Holomorphic distributions transverse to the sphere

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CIRM Marseille, December 1st, 2006

Date . . No.

Our fundamental question : Consider $E \subset T\mathbb{C}^n$ a holomorphic distribution in $T\mathbb{C}^n$, $n \geq 2$, which is transverse to $S^{2n-1}(1) = \partial D^{2n}(1)$ the boundary of the disc $D^{2n}(1)$. What happens E inside $D^{2n}(1)$?

§ Transversality between Z and $S^{2n-1}(1)$

§ Poincaré-Hopf Theorem and Poincaré-Bendixson Theorem

(The case of $\dim E = 1$)

§ The case of $\dim E = 2$

§ A Poincaré-Hopf type theorem for holomorphic one form

(The case of $\operatorname{codim} E = 1$)

§ Application (The case of $\operatorname{codim} E = 1$)

§ Transversality between $Z = z_1 \partial/\partial z_1 + \lambda z_2 \partial/\partial z_2$ and $S^3(1)$

Let $\lambda = a + \sqrt{-1}\, b$ and $z_j = x_j + \sqrt{-1}\, y_j$ $(j = 1, 2)$ be complex numbers. Consider a linear vector field Z on $\mathbb{C}^2$:

$$
\begin{aligned}
Z &= z_1 \partial/\partial z_1 + \lambda z_2 \partial/\partial z_2 \\
&= \frac{1}{2}\Big[x_1 \partial/\partial x_1 + y_1 \partial/\partial y_1 + (a x_2 - b y_2) \partial/\partial x_2 + (b x_2 + a y_2) \partial/\partial y_2 \\
&\qquad - \sqrt{-1}\, \big(-y_1 \partial/\partial x_1 + x_1 \partial/\partial y_1 - (b x_2 + a y_2) \partial/\partial x_2 + (a x_2 - b y_2) \partial/\partial y_2 \big) \Big] \\
&\underset{\text{put}}{=} \frac{1}{2}\left(X - \sqrt{-1}\, Y \right)
\end{aligned}
$$

where J is the almost complex structure of $\mathbb{C}^2$.

Let $\vec{R}$ be the radial vector field:

$$
\begin{aligned}
\vec{R} &= z_1 \partial/\partial z_1 + z_2 \partial/\partial z_2 \\
&= \frac{1}{2}\Big[\left(x_1 \partial/\partial x_1 + y_1 \partial/\partial y_1 + x_2 \partial/\partial x_2 + y_2 \partial/\partial y_2 \right) \\
&\qquad - \sqrt{-1} \left(-y_1 \partial/\partial x_1 + x_1 \partial/\partial y_1 - y_2 \partial/\partial x_2 + x_2 \partial/\partial y_2 \right) \Big] \\
&\underset{\text{put}}{=} \frac{1}{2}\left(\vec{n} - \sqrt{-1}\, J \vec{n} \right)
\end{aligned}
$$

We have an equation of estimation.

<u>Lemma</u> Z is tangent to $S^3(1)$ at $p = (z_1, z_2) \in S^3(1)$

$$\Updownarrow$$

$$\langle X, \vec{n} \rangle = 0 \quad \text{and} \quad \langle JX, \vec{n} \rangle = 0$$

$$\Updownarrow$$

$$\langle Z, \vec{R} \rangle_{\mathbb{C}} = z_1 \cdot \bar{z}_1 + \lambda z_2 \cdot \bar{z}_2 = \langle X, \vec{n} \rangle - \sqrt{-1}\langle JX, \vec{n} \rangle = 0$$

The solution of Z with the initial condition $w = (w_1, w_2) \in S^3(1)$ is $L_1 = \{(z_1, z_2) = (w_1 e^{T}, w_2 e^{\lambda T}) \mid T = s + \sqrt{-1}t \in \mathbb{C}\}$. Consider $L_1 \cap S^3(1) = \{ |w_1|^2 e^{2s} + |w_2|^2 e^{2(as-bt)} = 1 \}$.

<u>Case 1</u> $a = \frac{n}{m} > 0$, $b = 0$ (i.e. λ is positive rational)

$$L_1 \cap S^3(1) = \{ (w_1 e^{s_0} e^{\sqrt{-1}t}, w_2 e^{\frac{n}{m}s_0} \cdot e^{\sqrt{-1}\frac{n}{m}t}) \mid t \in \mathbb{R} \}$$

$\mathcal{F}(Z)|_{S^3(1)}$ is the Seifert fibration.

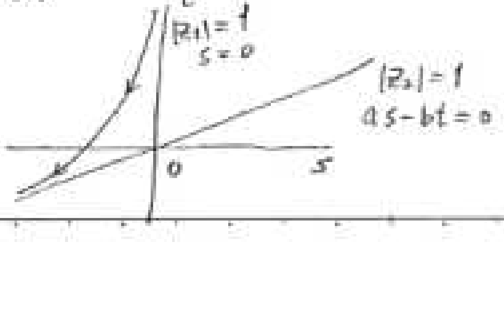
<u>Case 2</u> $b = 0$, $a > 0$ irrational (i.e. λ is positive irrational)

$$L_1 \cap S^3(1) = \{ (w_1 e^{s_0} \cdot e^{\sqrt{-1}t}, w_2 e^{a s_0} \cdot e^{\sqrt{-1}at}) \mid t \in \mathbb{R} \}$$

$\mathcal{F}(Z)|_{S^3(1)}$ is the irrational flow on torus $S^1(|w_1| e^{s_0}) \times S^1(|w_2| e^{a s_0}) \subset S^3(1)$.

<u>Case 3</u> $b \neq 0$, i.e. $\lambda \notin \mathbb{R}$

$\mathcal{F}(Z)|_{S^3(1)}$ is the Morse-Smale foliation.



Definition of transversality between Z and $S^{2n-1}(r)$

Let $\mathcal{F}(Z)$ be the foliation defined by orbits of a holomorphic vector field Z on $\mathbb{C}^n$, $n \geq 2$. $\mathcal{F}(Z)$ is transverse to $S^{2n-1}(r)$
$\sum_{j=1}^{n} f_j(z)\partial/\partial z_j$
if the following equation is satisfied for each point $p \in S^{2n-1}(r)$:

$$T_p(\mathcal{F}(Z)) + T_p S^{2n-1}(r) = T_p \mathbb{R}^{2n}$$

$$\left(\text{i.e.} \quad \sum_{j=1}^{n} f_j(\alpha)\cdot \bar{\alpha}_j \neq 0 \quad \text{for} \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in S^{2n-1}(r) \right)$$

<u>Example</u> Given non-zero complex numbers $\lambda_1, \cdots, \lambda_n \in \mathbb{C}^*$. If the origin $0 \in \mathbb{C}$ does not belong to the convex hull $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$ of the subset $\{\lambda_1, \cdots, \lambda_n\} \subset \mathbb{C}$. Take a linear vector field $Z = \sum_{j=1}^{n} \lambda_j z_j \partial/\partial z_j$, then $\mathcal{F}(Z)$ is transverse to $S^{2n-1}(r)$.

Problem We consider a non-singular vector field Z on $\mathbb{C}^2$:

$Z = (a_1(cz_1+z_2)+b_1)\partial/\partial z_1 + (a_2(cz_1+z_2)+b_2)\partial/\partial z_2$, a_i, b_i, c

$\in \mathbb{C}$ which satisfies the non-singular condition :

(i) if $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq 0$, $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is different from 0.

(ii) if $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$, $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is a non-zero vector.

Classify $\Sigma \cap S^3(r)$ where Σ is the total contact set of spheres and $\mathfrak{F}(Z)$.

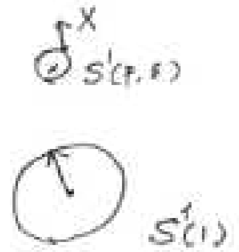
§ Poincaré-Hopf Theorem and Poincaré-Bendixson Theorem

First, we recall the classical theorem of Poincaré-Hopf. Given a smooth differential equation $X : \frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ on $\mathbb{R}^2$ or a smooth vector field $X = f(x,y)\partial/\partial x + g(x,y)\partial/\partial y$. Assume that (1) $\mathrm{Sing}(X) = \{f=0,\ g=0\} \cap D^2(1)$ is finite and (2) X is transverse to the boundary $\partial D^2(1) = S^1(1)$. Then we have the following equation:

$$\sum_{p \in \mathrm{Sing}(X) \cap D^2(1)} \mathrm{Ind}(X, p) = \chi(D^2(1)) = 1$$

where $\chi(D^2(1))$ is the Euler number of $D^2(1)$ and $\mathrm{Ind}(X, p)$ is defined by the degree of map F:

$$F : S^1(p:\varepsilon) \longrightarrow S^1(1)$$
$$(x,y) \longmapsto \frac{(f(x,y),\ g(x,y))}{\sqrt{f^2+g^2}}$$

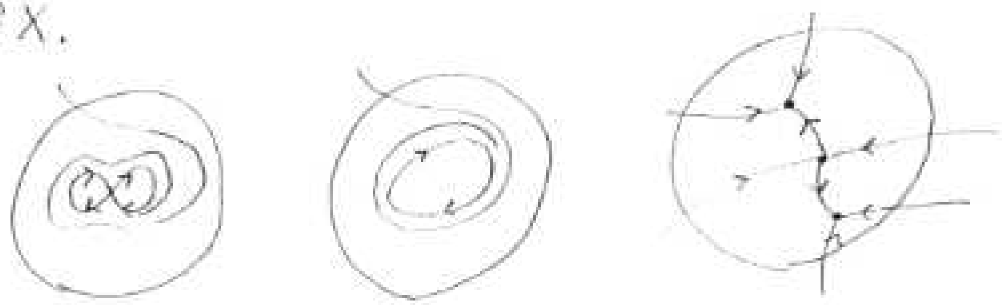


<u>Example</u>

$$\begin{aligned} &\mathrm{Ind}(X, p_1) = 1 \\ &\mathrm{Ind}(X, p_2) = -1 \\ +)\ &\mathrm{Ind}(X, p_3) = 1 \\ \hline &\sum_{i=1}^{3} \mathrm{Ind}(X : p_i) = 1 \end{aligned}$$

Secondly, I explain the Poincaré-Bendixson theorem.

Under the hypothesis, the curve of solution of X which crosses to $\partial D^2(1)$ tends to (i) figure 8 or (ii) closed curve or (iii) singular point of X.



Remark : In the case of $\mathbb{R}^n$, $n \geq 3$, this theorem is not true.

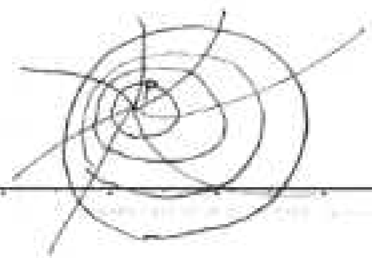
In the case of holomorphic vector field on $\mathbb{C}^n$, $n \geq 2$.

Theorem (A. Douady and T. Ito) Assume that Z is transverse to the boundary $\partial D^{2n}(1)$. Then,

(1) there exists only one singular point p of Z inside $D^{2n}(1)$,

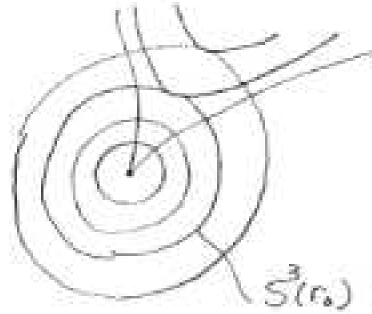
$\mathrm{Ind}(Z, p) = 1$, $\det(D(Z)(p)) \neq 0$

(2) each solution of Z which crosses to $D^{2n}(1)$ tends to the singular point p. More particulars, by Möbius



transformation we map p to the origin O, each sphere $S^{2n-1}(r)$, $0<r\leq 1$, is transverse to Z.

<u>Example</u> Take $Z = (2z_1 + az_2^2)\partial/\partial z_1 + z_2\partial/\partial z_2$. $\mathrm{Sing}(Z)$ consists of a singular point O. There exists a number $r_0>0$ such that (i) if $0<r<r_0$, Z is transverse to $S^3(r)$, (ii) if $r\geq r_0$, Z is not transverse to $S^3(r)$.



$S^3(r_0)$

We set $\Sigma_1 = \{ z\in\mathbb{C}^n \mid (2z_1 + az_2^2)\bar{z}_1 + z_2\cdot\bar{z}_2 = 0\}$.

(a) $\Sigma_1\cap S^3(r_0)$ is diffeomorphic to the circle S^1 and consists of degenerate critical points.

(b) $\Sigma_1\cap S^3(r)$, $r>r_0$, is diffeomorphic to the disjoint union $S^1\amalg S^1$ of two copies of the circle S^1. One circle of $\Sigma_1\cap S^3(r)$ consists of minimal points and the other consists of saddle points.

Example We consider $Z = z_1(1+\sqrt{1-z_1z_2})\partial/\partial z_1 + z_2(1-\sqrt{1-z_1z_2})\partial/\partial z_2$ on $\mathbb{C}^2$. The singular set consists of a single point 0. If $0<r<\sqrt{2}$, $\Sigma\cap S^3(r)$ is empty. If $r=\sqrt{2}$, $\Sigma\cap S^3(\sqrt{2})$ is diffeomorphic to the circle S^1. Indeed $\Sigma\cap S^3(\sqrt{2})$ belongs to the solution $z_1z_2=1$ of Z. If $r>\sqrt{2}$, $\Sigma\cap S^3(r)$ is diffeomorphic to the disjoint union $S^1\amalg S^1$ of two copies of the circle S^1 and consists of saddle points.

Problem Let Z be a holomorphic vector field (on $\mathbb{C}^n$) with two properties : (i) $\mathrm{Sing}(Z)=\{0\}$, (ii) if $0<r<r_0$, $\Sigma\cap S^{2n-1}(r)$ is empty. In this situation, if $\Sigma\cap S^{2n-1}(r_0)$ is not empty, then is Σ connected in a neighborhood U of $z\in\Sigma\cap S^{2n-1}(r_0)$?

§3 The case of $\dim E = 2$

In this section, we give an example.

Example. (T. Ito and M. Yoshino) Take complex numbers $\lambda_1, \cdots, \lambda_n, \mu_1, \cdots, \mu_n \in \mathbb{C}^*$ and assume that the origin 0 belongs to $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$ and $\mathcal{H}(\mu_1, \cdots, \mu_n)$. We make the following assumption: There exist real numbers c_1 and c_2 such that $\mathcal{H}(c_1\lambda_1 + c_2\mu_1, \cdots, c_1\lambda_n + c_2\mu_n) \not\ni 0$. Consider linear vector field $X = \sum_{j=1}^{n} \lambda_j z_j \partial/\partial z_j$ and $Y = \sum_{j=1}^{n} \mu_j z_j \partial/\partial z_j$. Then it is clear that $[X, Y] = 0$ so that X and Y span a foliation $\mathcal{F}$ of complex dimension two on $\mathbb{C}^n$. Also $\mathcal{F}$ has as singular set $\mathrm{Sing}(\mathcal{F})$ the union of the coordinate axis. Denote by $\Sigma(X)$ the set of tangent points of X with the spheres $S^{2n-1}(r) \subset \mathbb{C}^n$, any $r \geqq 0$, then we have $\Sigma(X)$ given by the equation $\sum_{j=1}^{n} \lambda_j |z_j|^2 = 0$. This is a real cone. Analogously we define $\Sigma(Y)$ and describe it by the equation $\sum_{j=1}^{n} \mu_j |z_j|^2 = 0$. Under the assumption, we have $\Sigma(X) \cap \Sigma(Y) = \{0\}$.

$\mathcal{F}$ is transverse to $S^{2n-1}(r) \setminus (\mathrm{Sing}(\mathcal{F}) \cap S^{2n-1}(r))$, $r > 0$.

Moreover each leaf of $\mathcal{F}$ accumulates the origin.

§ A Poincaré-Hopf type theorem for holomorphic one-form

Let $\omega = \sum_{j=1}^{n} f_j(z)\,dz_j$ be a holomorphic one form on $\mathbb{C}^n$, $n \geq 2$, and $P_\omega = \{ z \in T\mathbb{C}^n \mid \omega(z) = 0 \} \subset T\mathbb{C}^n$ the corresponding holomorphic distribution. Denote by $Sing(\omega) = \{ f_1 = 0, \cdots, f_n = 0 \}$ the singular set of ω.

<u>Definition</u> P_ω is transverse to the sphere $S^{2n-1}(1)$ if P_ω satisfies (1) and (2) : (1) $Sing(\omega) \cap S^{2n-1}(1)$ is empty and (2) $(P_\omega)_z + T_z S^{2n-1}(1) = T_z R^{2n}$ for all $z \in S^{2n-1}(1)$.

For $p \in Sing(\omega)$, $Ind(\omega : p)$ means the degree of the map F: $S^{2n-1}(p:\varepsilon) \longrightarrow S^{2n-1}(1)$ defined by $F(z) = \dfrac{(f_1(z), \cdots, f_n(z))}{\sqrt{|f_1(z)|^2 + \cdots + |f_n(z)|^2}}$.

<u>Theorem</u> (T. Ito and B. Scárdua) If P_ω is transverse to sphere $S^{2n-1}(1)$, we have the following equation :

$$\sum_{p \in Sing(\omega) \cap D^{2n}(1)} Ind(\omega : p) = (-1)^n \chi(D^{2n}(1)) = (-1)^n$$

<u>Corollary</u> (i) n is even.

(ii) ω has exactly one singular point $p \in D^{2n}(1)$

(iii) $\det\left(\frac{\partial f_i}{\partial z_k}(p)\right) \neq 0$

<u>Example</u> Take $\omega = \sum_{j=1}^{n} (z_{2j}\, dz_{2j-1} - z_{2j-1}\, dz_{2j})$ on $\mathbb{C}^{2n}$.
ω is not integrable because $\omega \wedge d\omega = -2z_4\, dz_1 \wedge dz_2 \wedge dz_3 - \cdots \neq 0$. Since $\omega(\vec{R}) = 0$, P_ω is transverse to $S^{4n-1}(1)$ and $\mathrm{Sing}(\omega) = \{0\}$.

<u>Example</u> Let $\omega = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} z_j\right) dz_i$ be a linear one form on $\mathbb{C}^n$, $n \geq 3$, satisfied with (1) $\mathrm{Sing}(\omega) \cap S^{2n-1}(1) = \phi$ and (2) $\omega \wedge d\omega = 0$. Then P_ω is not transverse to $S^{2n-1}(1)$.

<u>Example</u> Take a logarithmic one form $\omega = z_1 z_2 z_3 \left(\sum_{j=1}^{3} \lambda_j \frac{dz_j}{z_j}\right)$ on $\mathbb{C}^3$, where $\lambda_j \in \mathbb{C}^*$, $j = 1, 2, 3$, $\lambda_i / \lambda_j \notin \mathbb{R}$, $i \neq j$, and $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$. $\mathcal{F}(\omega)$ satisfies

(i) $\mathrm{Sing}(\omega) = \bigcup_{i \neq j} \{z_i = 0,\ z_j = 0\}$

(ii) $\mathcal{F}(\omega)$ is transverse to $S^5(1)$ outside $\mathrm{Sing}(\omega) \cap S^5(1)$.

<u>Problem</u> Let ω be a holomorphic (integrable) one form on $\mathbb{C}^3$. Assume $\mathrm{Codim}(\mathrm{Sing}(\omega)) \geq 2$. If $\mathcal{F}(\omega)$ is transverse to $S^5(1)$ outside $\mathrm{Sing}(\omega) \cap S^5(1)$, what happens $\mathcal{F}(\omega)$ in $D^6(1)$?

§ Applications

Let $\omega = \sum_{j=1}^{2m} f_j(z) dz_j$ be a holomorphic one form on $\mathbb{C}^{2m}$ (a neighborhood U of $D^{4m}(1) \subset$). Assume $P\omega$ is transverse to $S^{4m-1}(1)$.

<u>Theorem</u> (Ito and Scárdua) Under the above hypothesis, if there exists a holomorphic vector field ξ on U such that ξ is transverse to $S^{4m-1}(1)$ and $\omega(\xi) = 0$ in a neighborhood of $S^{4m-1}(1)$, then ω is not integrable.

Denote by $\mathcal{A}(2m)$ the set of all $2m \times 2m$ skew-symmetric complex matrices and $\mathcal{A}^*(2m)$ the subset of non-singular elements in $\mathcal{A}(2m)$. If $A = (a_{ij})$ belongs to $\mathcal{A}^*(2m)$, then for $m \geqq 2$, the one form $\Omega_A = \sum_{i,j=1}^{2m} a_{ij} z_i dz_j$ defines a non-integrable distribution transverse to the spheres $S^{4m-1}(r) \subset \mathbb{C}^{2m}$, $r > 0$. A particular case is the one-form $\Omega_{\mathbb{J}(2m)} = \sum_{j=1}^{m} (z_{2j-1} dz_{2j} - z_{2j} dz_{2j-1})$.

<u>Theorem</u> (Ito and Scárdua) Let $m \geqq 2$. Given $A \in \mathcal{A}^*(2m)$, $\mathrm{Ker}(\Omega_{J(2m)})$ and <u>$\mathrm{Ker}(\Omega_A)$ admit no integral manifold through the origin.</u>

Theorem (Ito and Scárdua) Let ω be a holomorphic one-form in a neighborhood U of $D^{4m}(1) \subset \mathbb{C}^{2m}$ and such that (1) $\omega(\vec{R}) = 0$, where $\vec{R}$ is the radial vector field in $\mathbb{C}^{2m}$ and (2) $\mathrm{Sing}(\omega) \cap S^{4m-1}(1) = \phi$. Then $\mathrm{Ker}(\omega)$ is homotopic to the linear distribution $\mathrm{Ker}(\Omega_{\mathbb{J}(2m)})$ by distribution $\mathrm{Ker}(\omega_s)$, $0 \le s \le 1$, such that $\omega_0 = \omega$ and $\omega_1 = \Omega_{\mathbb{J}(2m)}$, where ω_s is holomorphic and satisfies (1) and (2) above.

Theorem (Ito and Scárdua) Let ω be a holomorphic, integrable one form in a neighborhood U of $D^{4m}(1)$, $m \geqq 2$ and transverse to the boundary $S^{4m-1}(1)$. If $\mathcal{F}(\omega)$ has some leaf L_0 with $0 \in \overline{L_0}$ and which is closed in $U \setminus \mathrm{Sing}(\omega)$ and transverse to every sphere $S^{4m-1}(r)$, $0 < r \le 1$, then $n = 2$.