Some rigidity results for integrable systems and Hamiltonian actions

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Outline

Rigidity for smooth group actions

Let G be a compact Lie group and let $\rho : G \times M \longrightarrow M$ stand for a smooth action.

Theorem

(Bochner) A local smooth action with a fixed point is locally equivalent to the linearized action.

Theorem

(Palais) Two (C^1)-close group actions of compact Lie groups on a compact manifold are equivalent.

By equivalence we mean the existence of a C^k-diffeomorphism intertwining both actions i.e

$$\rho_0(g) \circ \phi = \phi \circ \rho_1(g)$$

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General Philosophy

Assume we have additional geometrical structures on the manifold

Goal

We want to prove rigidity for the group action and also for the additional geometrical structures.

We will consider the following cases:

- Symplectic structure + symplectic group action
- Symplectic structure + Integrable system + Symplectic group action
- Poisson structure + Poisson action (local)
- Poisson structure + Hamiltonian action (global)

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Rigidity for Symplectic group actions

Theorem (Equivariant Darboux Theorem)

Let $\rho : G \times M \longrightarrow M$ stand for a symplectic group action of a compact Lie group G on (M, ω) and let p be a fixed point for the action. Then there exists coordinates $(x_1, y_1, \dots, x_n, y_n)$ in a neighbourhood of p such that the group action is linear and the symplectic form can be written as $\omega_0 = \sum_i dx_i \wedge dy_i$.

Theorem

Let ρ_0 and ρ_1 be two C^1-close symplectic actions of a compact Lie group G on a compact symplectic manifold (M, ω) then there exists a symplectomorphism ϕ satisfying $\rho(g) \circ \phi = \phi \circ \rho_1(g), \forall g \in G$

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Equivariant normal forms for integrable systems on symplectic manifolds

General principle

Normal forms can lead to structural stability

Integrable system in the sense of Liouville: $F = (f_1, \dots, f_n)$, $\{f_i, f_j\} = 0$ on (M, ω) symplectic manifold.

Goal

Look for equivariant normal forms for these systems.

Strategy

First we look for normal forms and try to make them equivariant by studying properties of the group of local automorphisms of the system.

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Normal forms for non-degenerate singular compact orbits

Case I- The orbit is a non-degenerate point $d_pF = 0$. We can associate a Cartan Subalgebra. Cartan subalgebras were classified by Williamson.

Theorem (Williamson)

For any Cartan subalgebra $\mathcal{C}$ of $Q(2n, \mathbb{R})$ there is a symplectic system of coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ in $\mathbb{R}^{2n}$ and a basis $f_1, \ldots, f_n$ of $\mathcal{C}$ such that each f_i is one of the following:

$$f_i = x_i^2 + y_i^2 \qquad \text{for} \quad 1 \leq i \leq k_e \ , \qquad \text{(elliptic)}$$

$$f_i = x_i y_i \qquad \text{for} \quad k_e + 1 \leq i \leq k_e + k_h \ , \qquad \text{(hyperbolic)}$$

$$\begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i, & \text{(focus-focus pair)} \\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} & \text{for} \quad i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f \end{cases}$$

Normal forms for a fixed point

The triple (k_e, k_h, k_f) is called the Williamson type of the point.

Williamson's theorem gives a linear model for non-degenerate fixed points of the integrable system.

Theorem (Eliasson)

The integrable system at a point of Williamson type (k_e, k_h, k_f) is locally equivalent to its linear model.

This result gives normal forms for the integrable system and the symplectic structure.

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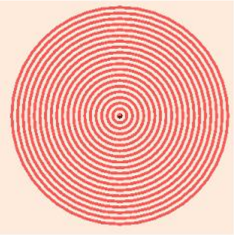
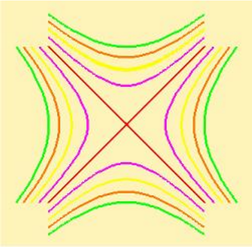
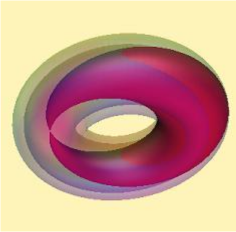
A picture

Eliasson's theorem establishes that locally the integrable system is symplectically equivalent to a product of k_e elliptic components , k_h hyperbolic components and k_f focus-focus components







k_e elliptic comp. k_h hyperbolic comp. k_f focus-focus comp.

Equivariant normal forms for fixed points

Let ρ be an action of a compact Lie group G fixing the non-degenerate point p and preserving ω and F.

Goal

Find normal forms for ω, F and ρ. We call this equivariant normal forms for the integrable system.

Idea: Try to make the normal forms equivariant.

Recall

Bochner's trick for linearizing compact group actions:

$$\Phi_G(x) = \int_G \rho(g)^{(1)} \circ \rho(g)^{-1}(x) d\mu$$

μ is a Haar measure on G

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We can assume $F = (f_1, \ldots, f_n)$ where f_i are a Williamson basis of the associated Cartan subalgebra and that ω is in Darboux form (Eliasson's coordinates).

Now we try to apply Bochner's trick for linearizing the group action.

Try to see that $\Phi_G = \int_G \rho(g)^{(1)} \circ \rho(g)^{-1}(x)$ preserves F and ω.

Key point

Prove that $\forall g \in G$ the diffeomorphism $\rho(g)^{(1)} \circ \rho(g)^{-1}$ is the time 1-map of a Hamiltonian vector field $X_{\Psi(g)}$ where $\Psi(g)$ is a first integral for the integrable system.

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Then

- The diffeomorphism given by Bochner Φ_G coincides with the time-1-map of the vector field $X_G = \int_G X_{\Psi(g)} d\mu$.

$$\Phi_G(x) = \phi^1_{X_G}(x) = \int_G \phi^1_{X_{\Psi(g)}}(x) d\mu.$$

- This diffeomorphism takes the initial action to the linear action.
- Preserves ω since it is the time 1-map of a Hamiltonian vector field.
- Preserves the integrable system since X_G is tangent to the foliation given by F.

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Equivariant normal forms at a fixed point

Consider
$\mathcal{G} = \{\phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0),$ such that $\phi^*(\omega) = \omega, \quad \mathbf{h} \circ \phi = \mathbf{h}\}$ the group of local automorphisms of the system.

$\mathcal{G}_0$= path-component of the identity of $\mathcal{G}$.

$\mathfrak{g}$ the Lie algebra of germs of Hamiltonian vector fields tangent to the fibration defined by F.

Theorem (M-Zung)

The exponential $\exp : \mathfrak{g} \longrightarrow \mathcal{G}_0$ *is a surjective group homomorphism, and moreover there is an explicit right inverse given by*

$$\phi \in \mathcal{G}_0 \longmapsto \int_0^1 X_t dt \in \mathfrak{g}$$

where $X_t \in \mathfrak{g}$ *is defined by* $X_t(R_t) = \frac{dR_t}{dt}$ *for any* C^1 *path* R_t *contained in* $\mathcal{G}_0$ *connecting the identity to* ϕ.

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Equivariant normal forms at a fixed point

In order to conclude with the proof of the equivariant normal forms it suffices to apply the Theorem.

In order to do so we need to check that the integrand in Bochner's formula belongs to $\mathcal{G}_0$.

The path

$$S_t^{\psi}(x) = \begin{cases} \dfrac{\psi \circ g_t}{t}(x) & t \in (0, 2] \\ \psi^{(1)}(x) & t = 0 \end{cases}$$

being g_t the homothecy $g_t(x_1, \ldots, x_n) = t(x_1, \ldots, x_n)$.

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This proves,

Corollary

Suppose that $\psi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ is a local symplectic diffeomorphism of $\mathbb{R}^{2n}$ which preserves the quadratic moment map $\mathbf{h} = (h_1, ..., h_n)$. Then,

1. The linear part $\psi^{(1)}$ is also a system-preserving symplectomorphism.
2. There is a vector field contained in $\mathfrak{g}$ such that its time-1-map is $\psi^{(1)} \circ \psi^{-1}$. Moreover, for each vector field X fulfilling this condition there is a unique local smooth function $\Psi : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ vanishing at 0 which is a first integral for the linear system given by $\mathbf{h}$ and such that $X = X_\Psi$. If ψ is real analytic then Ψ is also real analytic.

Theorem (M, Zung)

In a neighbourhood of a non-degenerate point, the integrable system is equivariantly equivalent to the linear system with the linear action.

The linear model for higher-dimensional orbits in a covering

Denote by $(p_1, ..., p_m)$ a linear coordinate system of a small ball D^m, $(q_1(mod1), ..., q_m(mod1))$ a standard periodic coordinate system of the torus $\mathbb{T}^m$, and $(x_1, y_1, ..., x_{n-m}, y_{n-m})$ a linear coordinate system of a small ball $D^{2(n-m)}$. Consider the manifold

$$V = D^m \times \mathbb{T}^m \times D^{2(n-m)} \quad (1)$$

with the standard symplectic form $\sum dp_i \wedge dq_i + \sum dx_j \wedge dy_j$, and the following moment map:

$$(\mathbf{p}, \mathbf{h}) = (p_1, ..., p_m, h_1, ..., h_{n-m}) : V \to \mathbb{R}^n \quad (2)$$

where

$$\begin{aligned} h_i &= x_i^2 + y_i^2 \ \text{ for } \ 1 \le i \le k_e \ , \\ h_i &= x_i y_i \ \text{ for } \ k_e + 1 \le i \le k_e + k_h \ , \\ h_i &= x_i y_{i+1} - x_{i+1} y_i \ \text{ and} \\ h_{i+1} &= x_i y_i + x_{i+1} y_{i+1} \ \text{ for } \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f \end{aligned} \quad (3)$$

Normal forms for non-degenerate compact orbits

We can define a notion of non-degenerate orbits using reduction.
For these orbits we have the following result which determines existence of normal forms for the symplectic form and the integrable system in the neighbourhood of a nondegenerate orbit.
The linear model is V/Γ, with an integrable system on it given by the same moment map:

$$(\mathbf{p}, \mathbf{h}) = (p_1, ..., p_m, h_1, ..., h_{n-m}) : V/\Gamma \to \mathbb{R}^n \quad (4)$$

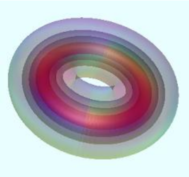
(Γ is a finite group that acts freely and symplectically on the linear model in a covering and comes from the hyperbolic twists)

Theorem (Eliasson-M-Zung)

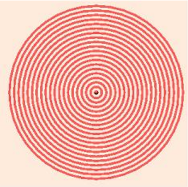
The integrable Hamiltonian system is equivalent to the linear model for Williamson type (k_e, k_h, k_f) in a neighbourhood of non-degenerate compact orbit.

A picture

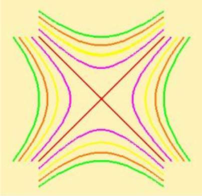
This theorem establishes equivalence with



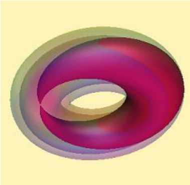
Liouville's tori



k_e elliptic comp.



k_h hyperbolic comp.



k_f focus-focus comp.

Equivariant normal forms

Let G be a compact Lie group preserving ω and F we can make this result equivariant.
The linear action of the group is G *acts on the product* $V = D^m \times \mathbb{T}^m \times D^{2(n-m)}$ *componentwise; the action of* G *on* D^m *is trivial, its action on* $\mathbb{T}^m$ *is by translations and its action on* $D^{2(n-m)}$ *is linear with respect to the coordinate system* $(x_1, y_1, ..., x_{n-m}, y_{n-m})$.

Theorem (M-Zung)

The integrable system is equivariantly equivalent in a neighbourhood of a compact orbit of dimension k *and Williamson type* (k_e, k_h, k_f) *to the integrable system given by the linear model endowed with the linear group action.*

Infinitesimal stability for integrable systems

Fix an integrable system **F** and introduce a deformation complex as follows. Denote by, $X = (\mathcal{C}^\infty, \{,\})$.

$L_0 \simeq \mathbb{R}^n$ acts on X by the adjoint representation:

$$L_0 \times X \ni (e_i, g) \mapsto \{f_i, g\} \in X.$$

Hence X is an L_0-module, in the Lie algebra sense, and we can introduce the corresponding Chevalley-Eilenberg complex:
$q \in \mathbb{N}$, $C^q(L_0, X) = \mathrm{Hom}({L_0}^{\wedge q}, X)$, $(C^0(L_0, X) = X)$.
We denote the associated differential $d_{\mathbf{f}}$.

Infinitesimal stability for integrable systems

Consider $\mathcal{C}_{\mathbf{f}} = \{h \in X, \{f_i, h\} = 0, \forall i\}$.

Definition

Two completely integrable systems **F** and **G** are *equivalent* if and only if

$$\mathcal{C}_{\mathbf{f}} = \mathcal{C}_{\mathbf{g}}$$

L_0 acts trivially on $\mathcal{C}_{\mathbf{f}} \rightsquigarrow X/\mathcal{C}_{\mathbf{f}}$ is a L_0-module, and we can define the corresponding Chevalley-Eilenberg complex: for $q \in \mathbb{N}$,
$C^q(L_0, X/\mathcal{C}_{\mathbf{f}}) = \mathrm{Hom}({L_0}^{\wedge q}, X/\mathcal{C}_{\mathbf{f}})$, with differential $\bar{d}_{\mathbf{f}}$.

Infinitesimal stability for integrable systems

Finally we define the *deformation complex* $C^\bullet(\mathbf{f})$ as follows:

$$0 \longrightarrow X/\mathcal{C}_{\mathbf{f}} \xrightarrow{\bar{d}_{\mathbf{f}}} C^1(L_0, X/\mathcal{C}_{\mathbf{f}}) \xrightarrow{\partial_{\mathbf{f}}} C^2(L_0, X) \xrightarrow{d_{\mathbf{f}}} C^3(L_0, X) \xrightarrow{d_{\mathbf{f}}} \cdots$$

where $\partial_{\mathbf{f}}$ is defined by the following diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & X & \xrightarrow{d_{\mathbf{f}}} & C^1(L_0, X) & \xrightarrow{d_{\mathbf{f}}} & C^2(L_0, X) & \xrightarrow{d_{\mathbf{f}}} & \cdots \\
& \downarrow \pi & \nearrow \partial_{\mathbf{f}} & \downarrow \pi & \nearrow \partial_{\mathbf{f}} & \downarrow \pi & \nearrow \partial_{\mathbf{f}} & \\
0 \longrightarrow & X/\mathcal{C}_{\mathbf{f}} & \xrightarrow[\bar{d}_{\mathbf{f}}]{} & C^1(L_0, X/\mathcal{C}_{\mathbf{f}}) & \xrightarrow[\bar{d}_{\mathbf{f}}]{} & C^2(L_0, X/\mathcal{C}_{\mathbf{f}}) & \xrightarrow[\bar{d}_{\mathbf{f}}]{} & \cdots
\end{array}$$

For all cochain complexes, cocycles and coboundaries are denoted the standard way: $Z^q(\mathbf{f})$ and $B^q(\mathbf{f})$.

1-Cocycles:

Definition

$Z^1(\mathbf{f})$ is the space of infinitesimal deformations of $\mathbf{f}$ modulo equivalence.

If we fix a basis $(e_1, \ldots, e_n)$ of L_0, a cocycle $\alpha \in Z^1(\mathbf{f})$ is just a set of functions $g_1 = \alpha(e_1), \ldots, g_n = \alpha(e_n)$ (defined modulo $\mathcal{C}_\mathbf{f}$) such that

$$\forall i,j \qquad \{g_i, f_j\} = \{g_j, f_i\}. \tag{5}$$

It is an infinitesimal deformation of $\mathbf{f}$ in the sense that, modulo ϵ^2,

$$\{f_i + \epsilon g_i, f_j + \epsilon g_j\} \equiv 0.$$

Infinitesimal stability for integrable systems

1-Coboundaries: $B^1(\cdot)$

$\alpha \in Z^1(\mathbf{f})$ is a coboundary if $\alpha = d_{\mathbf{f}}(h)$, $h \in X$ by

$$L_0 \ni \ell \mapsto \alpha(e_i) = \{h, f_i\} \mod \mathcal{C}_{\mathbf{f}}. \tag{6}$$

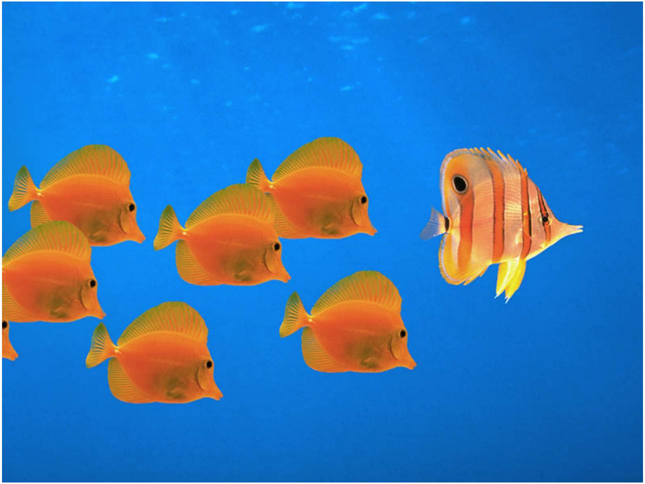
A smooth singular Poincare lemma (M. San Vu Ngoc) for non-degenerate singularities proves that every deformation cocycle of a non-degenerate integrable system is indeed a deformation coboundary,

Theorem (M, San Vu Ngoc)

Let $q_1, \ldots, q_n$ be a standard basis (in the sense of Williamson) of a Cartan subalgebra of $\mathcal{Q}(2n, \mathbb{R})$. Then the corresponding completely integrable system $\mathbf{q}$ in $\mathbb{R}^{2n}$ is $\mathcal{C}^\infty$-infinitesimally stable at $m = 0$: that is,

$$H^1(\mathbf{q}) = 0.$$

The Poisson case



The Poisson case

Assume now that ρ stands for the action of a compact Lie group on a Poisson manifold (M, Π) that preserves the Poisson structure Π.

Goal

We want to know if these actions are rigid.

Theorem (Ginzburg, Rigidity by deformation)

Let ρ_t be a family of Π-preserving actions smoothly parametrized by $t \in [0,1]$. Then there exists a family of Poisson diffeomorphisms $\phi_t : M \longrightarrow M$ which sends ρ_0 to ρ_t such that $\rho_t(g)x = \phi_t(\rho_0(g)\phi_t^{-1}(x))$ for all $x \in M$, $g \in G$ and $\phi_0 = Id$.

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The Poisson case. Our plan.

We will distinguish two aspects:

- The local aspect. Assume that ρ stands for an action and p is a fixed point.
  Like in the symplectic case, we want to prove that there exist a linearization theorem for the action in priviledged coordinates for the Poisson structure.
  The local structure theorem for Poisson manifolds is given by Weinstein's splitting theorem.
  We will prove an equivariant version of the splitting theorem.
- The global aspect. We will consider close Hamiltonian actions of compact semisimple groups on compact symplectic manifolds. We will outline the main ideas for rigidity in this case. .

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The local Poisson case. Splitting Theorem.

The local structure for Poisson manifolds is given by the following:

Theorem (Weinstein)

Let (P^n, Π) be a smooth Poisson manifold and let p be a point of P of rank $2k$, then there is a smooth local coordinate system $(x_1, y_1, \dots, x_{2k}, y_{2k}, z_1, \dots, z_{n-2k})$ near p, in which the Poisson structure Π can be written as

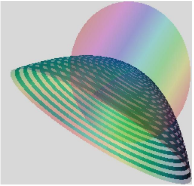
$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where f_{ij} vanish at the origin.

Local Structure

The Poisson manifold is locally a product of a symplectic manifold with a Poisson manifold with vanishing Poisson structure at the point.

$$(P^n, \Pi, p) \approx (M^{2k}, \omega, p_1) \times (P_0^{n-2k}, \Pi_0, p_2)$$



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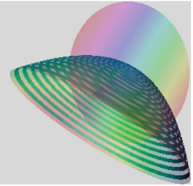

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Conn's Theorem

Theorem (Conn)

Let Π be a Poisson structure vanishing at a point with linear part of compact semisimple type then there exists ϕ such that $\phi_*(\Pi) = \Pi^{lin}$.

The linear Poisson structure can be written in terms of the structure constants c_{ij} of the Lie algebra $\mathfrak{g}$ associated to the linear part.

$$\frac{1}{2}\sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

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The formal linearization had been established by Weinstein.

Theorem

Let (P^n, Π) be a smooth Poisson manifold, p a point of P of rank $2r$, Assume that the linear part of transverse Poisson structure of Π at p corresponds to a semisimple compact Lie algebra $\mathfrak{k}$. Then there is a smooth local coordinate system $(x_1, y_1, \dots, x_{2r}, y_{2r}, z_1, \dots, z_{n-2r})$ near p, in which the Poisson structure Π can be written as

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

where c_{ij}^k are structural constants of $\mathfrak{k}$.

Looking for an equivariant counterpart

Question:

What happens if there is a compact Lie group G preserving Π and fixing the point p?

Do we have an *equivariant splitting theorem*?

Do we have an *equivariant linearization result*?

This result can be seen as a local rigidity result.

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A Poisson manifold can be seen as a space foliated by symplectic manifolds.

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Equivariant splitting theorem

Theorem (M.-Nguyen Tien Zung)

Let (P^n, Π) be a smooth Poisson manifold, $p \in P$ a point of rank $2k$, and G a compact Lie group which acts on P in such a way that the action preserves Π and fixes the point p. Assume that the Poisson structure Π is tame at p. Then there is a smooth canonical local coordinate system $(x_1, y_1, \ldots, x_{2k}, y_{2k}, z_1, \ldots, z_{n-2k})$ near p, in which the Poisson structure Π can be written as

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

, where f_{ij} vanish at the origin and in which the action of G is linear and preserves the subspaces $\{x_1 = y_1 = \ldots x_k = y_k = 0\}$ and $\{z_1 = \ldots = z_{n-2k} = 0\}$.

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Tame Poisson Structures

Let (P^n, Π) be a smooth Poisson manifold and p a point in P.

Definition

We will say that Π is tame at p if for any pair X_t, Y_t of germs of smooth Poisson vector fields near p which are tangent to the symplectic foliation of (P^n, Π) and which may depend smoothly on a (multi-dimensional) parameter t, then the function

$$\Pi^{-1}(X_t, Y_t)$$

is smooth and depends smoothly on t.

Examples of tame Poisson structures

Assume that $X_t = X_{h_t}$ where h_t is a germ of smooth function near p which depends smoothly on the parameter t. Then $\Pi^{-1}(X_t, Y_t) = -Y_t(h_t)$ is smooth for any Y_t.

Consequence

In particular $H^1_\Pi(P^n, p) = 0 \rightsquigarrow \Pi$ is tame at p.

Examples

Let $\mathfrak{g}$ be a compact semi-simple Lie algebra, then $(\mathfrak{g}^*, \Pi_{lin})$ is tame at the origin.

By virtue of Conn's thm this also holds for Poisson structures with linear part of compact semi-simple type.

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Example of non-tame Poisson structure

Non-tame Poisson structure

Consider $\Pi = x^4 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.
The vector fields

$$X = x^2 \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}, \quad Y = x \frac{\partial}{\partial y},$$

are Poisson and tangent to the symplectic foliation but $\Pi^{-1}(X, Y) = \dfrac{1}{x}$ is not smooth at the origin. So this Poisson structure is not tame.

Tameness and a division property

If X is not Hamiltonian (and maybe not even Poisson) but can be written as $X = \sum_{i=1}^{m} f_i X_{g_i}$ where f_i, g_i are smooth functions, then $\Pi^{-1}(X, Y) = -\sum_{i=1}^{m} f_i Y(g_i)$ is still smooth.

Definition

We say that a smooth (resp real analytic) Poisson structure Π satisfies the *smooth division property* (resp *analytic division property*) at a point p if for any germ of smooth (resp. analytic) vector field Z -which may depend smoothly (resp. analytically) on some parameters- which is tangent to the symplectic foliation there exists a finite number of germs of smooth (resp. analytic) functions $f_1, \ldots, f_m, g_1, \ldots, g_m$ -which depend smoothly (resp. analytically) on the same parameters as Z- such that $Z = \sum f_i X_{g_i}$.

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Do all linear structures satisfy the division property?

Proposition

Any linear Poisson structure in dimension 2 or 3 has the division property at the origin and in particular are tame.

Higher dimensions: Dixmier constructs examples of non-semisimple Lie algebras which do not satisfy the division property and proves that all semisimple Lie algebras satisfy the analytic division property.

Conjecture

All semisimple Lie algebras satisfy the smooth division property.

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All semisimple Lie algebras satisfy the smooth division property.

Proof of the equivariant linearization assuming the equivariant splitting

Since the transverse Poisson structure is of compact semi-simple type it satisfies the tameness condition at the origin. Apply the equivariant splitting $\rightsquigarrow$ there exists an equivariant local splitting.

Theorem (Ginzburg)

Assume that a Poisson structure Π vanishes at a point p and is smoothly linearizable near p. If there is an action of a compact Lie group G which fixes p and preserves Π, then Π and this action of G can be linearized simultaneously.

Since the transverse Poisson structure is linearizable (Conn) we may conclude by applying Ginzburg's theorem.

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A 3-step proof for the equivariant splitting theorem

1. Take the initial symplectic foliation to the splitted symplectic foliation and the initial group action to the linear one.

   Here we need the coupling method that reconstructs the Poisson structure via the geometric data $(\Pi_{Vert}, \Gamma, \mathbb{F})$. We take the initial symplectic foliation to the final one using the parallel transport associated to the Ehresman connection Γ.

2. Use the path method and the tameness condition to prove that we can take the Poisson structure to the splitted one.

   The key point is the construction of a path of geometric data geometric data $(\Pi_{Vert}, \Gamma_t, \mathbb{F}_t)$ connecting the initial to the splitted one.

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The Hamiltonian Case.

Assume that ρ_0 and ρ_1 are two Hamiltonian actions with moment maps $\mu_0 : M \longrightarrow \mathfrak{g}^*$ and $\mu_1 : M \longrightarrow \mathfrak{g}^*$ where $\mathfrak{g}$ is semisimple of compact type then,

Theorem (Monnier, M, Zung)

Two close Hamiltonian group actions of semisimple compact type on a compact Poisson manifold are equivalent.

The Hamiltonian Case. Sketh of proof.

Given a Hamiltonian group action ρ_0 we can associate to it a moment map $\mu : M \longrightarrow \mathfrak{g}^*$.
Let ξ_i be a basis of $\mathfrak{g}$.

$\mathfrak{g}$ acts on $X = C^\infty(M)$ by the adjoint representation:

$$\mathfrak{g} \times X \ni (\xi_i, g) \mapsto \{\mu_i, g\} \in X.$$

Hence X is an $\mathfrak{g}$-module, in the Lie algebra sense, and we can introduce the corresponding Chevalley-Eilenberg complex.
For $q \in \mathbb{N}, C^q(\mathfrak{g}, X) = \mathrm{Hom}(\mathfrak{g}^{\wedge q}, X)$ is the space of alternating q-linear maps from $\mathfrak{g}$ to X with the convention $C^0(\mathfrak{g}, X) = X$. The associated differential is denoted by d_μ.

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Homotopy operators

In the case $\mathfrak{g}$ is a semisimple Lie algebra of compact type, Conn proves that we can construct homotopy operators h_i satisfying:

$$h \circ d_\mu + d_\mu \circ h = Id$$

Assume now that ρ_0 and ρ_1 are Hamiltonian group actions. We have two moment maps $\mu_0 : M \longrightarrow \mathfrak{g}^*$ and $\mu_1 : M \longrightarrow \mathfrak{g}^*$.
The difference $\phi = \mu_0 - \mu_1$ defines a 1-cochain. If the actions are close this 1-cochain is a near 1-cocycle.

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Idea of the proof

We define the diffeomorphism by an iterative method based on the following idea.

Even if ϕ is not a 1-cocycle, we will apply to it the homotopy operator h introduced by Conn.

Given a 1-cochain ϕ we define $\Phi = S \circ \varphi$ where φ is the time-1-map of the Hamiltonian vector field $X_{h(\phi)}$ and S is a smoothing operator introduced by Conn.
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