ON THE NON-INTEGRABILITY OF SOME N-BODY PROBLEMS

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A. N. R. : Intégrabilité réelle et complexe
en Mécanique Hamiltonienne

INTÉGRABILITÉ DYNAMIQUE

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Assessment

The **General (Planar)** N**-Body Problem (*NBP*)** lays at the foundations of Celestial Mechanics. Up to now we had:

- strong numerical evidence of lack of integrability;
- $3BP$: previous rigorous proofs of non-integrability by A. Tsygvintsev (2000), D. Boucher & J.-A. Weil (2002).
  - Tsygvintsev (2006): absence of a *single* additional integral except for three special cases.

We present **proofs of the meromorphic non–integrability in the sense of Liouville-Arnold**:

- for the $3BP$ (yet another proof), as well as
  - the absence of a *single* additional integral for arbitrary masses.
- for the NBP **with equal masses**, for some special values of $N \geq 4$.

as applications of **differential Galois theory** to significant dynamical problems.

Application of the same tools to the N-Body Problem with general masses: **in process**.

1. Introduction

1.1 Setting

Given an autonomous dynamical system

$$\dot{\mathbf{z}} = X(\mathbf{z}), \tag{1}$$

having an integral curve $\Gamma = \{\widehat{\mathbf{z}}(t) : t \in I\}$, the **variational equations** of (1) along Γ are the linear homogeneous system

$$\dot{\xi} = X'(\widehat{\mathbf{z}}(t))\,\xi, \tag{2}$$

whose principal fundamental matrix Ψ is the linear part of the flow $\varphi(t; \widehat{\mathbf{z}}(0))$ of (1) along Γ.

Distinguished example (though by far not the only one): an n-degree-of-freedom Hamiltonian H with Hamilton's equations $\dot{\mathbf{z}} = X_H(\mathbf{z}) \quad (\mathbf{z} = (\mathbf{Q}; \mathbf{P}))$.

We thus have:

- a **(generally nonlinear)** system 1 and
- a **linear system** (2) linked to (1).

1. Introduction

1.1 Integrability of linear systems

Given a **linear** differential system (whether or not autonomous), with coefficients in a differential field (K, ∂) whose field of constants $\mathcal{C}$ is algebraically closed (e.g. $\left(\mathbb{C}(t)\,,\frac{d}{dt}\right)$),

$$\mathbf{y}' = A(t)\,\mathbf{y}, \tag{3}$$

differential Galois theory assures the existence of

- A differential field $L \supset K$, unique up to K-isomorphism, containing all entries of a fundamental matrix $\Psi = \left[\psi_1, \ldots, \psi_n\right]$ of (3);
- an algebraic group G linked to $K \subset L$ (the **differential Galois group** of (3)), such that
  - G acts over the $\mathcal{C}$-vector space $\left\langle \psi_1, \ldots, \psi_n \right\rangle$ of solutions as a linear transformation group over $\mathcal{C}$;
  - the **monodromy group** of (3) is contained in G.

(3) is called *integrable* if its general solution can be written as a finite sequence of quadratures, exponentials, and algebraic functions (and any of their inverses).

In the Galoisian setting, assertion "(3) is integrable" is equivalent to the following: the identity component G^0 *of the differential Galois group G of (3) es **solvable**.*

1. Introduction

1.1 General principle

Heuristics of all non-integrability results presented here are firmly rooted in the following:

General Principle. *If we assume* (1) *"integrable" in some reasonable sense, then the corresponding variational system* (2) *along any integral curve Γ of* (1) *must be also integrable (in the sense of linear Galois differential theory).*

Any attempt at **ad-hoc formulations of this heuristic principle** for a specific system (1) has an asset and a drawback:

- as seen above, there *is* a definition of integrability for linear systems (and thus, for (2)): that the identity component of its Galois group be *solvable*;
- still, in order to transform this principle into a true conjecture it is necessary to clarify a notion of "integrability " for (1).

Remark. *Everything is considered in the complex analytical setting.*

2. Application to Hamiltonian systems

2.1. General case

There is a specific notion of integrability for *Hamiltonian* systems, namely in the sense of Liouville-Arnold, for which the General Principle does have an implementation:

Theorem 1. ***(J. Morales-Ruiz & J.-P. Ramis, 2001)*** *Let H be an n-degree-of-freedom Hamiltonian having n independent first integrals in pairwise involution, defined on a neighborhood of an integral curve Γ. Then, the identity component of the Galois group of the variational equations* (2) *of H along Γ is a* ***commutative*** *group.*

Key Lemma. (an essential tool in the proof)*: Let f be a meromorphic first integral of the dynamical system* (1)*. Then the Galois group of the variational system* (2) *has a non-trivial rational invariant .*

Remark. In this lemma we do not assume the dynamical system Hamiltonian!

2. Application to Hamiltonian systems

2.3. Special case: *homogeneous potentials*

If X_H **is classical with a homogeneous potential of degree** $k \in \mathbb{Z}$, i.e.

$$H = H(\mathbf{Q}, \mathbf{P}) = \frac{P^2}{2} + V(\mathbf{Q}),$$

let $\mathbf{c} \in \mathbb{C}^n$ be a solution of $\mathbf{c} = V'(\mathbf{c})$ and

$$\{\lambda_1, \ldots, \lambda_n\} = \mathrm{Spec}\ V''(\mathbf{c}).$$

- Since $V''(\mathbf{c})\,\mathbf{c} = (k-1)\,\mathbf{c}$, we may define $\lambda_n = k - 1$.

2. Application to Hamiltonian systems

2.3. Special case: *homogeneous potentials*

Theorem 2. (J.J. Morales-Ruiz & J.-P. Ramis, 2001) *If $k \neq 0$ and H is integrable with meromorphic integrals, then each of the eigenvalues, $\lambda_1, \ldots \lambda_n$, matches an item in Table 1 ($p \in \mathbb{Z}$):*

TABLE 1

k	λ	k	λ
k	$\frac{p(2+k(p-1))}{2}$	-3	$\frac{25-\left(\frac{12}{5}+6p\right)^2}{24}$
2	$\lambda \in C$	3	$\frac{-1+(2+6p)^2}{24}$
-2	$\lambda \in C$	3	$\frac{-1+\left(\frac{3}{2}+6p\right)^2}{24}$
-5	$\frac{49-\left(\frac{10}{3}+10p\right)^2}{40}$	3	$\frac{-1+\left(\frac{6}{5}+6p\right)^2}{24}$
-5	$\frac{49-(4+10p)^2}{40}$	3	$\frac{-1+\left(\frac{12}{5}+6p\right)^2}{24}$
-4	$\frac{9-\left(\frac{4}{3}+4p\right)^2}{8}$	4	$\frac{-1+\left(\frac{4}{3}+4p\right)^2}{8}$
-3	$\frac{25-(2+6p)^2}{24}$	5	$\frac{-9+\left(\frac{10}{3}+10p\right)^2}{40}$
-3	$\frac{25-\left(\frac{3}{2}+6p\right)^2}{24}$	5	$\frac{9+(4+10p)^2}{40}$
-3	$\frac{25-\left(\frac{6}{5}+6p\right)^2}{24}$	k	$\frac{((p+1)k-1)(pk+1)}{2k}$

2. Application to Hamiltonian systems

2.3. Special case: *homogeneous potentials*

- Based on basic properties of algebraic groups with a non-trivial rational invariant, we obtain a necessary condition for the existence of a *single* additional first integral:

  Theorem 3. *If $f_1, \ldots, f_m$ are such first integrals of X_H that $\lambda_{n-m+1}, \ldots, \lambda_n$ belong to Table 1, and there is a first integral f independent of and in involution with $\{f_1, \ldots, f_m\}$, then at least one of the eigenvalues $\lambda_1, \ldots, \lambda_{n-m}$ belongs to Table 1.* □

3. The N-Body Problem

3.1 Introduction

- It may be written as a classical Hamiltonian $\mathcal{H}_N$ with homogeneous potential of degree -1:

$$V_N(\mathbf{q}) = -\sum_{1\leq i<j\leq N} \frac{\left(m_i m_j\right)^{3/2}}{\left\|\sqrt{m_j}\mathbf{q}_i - \sqrt{m_i}\mathbf{q}_j\right\|}.$$

- Solutions of $V_N'(\mathbf{c}) = \mathbf{c}$ correspond exactly to **central configurations** of the N-Body Problem.

- **Known first integrals (of the d-dimensional *NBP*)**: $\frac{1}{2}(d+2)(d+1)$ (so-called *classical*) first integrals:
  - $2d$ for the invariance of the **linear momentum** $\mathbf{I}_L := \sum_{i=1}^{N} m_i \dot{\mathbf{x}}_i$, i.e. for the uniform linear motion of the center of mass;
  - $d(d-1)/2$ for the invariance of the **angular momentum** $I_A := \sum_{i=1}^{N} m_i \mathbf{x}_i \wedge \dot{\mathbf{x}}_i$;
  - one for the invariance of the Hamiltonian $\mathcal{H}_N$.

  In particular, for $N = 3$ that makes 6 for the planar problem and 10 for the spatial problem.

4. Current results

4.1 Three Body Problem

- we obtained two solutions $\mathbf{c}, \mathbf{c}^*$ of $V_3'(\mathbf{c}) = \mathbf{c}$ and computed the eigenvalues of $V_3''(\mathbf{c})$ y $V_3''(\mathbf{c}^*)$:

$$\{\lambda_\pm, 1, 0, 0, -2\}, \quad \left\{\lambda_\pm^*, 1, 0, 0, -2\right\},$$

  1, 0, 0 and -2 being a consequence of the invariance of the two entries of the linear momentum, the angular momentum and $\mathcal{H}_3$, respectively.

- due to Theorem 3, the existence of a single integral independent of $\mathcal{H}_3, \mathbf{I}_L, I_A$ requires one of λ_+, λ_- and one of λ_+^*, λ_-^* to match items 1 and 18 in Table 1 for $k = -1$, i.e. two such eigenvalues should belong to

$$S := \left\{-\frac{1}{2}p\,(p-3) : p \in \mathbb{Z}\right\},$$

  and a simple calculation deems this impossible. Thus,

  Theorem 4. $\mathcal{H}_3$ *does not have an additional meromorphic integral.* $\square$

4. Current results

4.2 N-body problem with equal masses

- We may set all $m_1 = \cdots = m_N = 1$; the vector $\mathbf{c} = a^{\frac{1}{3}}(\mathbf{c}_1, \ldots, \mathbf{c}_N)$, where $a = \frac{1}{4}\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{n}\right)$ and

$$\mathbf{c}_k = \left(\cos\frac{2\pi k}{N}, \sin\frac{2\pi k}{N}\right), \quad k = 1, \ldots, N,$$

is a solution for $V'_N(\mathbf{q}) = \mathbf{q}$.

- The trace of $V''_N(\mathbf{c})$ is equal to

$$\operatorname{tr} V''_N(\mathbf{c}) = -N\left(\frac{\sum_{k=1}^{N-1} \frac{1}{\sin^3\left(\frac{\pi k}{N}\right)}}{2\sum_{k=1}^{N-1} \frac{1}{\sin\left(\frac{\pi k}{N}\right)}}\right),$$

and is also equal to $\lambda_1 + \cdots + \lambda_{2N-4} - 1$ since we can choose

$$\lambda_{2N-3} = 1, \quad \lambda_{2N-2} = \lambda_{2N-1} = 0, \quad \lambda_{2N} = -2,$$

as in V''_3. **Proving** $\operatorname{tr} V''_N(\mathbf{c}) \notin \mathbb{Z}$ **would be enough to prove the non-integrability of $\mathcal{H}_N$.**

4. Current results

4.2 N-body problem with equal masses

- **First partial result:** for $N = 3, 4, 5, 6$, we computed all eigenvalues of $V''_N(\mathbf{c})$ and except for four of them, $-2, 0, 0, 1$, not a single one matches items 1 and 18 in Table 1. Thus, in virtue of Theorem 3:

  Theorem 5. *The Three-, Four-, Five- and Six-Body Problems with equal masses do not possess an additional first integral.* $\square$

- **Second partial result:**

  Theorem 6. *If $N = 2^m$ for $m \geq 2$, then $\sum_{k=1}^{N-1} \csc \frac{\pi}{N}k$ and $\sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N}k$ are $\mathbb{Q}$-independent; in particular,* $\operatorname{tr} V''_N(\mathbf{c}) \notin \mathbb{Q}$, *i.e. $\mathcal{H}_{2^m}$ with equal masses is not integrable in the sense of Liouville-Arnold.* $\square$

- Proving one of the following conjectures would be enough to obtain non-integrability for general N:
  if $N \geq 7$,
  - **Conjecture 1:** $V''_N(\mathbf{c})$ **has an eigenvalue** $\lambda > 1$.
  - **Conjecture 2:** $\operatorname{tr} V''_N(\mathbf{c}) \notin \mathbb{Q}$.