\frac{1}{2}(a_3 + b_3) + \cdots :$$

A nondegenerate maximum at $(\mathbf{a}, \mathbf{b}) = (0, 0, -L, 0, 0, -L)$

A nondegenerate minimum at $(\mathbf{a}, \mathbf{b}) = (0, 0, L, 0, 0, L)$.

$\downarrow$

By Reeb's Theorem 2 and Corollary about stability they correspond to elliptic periodic solutions of the spatial RTBP of period $T(\varepsilon) = T + O(\varepsilon^3)$.

These are the **circular equatorial motions** also detected in the planar case.

# Relative Equilibria #2

Two nondegenerate critical points of index 2 at

$$(\mathbf{a},\mathbf{b}) = (0,0,\pm L,0,0,\mp L).$$

- **Rectilinear motions** of the spatial RTBP corresponding with periodic orbits in the vertical axis $x_3$.
- They generalise the rectilinear trajectories found by Belbruno in 1981 for small $\mu$.
- Refining the computation we find out that these orbits are not longer rectilinear:

$$\begin{aligned} e &= 1 - \tfrac{11025}{512}\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\,L^{10} + \cdots\,, \\ G &= \tfrac{105}{16}\varepsilon^{5}\,\mu\,(1-\mu)^{1/3}\,L^{6} + \cdots\,. \end{aligned}$$

# Relative Equilibria #3

**$\bar{\mathcal{H}}$ is a Morse function**

The Betti numbers of $S^2 \times S^2$ are $\beta_0 = \beta_4 = 1$, $\beta_2 = 2$ and all the others are zero: the minimum number of critical points is consistent with the Morse inequalities.

The characteristic exponents of all the four critical points of $Y_\varepsilon$ at the four equilibria are $\pm i$ (double).

$\downarrow$

Corollary about Stability

$\downarrow$

The characteristic multipliers of the associated periodic solutions are:

$$1,\ 1,\ 1 + \varepsilon^3 T\, i,\ 1 + \varepsilon^3 T\, i,\ 1 - \varepsilon^3 T\, i,\ 1 - \varepsilon^3 T\, i$$

plus terms factored by $\varepsilon^6$.

# Kam Tori around $(0,0,\pm L,0,0,\mp L)$

- Move the critical point to the origin:

$$a_1=\bar a_1,\quad a_2=\bar a_2,\quad a_3=\bar a_3\pm L,\quad b_1=\bar b_1,\quad b_2=\bar b_2,\quad b_3=\bar b_3\mp L.$$

- Introduce the local canonical change:

$$Q_1=\frac{\bar a_2}{\sqrt{\pm\bar a_3+2L}},\qquad Q_2=\frac{\bar b_2}{\sqrt{\mp\bar b_3+2L}},$$
$$P_1=\mp\frac{\bar a_1}{\sqrt{\pm\bar a_3+2L}},\qquad P_2=\pm\frac{\bar b_1}{\sqrt{\mp\bar b_3+2L}}.$$

- Scale through $\bar Q_j=\varepsilon^{-3/2}\,Q_j$ and $\bar P_j=\varepsilon^{-3/2}\,P_j$ for $j\in\{1,2\}$; make the change canonical and expand this Hamiltonian in powers of $\varepsilon$.

# Kam Tori around $(0, 0, \pm L, 0, 0, \mp L)$ #2

$$
\begin{aligned}
\bar{\mathcal{H}} &= \pm\tfrac{1}{2}(\bar{P}_1^2 + \bar{Q}_1^2) \mp \tfrac{1}{2}(\bar{P}_2^2 + \bar{Q}_2^2) \\
&\quad - \tfrac{3}{4}\varepsilon^3\,\mu\,L^3\left(3(\bar{P}_1^2 + \bar{P}_2^2) + 4\bar{P}_1\,\bar{P}_2 + \bar{Q}_1^2 + \bar{Q}_2^2 + 2\bar{Q}_1\,\bar{Q}_2\right) + \cdots .
\end{aligned}
$$

The eigenvalues associated with the linear differential equation given through the quadratic part of $\bar{\mathcal{H}}$ are:

$$
\pm\sqrt{1 + 20\bar{\varepsilon}^2 + 2\sqrt{5}\bar{\varepsilon}\sqrt{3 + 20\bar{\varepsilon}^2}}\,i = \pm\omega_1\,i,
$$

$$
\pm\sqrt{1 + 20\bar{\varepsilon}^2 - 2\sqrt{5}\bar{\varepsilon}\sqrt{3 + 20\bar{\varepsilon}^2}}\,i = \pm\omega_2\,i
$$

where

- $\bar{\varepsilon} = \frac{3}{4}\varepsilon^3\,\mu\,L^3, \quad \omega_1 > 1 > \omega_2 > 0,$
- $\omega_1 = \omega_2 = 1$ when $\varepsilon = 0$ and the quadratic part of $\bar{\mathcal{H}}$ is in 1-1 resonance.

# Kam Tori around $(0, 0, \pm L, 0, 0, \mp L)$ #3

- Bring the quadratic part of $\bar{\mathcal{H}}$ into normal form through a linear canonical change of variables.
- The quadratic part of $\bar{\mathcal{H}}$ is:

$$\pm\omega_1\, i\, q_1\, p_1 \mp \omega_2\, i\, q_2\, p_2,$$

  $(q_1, q_2, p_1, p_2)$ being the new variables.

- Introduce action-angle variables $(I_1, I_2, \varphi_1, \varphi_2)$:

$$q_1 = \sqrt{I_1/\omega_1}\,(\cos\,\varphi_1 - i\,\sin\,\varphi_1), \quad q_2 = \sqrt{I_2/\omega_2}\,(\cos\,\varphi_2 - i\,\sin\,\varphi_2),$$
$$p_1 = \sqrt{\omega_1\, I_1}\,(\sin\,\varphi_1 - i\,\cos\,\varphi_1), \quad p_2 = \sqrt{\omega_2\, I_2}\,(\sin\,\varphi_2 - i\,\cos\,\varphi_2).$$

# Kam Tori around $(0,0,\pm L,0,0,\mp L)$ #4

- Average $\bar{\mathcal{H}}$ over $\varphi_1$ and $\varphi_2$ arriving in both cases at

$$\bar{\mathcal{H}} \;=\; \pm\omega_1\, I_1 \mp \omega_2\, I_2 + \bar{\varepsilon}\, F(I_1, I_2) + \cdots .$$

- Compute:

$$\det\begin{bmatrix} \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_1^2} & \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_1 \partial I_2} \\ \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_2 \partial I_1} & \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_2^2} \end{bmatrix} = \frac{(\omega_1^2-1)^6\,(7\omega_1^8-28\omega_1^6-534\omega_1^4-604\omega_1^2-137)}{225\mu^2\, L^8\,\omega_1^2\,(\omega_1^2-4)\,(\omega_1^2+2)^4\,(2\omega_1^2+1)^2} + \cdots$$

- KAM theory hypotheses hold.
- There are families of invariant 3-tori around these relative equilibria.

# Nonlinear Stability

Apply Arnold's Theorem:

- Compute $\bar{\mathcal{H}}_4$, the quartic terms of $\bar{\mathcal{H}}$, evaluate it at $I_1 = -\omega_2$ and $I_2 = \omega_1$ (i.e., compute $\bar{\mathcal{H}}_4(-\omega_2, \omega_1)$) and ensure that it does not vanish for $\bar{\varepsilon}$ positive and small.
- The points $(0, 0, \pm L, 0, 0, \mp L)$ are **nonlinearly stable** in the space $S^2 \times S^2$.

# Kam Tori around $(0, 0, \pm L, 0, 0, \pm L)$

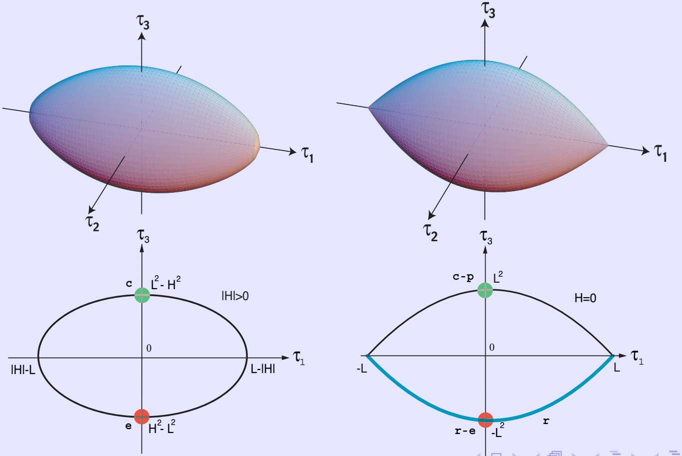
- Similar results hold for these points and there are (stable) periodic orbits as they correspond to nondegenerate maximum or minimum of the Hamiltonian.

- Besides, one can find KAM 3-tori around these periodic orbits.

# Second Reduction

**Melons and lemons** (Cushman, 1983)

$$\mathcal{U}_{L,H} = \{\tau \in \mathbb{R}^3 \mid \tau_2^2 + \tau_3^2 = [(L+\tau_1)^2 - H^2]\,[(L-\tau_1)^2 - H^2]\}$$



# Second Reduction #2

1. We find rectilinear, circular and equatorial relative equilibria for all cases.
2. There are up to six different equilibria.
3. A pitchfork bifurcation takes place for $|H|/L = \sqrt{3/5}$.
4. The equilibria are reconstructed into (approximate) invariant 2-tori of the restricted three body problem.

So far it is not clear how to apply a KAM theorem to conclude the existence of invariant 3-tori and how to reconstruct the pitchfork bifurcation of invariant tori.