A class of Calogero type reductions of free motion on a simple Lie group

L. FEHÉR[a] and B.G. PUSZTAI[b]

[a]Department of Theoretical Physics, MTA KFKI RMKI
1525 Budapest 114, P.O.B. 49, Hungary, and
Department of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, H-6720 Szeged, Hungary
e-mail: lfeher@rmki.kfki.hu

[b]Centre de recherches mathématiques, Université de Montréal
C.P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7, and
Department of Mathematics and Statistics, Concordia University
1455 de Maisonneuve Blvd. West, Montréal, Québec, Canada H3G 1M8
e-mail: pusztai@CRM.UMontreal.CA

Abstract

The reductions of the free geodesics motion on a non-compact simple Lie group G based on the $G_+ \times G_+$ symmetry given by left- and right-multiplications for a maximal compact subgroup $G_+ \subset G$ are investigated. At generic values of the momentum map this leads to (new) spin Calogero type models. At some special values the 'spin' degrees of freedom are absent and we obtain the standard BC_n Sutherland model *with three independent coupling constants* from $SU(n+1,n)$ and from $SU(n,n)$. This generalization of the Olshanetsky-Perelomov derivation of the BC_n model with two independent coupling constants from the geodesics on G/G_+ with $G = SU(n+1,n)$ relies on fixing the right-handed momentum to a non-zero character of G_+. The reductions considered permit further generalizations and work at the quantized level, too, for non-compact as well as for compact G.

1 Introduction

The 'Calogero type' integrable models of interacting particles on the line are interesting on account of their physical applications and relationships to important fields of mathematics. Generalizations of the original model [1] can be associated with root systems in correspondence with various admissible interaction potentials and possible couplings to internal 'spin' degrees of freedom and to external fields. The richness of these models is demonstrated by the growing number of reviews devoted to them [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. One of the basic models of the family is the hyperbolic BC_n Sutherland model defined classically by the Hamiltonian

$$\mathcal{H}_{BC_n} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{1\leq j<k\leq n} \left(\frac{g^2}{\sinh^2(q^j - q^k)} + \frac{g^2}{\sinh^2(q^j + q^k)}\right) + \sum_{k=1}^{n}\left(\frac{g_1^2}{\sinh^2(q^k)} + \frac{g_2^2}{\sinh^2(2q^k)}\right) \tag{1.1}$$

with arbitrary coupling constants g, g_1, g_2. Olshanetsky and Perelomov [14, 15, 2] showed that this model can be viewed as a 'projection' of the geodesic system on the symmetric space $SU(n+1,n)/(S(U(n+1)\times U(n))$ if the coupling constants obey the quadratic relation $g_1^2 - 2g^2 + \sqrt{2}gg_2 = 0$. For arbitrary coupling constants, classical and quantum solvability of the model was established by means of different, rather algebraic, methods [16, 17, 18, 19].

Since Hamiltonian reduction is a very effective and general approach to integrable systems, it would be interesting to lift the above quadratic relation of Olshanetsky and Perelomov sticking to this method. Motivated partly by this problem, recently we undertook a systematic study of reductions of the free geodesic motion on Riemannian symmetric spaces, which led to new spin Calogero models as well as to an understanding of the geometric origin of the quadratic relation [20]. Here, we extend this work by going one stage up and explore the reductions of the geodesic system defined on the isometry group of the symmetric space. We shall demonstrate that the classical BC_n model (1.1) with three independent coupling constants can be obtained by Hamiltonian reduction in this extended framework.

The geodesic system on a symmetric space, realized as a coset space G/G_+, is a reduction of the geodesic system on the isometry group G, belonging to the zero value of the momentum map for the action of the little group G_+ on T^*G generated by right-multiplications. This system then can be reduced to spin Calogero models using the residual symmetry generated by the left-multiplications associated with G_+. It is clear that more general reduced systems result if one fixes the right-handed momentum to some non-zero value. First, we shall describe the most general reductions of T^*G that rely on the action of $G_+ \times G_+$ through left- and right multiplications. In fact, one obtains (new) spin Calogero type models in general, with the spin degrees of freedom restricted to a trivial one-point space in certain very special cases. Second, we observe that if the space of spin degrees of freedom is trivial for the zero value of the 'right-handed' momentum map, then this feature can be ensured also for any non-zero character of G_+. Such characters, forming one-point coadjoint orbits of G_+, exist precisely if G/G_+ is a Hermitian symmetric space. Taking advantage of this observation, we can derive the BC_n model with three independent coupling constants from the geodesic motion on $SU(m,n)$ both for $m = n$ and for $m = n+1$. The model with two independent coupling constants is obtained from $SU(m,n)$ for any $m \geq (n+2)$.

The main results of this letter are the characterization of the reductions of the geodesic system on a real simple Lie group G under the $G_+ \times G_+$ symmetry presented in Section 2, where G is non-compact and G_+ is a maximal compact subgroup, and the derivation of the model (1.1) contained in Section 3. Our derivation of the classical BC_n models should be compared with the work of Oblomkov [21], where a three parameter family of the BC_n type Jacobi polynomials of Heckman and Opdam (see e.g. [5]), which underlie the eigenstates of the quantum Hamiltonian corresponding to the trigonometric version of (1.1), is interpreted in terms of generalized spherical functions on the coset space $GL(m+n,\mathbb{C})/(GL(m,\mathbb{C})\times GL(n,\mathbb{C}))$. The investigation of the quantum mechanical analogue of our construction is currently in progress. In general it seems to work differently from the corresponding construction of [21], detailed comparison will be given elsewhere. See also Section 4 for a sketch of the quantization and for further remarks.

2 From free motion to spin Calogero type models

Next we briefly recall some group theoretic background material and introduce our notations, then describe the Hamiltonian reductions of the free particle on a Lie group to spin Calogero type models in a convenient framework. The relevant Lie theoretic results are treated in detail in [22, 23], and we refer to [4, 24] for reviews of symplectic geometry and Hamiltonian reduction.

Let G be a non-compact real simple Lie group with finite centre and $\mathcal{G}$ its Lie algebra. Up to conjugation there exists a unique Cartan involution[1] Θ of G, for which the associated automorphism θ of $\mathcal{G}$ induces the decomposition

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-, \qquad \theta(X_\pm) = \pm X_\pm \quad \forall X_\pm \in \mathcal{G}_\pm, \tag{2.1}$$

where the restriction of the Killing form $\langle\ ,\ \rangle$ of $\mathcal{G}$ is negative (resp. positive) definite on $\mathcal{G}_+$ (resp. on $\mathcal{G}_-$). The fixed point set of Θ is a maximal compact subgroup $G_+ \subset G$ with Lie algebra $\mathcal{G}_+$. The elements of $\mathcal{G}_-$ are diagonalizable, with real eigenvalues, in the adjoint representation of $\mathcal{G}$ and it is useful to fix a maximal Abelian subspace $\mathcal{A} \subset \mathcal{G}_-$. The choice of $\mathcal{A}$ leads to the refined decomposition

$$\mathcal{G}_- = \mathcal{A} + \mathcal{A}^\perp, \qquad \mathcal{G}_+ = \mathcal{M} + \mathcal{M}^\perp, \tag{2.2}$$

with

$$\mathcal{M} := \{ X \in \mathcal{G}_+ \,|\, [X,Y] = 0 \ \ \forall Y \in \mathcal{A} \} \tag{2.3}$$

and the complementary spaces $\mathcal{A}^\perp$, $\mathcal{M}^\perp$ defined with the aid of $\langle\ ,\ \rangle$. We may write any $X \in \mathcal{G}$ as $X = X_- + X_+ = X_{\mathcal{A}} + X_{\mathcal{A}^\perp} + X_{\mathcal{M}} + X_{\mathcal{M}^\perp}$ according to (2.1) and (2.2). We also need the group corresponding to $\mathcal{M}$, the centralizer of $\mathcal{A}$ in G_+,

$$M := \{ m \in G_+ \,|\, mYm^{-1} = Y \quad \forall Y \in \mathcal{A} \}. \tag{2.4}$$

We remind in passing that the Weyl group of the Riemannian symmetric space G/G_+ is $W := \hat{M}/M$, where $\hat{M}$ is the normalizer of $\mathcal{A}$ in G_+.

Let us call an element of $\mathcal{A}$ *regular* if its kernel in the adjoint representation of $\mathcal{G}$ is $\mathcal{A}+\mathcal{M}$. The set of regular elements, denoted as $\hat{\mathcal{A}} \subset \mathcal{A}$, is the union of its connected components and

[1] Notationwise we pretend that G is a matrix group. One may also think of Θ concretely as $\Theta(g) = (g^{-1})^\dagger$.

we choose an open Weyl chamber $\check{\mathcal{A}} \subset \hat{\mathcal{A}}$ to be such a connected component. The regular elements of G can be characterized by admitting a decomposition of the form

$$g = g_+ e^q h_+ \qquad q \in \check{\mathcal{A}}, \quad g_+, h_+ \in G_+, \tag{2.5}$$

and this decomposition is unique up to replacing (g_+, h_+) by $(g_+ m, m^{-1} h_+)$ for any $m \in M$.

Denote by $\check{G} \subset G$ the open dense submanifold formed by the regular elements. From now on identify the dual space $\mathcal{G}^*$ with $\mathcal{G}$ by means of the Killing form $\langle\ ,\ \rangle$. Then, using the trivialization defined by right-translations on G, consider the cotangent bundle $T^*\check{G}$,

$$P := T^*\check{G} \simeq \check{G} \times \mathcal{G} = \{(g, J^l) \,|\, g \in \check{G},\ \ J^l \in \mathcal{G}\}, \tag{2.6}$$

equipped with the symplectic form Ω and the Hamiltonian $\mathcal{H}$ of the geodesic system,

$$\Omega = d\langle J^l, (dg)g^{-1}\rangle, \qquad \mathcal{H}(g, J^l) = \frac{1}{2}\langle J^l, J^l\rangle. \tag{2.7}$$

J^l generates the left-translations and $J^r = -g^{-1} J^l g$ generates the right-translations on T^*G. We shall perform Hamiltonian symmetry reduction relying on the subgroup $G_+ \times G_+$ of $G \times G$.

To study the reductions of the geodesic system, it is convenient to first extend it as follows. Take two arbitrary coadjoint orbits of G_+, say $(\mathcal{O}^l, \omega^l)$ and $(\mathcal{O}^r, \omega^r)$. The orbits are realized as submanifolds of $\mathcal{G}_+ \simeq \mathcal{G}_+^*$ and $\omega^{l,r}$ denote their own symplectic forms. The extended system is $(P^{\rm ext}, \mathcal{H}^{\rm ext}, \Omega^{\rm ext})$:

$$P^{\rm ext} := P \times \mathcal{O}^l \times \mathcal{O}^r = \{(g, J^l, \xi^l, \xi^r) \,|\, g \in \check{G},\ J^l \in \mathcal{G},\ \xi^l \in \mathcal{O}^l,\ \xi^r \in \mathcal{O}^r\}, \tag{2.8}$$

$$\Omega^{\rm ext} := \Omega + \omega^l + \omega^r, \qquad \mathcal{H}^{\rm ext}(g, J^l, \xi^l, \xi^r) := \mathcal{H}(g, J^l). \tag{2.9}$$

Using the Poisson bracket associated with $\Omega^{\rm ext}$, the corresponding equation of motion reads

$$\dot{g} = \{g, \mathcal{H}^{\rm ext}\} = J^l g, \quad \dot{J}^l = \{J^l, \mathcal{H}^{\rm ext}\} = 0, \quad \dot{\xi}^\lambda = \{\xi^\lambda, \mathcal{H}^{\rm ext}\} = 0 \quad \text{for} \quad \lambda = l, r. \tag{2.10}$$

The solution with initial value $(g(0), J^l, \xi^l, \xi^r)$ yields the geodesic $g(t) = e^{tJ^l} g(0)$.

Now we consider the reduction of the above system based on the symmetry group $G_+ \times G_+$. Any $(g_+^l, g_+^r) \in G_+ \times G_+$ operates by the transformation $T(g_+^l, g_+^r) \in \mathrm{Diff}(P^{\rm ext})$ defined by

$$T(g_+^l, g_+^r) : \left(g, J^l, \xi^l, \xi^r\right) \mapsto \left(g_+^l g (g_+^r)^{-1}, g_+^l J^l (g_+^l)^{-1}, g_+^l \xi^l (g_+^l)^{-1}, g_+^r \xi^r (g_+^r)^{-1}\right). \tag{2.11}$$

The equivariant momentum map, $\Psi = (\Psi^l, \Psi^r) : P^{\rm ext} \to \mathcal{G}_+^* \oplus \mathcal{G}_+^*$, for this Hamiltonian action is furnished by

$$\Psi(g, J^l, \xi^l, \xi^r) = (J_+^l + \xi^l, -(g^{-1} J^l g)_+ + \xi^r), \tag{2.12}$$

where the factors $\mathcal{G}_+^*$ are identified with $\mathcal{G}_+$ using the scalar product, the elements of $\mathcal{G}_+ \oplus \mathcal{G}_+$ are denoted as ordered pairs, and $J^l = J_+^l + J_-^l$ according to (2.1). We are interested in the reduced Hamiltonian system

$$(P_{\rm red}, \Omega_{\rm red}, \mathcal{H}_{\rm red}) \tag{2.13}$$

obtained from $(P^{\rm ext}, \Omega^{\rm ext}, \mathcal{H}^{\rm ext})$ at the zero value of the momentum map Ψ, i.e.,

$$P_{\rm red} := P^{\rm ext}_{\Psi=0} / (G_+ \times G_+). \tag{2.14}$$

It is easy to see that this is equivalent to the (singular) Marsden-Weinstein reduction of the original system $(P, \Omega, \mathcal{H})$ at an arbitrary value $(-\mu^l, -\mu^r)$ of the corresponding momentum map, (J^l_+, J^r_+), with $\mu^l \in \mathcal{O}^l$, $\mu^r \in \mathcal{O}^r$. (The last statement is known as the 'shifting trick' in the theory of Hamiltonian reduction [24].) We assume in what follows that $P^{\rm ext}_{\Psi=0}$ is *non-empty*, which is a condition on the pair of orbits $\mathcal{O}^l$, $\mathcal{O}^r$. In fact, the condition that $\Psi(g, J^l, \xi^l, \xi^r) = 0$ admits a solution on $P^{\rm ext}$ is equivalent to the consistency of (2.20) below for some $\xi^l \in \mathcal{O}^l$ and $\xi^r \in \mathcal{O}^r$.

Now we are ready to characterize the reduced Hamiltonian system defined above. The key step is to utilize that all $G_+ \times G_+$ orbits in the constrained manifold $P^{\rm ext}_{\Psi=0}$ intersect the following gauge slice:

$$S := \{(e^q, J^l, \xi^l, \xi^r) \in P^{\rm ext}_{\Psi=0} \,|\, q \in \check{\mathcal{A}}\}, \tag{2.15}$$

since every regular element of G can be transformed into $\exp(\check{\mathcal{A}})$ by means of the action (2.11). The gauge slice S represents only a partial gauge fixing of the gauge transformations defined by the $G_+ \times G_+$ action (2.11). The residual gauge transformations (the maps that transform an arbitrarily chosen point of S into S) are generated by the subgroup

$$M_{\rm diag} := \{(m, m) \in G_+ \times G_+ \,|\, m \in M\}. \tag{2.16}$$

$M_{\rm diag}$ is naturally isomorphic to, and is below often identified with, M. At this point we arrived at the model

$$P_{\rm red} = P^{\rm ext}_{\Psi=0}/(G_+ \times G_+) = S/M_{\rm diag}. \tag{2.17}$$

To describe $P_{\rm red}$ more explicitly, we use the orthogonal complement of the Lie algebra $\mathcal{M}_{\rm diag} \subset \mathcal{G}_+ \oplus \mathcal{G}_+$ of $M_{\rm diag}$,

$$\mathcal{M}^\perp_{\rm diag} = \{(X_1, X_2) \in \mathcal{G}_+ \oplus \mathcal{G}_+ \,|\, \langle X_1 + X_2, V\rangle = 0 \quad \forall V \in \mathcal{M}\}, \tag{2.18}$$

with respect to the scalar product $\langle (X_1, X_2), (Y_1, Y_2)\rangle_+ = \langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle$ on $\mathcal{G}_+ \oplus \mathcal{G}_+$. By decomposing $J^l \in \mathcal{G}$ and $\xi^\lambda \in \mathcal{O}^\lambda \subset \mathcal{G}_+$ $(\lambda = l, r)$ according to (2.2),

$$J^l = J^l_{\mathcal{A}} + J^l_{\mathcal{A}^\perp} + J^l_{\mathcal{M}} + J^l_{\mathcal{M}^\perp}, \qquad \xi^\lambda = \xi^\lambda_{\mathcal{M}} + \xi^\lambda_{\mathcal{M}^\perp}, \tag{2.19}$$

and using that ad_q $(\forall q \in \check{\mathcal{A}})$ yields a linear bijection between $\mathcal{M}^\perp$ and $\mathcal{A}^\perp$, the constraint $\Psi = 0$ on S can be solved as follows. In fact, the condition $\Psi = 0$ on S is equivalent to the equations

$$\xi^l_{\mathcal{M}} + \xi^r_{\mathcal{M}} = 0 \tag{2.20}$$

and

$$J^l = J^l_{\mathcal{A}} - F(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp} - w(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp} - \xi^l, \tag{2.21}$$

where $J^l_{\mathcal{A}} \in \mathcal{A}$ is arbitrary and F, w are the analytic functions

$$F(z) = \coth z, \qquad w(z) = \frac{1}{\sinh z}. \tag{2.22}$$

Equation (2.20) ensures that $(\xi^l, \xi^r) \in \mathcal{M}^\perp_{\rm diag}$ and, using $J^r = -e^{-\mathrm{ad}_q} J^l$, (2.21) could be alternatively expressed as

$$J^r = J^r_{\mathcal{A}} + F(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp} + w(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp} - \xi^r, \qquad J^r_{\mathcal{A}} = -J^l_{\mathcal{A}}. \tag{2.23}$$

Motivated by the parametrization (2.21), let us introduce the smooth one-to-one map

$$I : (\check{\mathcal{A}} \times \mathcal{A}) \times (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\rm diag} \to S,$$
$$I(q, p, \xi^l, \xi^r) := (e^q, \mathcal{L}(q, p, \xi^l, \xi^r), \xi^l, \xi^r) \tag{2.24}$$

with

$$\mathcal{L}(q, p, \xi^l, \xi^r) := p - F(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp} - w(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp} - \xi^l. \tag{2.25}$$

The pull-back of $\Omega^{\rm ext}|_S$ by I, where $\Omega^{\rm ext}|_S$ is the pull-back of $\Omega^{\rm ext}$ to the submanifold $S \subset P^{\rm ext}$, turns out to be

$$I^*(\Omega^{\rm ext}|_S) = d\langle p, dq\rangle + \left(\omega^l + \omega^r\right)|_{(\mathcal{O}^l \oplus \mathcal{O}^r)\cap \mathcal{M}^\perp_{\rm diag}}. \tag{2.26}$$

The first term is the canonical symplectic structure of $T^*\check{\mathcal{A}} \simeq \check{\mathcal{A}} \times \mathcal{A} = \{(q, p)\}$. The second term in (2.26) is the restriction of $\omega^l + \omega^r$ to the zero level set of the momentum map for the action of the group $M \simeq M_{\rm diag}$ on $\mathcal{O}^l \oplus \mathcal{O}^r$, provided by $(\xi^l, \xi^r) \mapsto (\xi^l_{\mathcal{M}} + \xi^r_{\mathcal{M}}) \in \mathcal{M} \simeq \mathcal{M}^*$. Notice that I is an M equivariant map, where M acts trivially on $T^*\check{\mathcal{A}}$. On account of its equivariance, the map I gives rise to the identification $S/M_{\rm diag} = T^*\check{\mathcal{A}} \times \mathcal{O}_{\rm red}$ with

$$\mathcal{O}_{\rm red} := (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\rm diag}/M_{\rm diag}. \tag{2.27}$$

In terms of the model of S in (2.24), the Hamiltonian of the geodesic motion takes the form

$$(\mathcal{H}^{\rm ext} \circ I)(q, p, \xi^l, \xi^r) = \frac{1}{2}\langle \mathcal{L}(q, p, \xi^l, \xi^r), \mathcal{L}(q, p, \xi^l, \xi^r)\rangle. \tag{2.28}$$

By collecting the above formulae and applying some algebra to spell out the Hamiltonian relying on the identity

$$F(z)w(z) = \frac{1}{2}w^2(\frac{z}{2}) - w^2(z), \tag{2.29}$$

we obtain our

Main result: *The reduced geodesic system* $(P_{\rm red}, \Omega_{\rm red}, \mathcal{H}_{\rm red})$ *defined above can be identified as*

$$P_{\rm red} = T^*\check{\mathcal{A}} \times \mathcal{O}_{\rm red}, \qquad \Omega_{\rm red} = d\langle p, dq\rangle + \omega_{\rm red}, \tag{2.30}$$

where q, p *are the natural variables on* $T^*\check{\mathcal{A}}$ *and* $(\mathcal{O}_{\rm red}, \omega_{\rm red})$ *(2.27) is the symplectic reduction of* $\mathcal{O}^l \oplus \mathcal{O}^r$ *by the subgroup* $M_{\rm diag} \subset G_+ \times G_+$ *at the zero value of its momentum map. The reduced Hamiltonian yields a hyperbolic spin Calogero type model in general, since as an* M *invariant function on* $T^*\check{\mathcal{A}} \times (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\rm diag}$ *it has the form*

$$\begin{aligned}\mathcal{H}_{\rm red}(q, p, \xi^l, \xi^r) = &\frac{1}{2}\langle p, p\rangle - \frac{1}{2}\langle \xi^l_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle - \frac{1}{2}\langle \xi^r_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp}\rangle \\ &+\frac{1}{2}\langle \xi^l_{\mathcal{M}}, \xi^l_{\mathcal{M}}\rangle + \langle \xi^r_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle - \frac{1}{2}\langle \xi^r_{\mathcal{M}^\perp}, w^2(\frac{1}{2}\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle,\end{aligned} \tag{2.31}$$

where $w(z) = \frac{1}{\sinh z}$ *and* $\xi^l_{\mathcal{M}} + \xi^r_{\mathcal{M}} = 0$.

Now some remarks are in order. First, note that our spin Calogero models enjoy Weyl group symmetry analogously to all the standard Calogero type models. This symmetry is not explicit in the above since the different Weyl chambers are permuted by the Weyl group W and we have gauge fixed the coordinate variable q to a single chamber $\check{\mathcal{A}}$. However, we could have used the larger gauge slice, $\hat{S}$, in our derivation, which differs from S (2.15) only in that q runs over the full set of regular elements $\hat{\mathcal{A}} \subset \mathcal{A}$. The corresponding residual gauge transformations belong to the normalizer $\hat{M}$, and it is easily seen that

$$P_{\rm red} = S/M = \hat{S}/\hat{M} = \hat{P}_{\rm red}/W \quad \text{with} \quad \hat{P}_{\rm red} := \hat{S}/M = T^*\hat{\mathcal{A}} \times \mathcal{O}_{\rm red}. \tag{2.32}$$

The point is that the spin Calogero model defined on $\hat{P}_{\rm red}$ is invariant with respect to the natural action of $W = \hat{M}/M$ induced by the action of $\hat{M} \simeq \hat{M}_{\rm diag} \subset G_+ \times G_+$ on $\hat{S}$.

The structure of the reduced phase space given by (2.30), (2.32) is consistent with general results on reduced cotangent bundles derived in [25] under the assumption that only one isotropy type appears for the action of the symmetry group on the configuration space. Indeed, the isotropy group of any element (2.5) of $\check{G}$ is conjugate to $M_{\rm diag}$ for the action of $G_+ \times G_+$. On account of the factor $\mathcal{O}_{\rm red}$ (2.27), $P_{\rm red}$ (2.30) is not a smooth manifold in general. This does not cause any difficulty, since one can define the smooth functions on $P_{\rm red}$ to be the smooth, gauge invariant functions on $P^{\rm ext}_{\Psi=0}$. For a review of singular symplectic reduction, see [24].

The solutions of the reduced system $(P_{\rm red}, \Omega_{\rm red}, \mathcal{H}_{\rm red})$ can be obtained algebraically, by projecting the obvious solution curves of $(P^{\rm ext}, \Omega^{\rm ext}, \mathcal{H}^{\rm ext})$ (2.10) that satisfy the constraint $\Psi = 0$. All spin Calogero models that arise by reduction are integrable in this direct sense. These models naturally possess many constants of motion, too. Indeed, J^λ and ξ^λ ($\lambda = l, r$) are conserved quantities for the dynamics (2.10) on $P^{\rm ext}$, and any combination of them that is invariant with respect to the $G_+ \times G_+$ symmetry transformations (2.11) induces a constants of motion for the reduced system. For example, consider the function on $P_{\rm red}$ induced by

$$h(K^\lambda(v)) \quad \text{with} \quad K^\lambda(v) := J^\lambda_- - v\xi^\lambda, \tag{2.33}$$

where v is any real parameter, $\lambda \in \{l, r\}$, and h is a G invariant real function on $\mathcal{G}$. A straightforward calculation, similar to Section 4 in [20], shows that all constants of motion of the form (2.33) are in involution on $P_{\rm red}$. (Notice that $K^\lambda(1)$ becomes equal to J^λ on $P^{\rm ext}_{\Psi=0}$.) The Liouville integrability of the reduced systems could be shown starting from these remarks.

If desired, one may also construct Lax pairs as follows. Let $\sigma \subseteq S$ (2.15) denote a gauge slice (of a partial or complete gauge fixing) and for any $v \in \mathbb{R}$ define $L^\lambda(v) : \sigma \to \mathcal{G}$ by

$$L^\lambda(v) := K^\lambda(v)|_\sigma. \tag{2.34}$$

With respect to the projection of the Hamiltonian vector field (2.10) to σ, $L^\lambda(v)$ is found to satisfy a Lax equation

$$\dot{L}^\lambda(v) = [\mathcal{Y}^\lambda, L^\lambda(v)], \qquad \lambda = l, r. \tag{2.35}$$

In fact, proceeding like in [20] we find that

$$\begin{aligned} \mathcal{Y}^l &= \mathcal{Y}_\mathcal{M} + \frac{1}{2}\xi^l_\mathcal{M} - w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp} - (wF)(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp}, \\ \mathcal{Y}^r &= \mathcal{Y}_\mathcal{M} + \frac{1}{2}\xi^r_\mathcal{M} - w^2(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp} - (wF)(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}, \end{aligned} \tag{2.36}$$

where w, F appear in (2.22) and $\mathcal{Y}_\mathcal{M} \in \mathcal{M}$ can be determined by the consistency of the gauge fixing conditions imposed on σ. Equation (2.10) also implies that $\dot{q} = p$ and by using this one can verify that the two Lax equations in (2.35) are actually equivalent to each other. Because of (2.34), the conserved quantities exhibited in the preceding paragraph can be expressed in terms of the Lax operators $L^\lambda(v)$ as well. Finally, note that $L^l(1)$ reproduces $\mathcal{L}$ in (2.25).

3 Spinless BC_n Sutherland models from $SU(m,n)$

Let us begin by noting that the symmetry reductions based on $G_+ \times G_+$ can be implemented also as a two step process, say imposing first the momentum map constraint on J^r_+. If one chooses $\mathcal{O}^r = \{0\}$ in this first step, then one obtains the geodesic system on the symmetric space G/G_+, which is subsequently reduced in the second step imposing the constraint on J^l_+. The $\mathcal{O}^r = \{0\}$ special case of the result given by (2.30), (2.31) reproduces a result in [20], where we studied the reductions of the geodesic system on G/G_+ taking an arbitrary orbit for $\mathcal{O}^l$. In this reference we also examined the cases for which the reduced phase space is isomorphic to $T^*\check{\mathcal{A}}$, which means that the reduced system gives a *spinless* Calogero model. Next we outline a mechanism whereby the models obtained in [20] can be deformed whenever G_+ admits a one-point coadjoint orbit consisting of a non-zero (infinitesimal) character.

Let $C \in \mathcal{G}^*_+ \simeq \mathcal{G}_+$ be a non-zero character, i.e., an element invariant under conjugation by G_+. Starting from $\mathcal{O}_{\rm red}$ (2.27), we can define a shifted space of spin degrees of freedom by

$$\mathcal{O}^y_{\rm red} := \big((\mathcal{O}^l - yC) \oplus (\mathcal{O}^r + yC)\big) \cap \mathcal{M}^\perp_{\rm diag}/M_{\rm diag}, \qquad \forall y \in \mathbb{R}, \tag{3.1}$$

where $(\mathcal{O}^r + yC)$ and $(\mathcal{O}^l - yC)$ are one parameter families of coadjoint orbits of G_+. This is possible since the constraint (2.20) is invariant under replacing (ξ^l, ξ^r) by $(\xi^l - yC, \xi^r + yC)$. A crucial point to notice is that if $\mathcal{O}_{\rm red} = \mathcal{O}^{y=0}_{\rm red}$ is a one-point space, then this feature holds for any $y \in \mathbb{R}$ with the reduced Hamiltonian $\mathcal{H}_{\rm red}$ (2.31) *acquiring a dependence on the 'deformation parameter'* y. It is well-known [22, 23] that non-trivial characters exist if and only if G/G_+ is a Hermitian symmetric space, which holds for example if $G = SU(m,n)$, and in these cases the space of characters is one-dimensional.

In [20] we explained that one-point reduced orbits (2.27) with $\mathcal{O}^r = \{0\}$ result if one takes $G = SU(m,n)$ and chooses $\mathcal{O}^l$ in a very special manner utilizing minimal coadjoint orbits of an $SU(k)$ factor of G_+. This is the essential point behind the derivation of the BC_n Sutherland model from the geodesic system of the symmetric space of $SU(n+1,n)$ due to Olshanetsky and Perelomov [14, 15, 2]. However, the three coupling constants of the model resulting from their procedure are necessarily subject to a quadratic relation. Here, we utilize the one parameter family of characters of G_+ to increase the number of independent coupling constants in the reduced Hamiltonian by one. In fact, we show below that in this way the classical BC_n Sutherland model with three independent coupling constants can be obtained as a reduction of the geodesics on $SU(n,n)$ and on $SU(n+1,n)$.

We need some further notations. Consider the joint eigensubspaces of the elements of $\mathcal{A}$,

$$\mathcal{G}_\alpha := \{X \in \mathcal{G} \,|\, [Y,X] = \alpha(Y)X \ \ \forall Y \in \mathcal{A}\}. \tag{3.2}$$

The linear functions $\alpha \subset \mathcal{A}^* \setminus \{0\}$ with $\dim(\mathcal{G}_\alpha) \neq 0$ are called restricted roots. They form a crystallographic root system, denoted by $\mathcal{R}$. The subspaces in (2.2) satisfy $\mathcal{M}^\perp + \mathcal{A}^\perp = \oplus_{\alpha\in\mathcal{R}}\mathcal{G}_\alpha$. We fix a polarization $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ and choose $E^a_\alpha \in \mathcal{G}_\alpha$ $(a = 1, \ldots, \nu_\alpha := \dim(\mathcal{G}_\alpha))$ so that

$$\theta(E^a_\alpha) = -E^a_{-\alpha}, \qquad \langle E^a_\alpha, E^b_\beta \rangle = \delta_{\alpha,-\beta}\delta^{a,b}. \tag{3.3}$$

Then $\mathcal{M}^\perp$ and $\mathcal{A}^\perp$ are spanned by

$$E^{+,a}_\alpha = \frac{1}{\sqrt{2}}(E^a_\alpha + \theta(E^a_\alpha)) \in \mathcal{M}^\perp, \qquad E^{-,a}_\alpha = \frac{1}{\sqrt{2}}(E^a_\alpha - \theta(E^a_\alpha)) \in \mathcal{A}^\perp \qquad \forall \alpha \in \mathcal{R}_+. \tag{3.4}$$

Let us now focus on $SU(m,n)$ and its Lie algebra $su(m,n)$, given by

$$SU(m,n) = \{g \in SL(m+n,\mathbb{C}) \,|\, g^\dagger I_{m,n} g = I_{m,n}\}, \tag{3.5}$$
$$su(m,n) = \{X \in \ sl(m+n,\mathbb{C}) \,|\, X^\dagger I_{m,n} + I_{m,n} X = 0\}, \tag{3.6}$$

where $I_{m,n} := \mathrm{diag}(\mathbf{1}_m, -\mathbf{1}_n)$, $m \geq n$ and $\mathbf{1}_k$ $(k = m,n)$ is the $k \times k$ identity matrix. A block matrix $X \in \mathcal{G} = su(m,n)$ reads

$$X = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}, \tag{3.7}$$

where $B \in \mathbb{C}^{m\times n}$, $A \in u(m)$, $D \in u(n)$ and $\mathrm{tr}\, A + \mathrm{tr}\, D = 0$. The Cartan involution of $G = SU(m,n)$ is $\Theta : g \mapsto (g^\dagger)^{-1}$. Thus

$$G_+ = S(U(m) \times U(n)) \tag{3.8}$$

and

$$\mathcal{G}_+ = su(m) \oplus su(n) \oplus \mathbb{R}C_{m,n} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + xC_{m,n} \,\middle|\, A \in su(m),\ D \in su(n),\ x \in \mathbb{R} \right\} \tag{3.9}$$

with the central element

$$C_{m,n} := \mathrm{diag}(\mathrm{i}n\mathbf{1}_m, -\mathrm{i}m\mathbf{1}_n), \tag{3.10}$$

which spans the space of characters. A maximal Abelian subspace of $\mathcal{G}_-$ is furnished by

$$\mathcal{A} := \left\{ q := \begin{pmatrix} \mathbf{0}_n & 0 & Q \\ 0 & \mathbf{0}_{m-n} & 0 \\ Q & 0 & \mathbf{0}_n \end{pmatrix} \,\middle|\, Q = \mathrm{diag}(q^1,\dots,q^n),\ q^j \in \mathbb{R} \right\}. \tag{3.11}$$

Taking $\chi := \mathrm{diag}(\chi_1,\dots,\chi_n)$ with any $\chi_j \in \mathbb{R}$, the centralizer of $\mathcal{A}$ in $\mathcal{G}_+$ is

$$\mathcal{M} = \{\mathrm{diag}(\mathrm{i}\chi, \gamma, \mathrm{i}\chi) \,|\, \gamma \in u(m-n),\ \mathrm{tr}\,\gamma + 2\mathrm{i}\mathrm{tr}\,\chi = 0\}, \tag{3.12}$$

and the subgroup M of G_+ is

$$M = \{\mathrm{diag}(e^{\mathrm{i}\chi}, \Gamma, e^{\mathrm{i}\chi}) \,|\, \Gamma \in U(m-n),\ (\det\Gamma)(\det e^{\mathrm{i}2\chi}) = 1\}. \tag{3.13}$$

One may define the functionals $e_k \in \mathcal{A}^*$ $(k = 1,\dots,n)$ by $e_k(q) := q^k$. The system of restricted roots is of BC_n type if $m > n$ and of C_n type if $m = n$. Indeed, we have

$$\mathcal{R}_+ := \{e_j \pm e_k\ (1 \leq j < k \leq n),\ 2e_k, e_k\ (1 \leq k \leq n)\} \quad \text{if} \quad m > n, \tag{3.14}$$

and

$$\mathcal{R}_+ := \{e_j \pm e_k\ (1 \leq j < k \leq n),\ 2e_k\ (1 \leq k \leq n)\} \quad \text{if} \quad m = n, \tag{3.15}$$

with the multiplicities

$$\nu_{e_j \pm e_k} = 2 \quad (1 \leq j < k \leq n), \quad \nu_{2e_k} = 1 \quad \text{and} \quad \nu_{e_k} = 2(m-n) \quad (1 \leq k \leq n). \tag{3.16}$$

We adopt the convention described explicitly in [20], where the basis vectors of $\mathcal{M}^\perp$ (3.4) are denoted as

$$E^{+,\mathrm{r}}_{e_j \pm e_k}, \quad E^{+,\mathrm{i}}_{e_j \pm e_k}, \quad E^{+,\mathrm{i}}_{2e_k}, \quad E^{+,\mathrm{r},d}_{e_k}, \quad E^{+,\mathrm{i},d}_{e_k} \quad \text{for} \quad 1 \leq d \leq (m-n). \tag{3.17}$$

The superscripts r or i refer to purely real or imaginary matrices.

Since G_+ (3.8) contains factors of $SU(k)$ type, we can use the minimal coadjoint orbits of $SU(k)$ in our reduction procedure, which underlie also the derivation [26] of the k-particle Sutherland model from the geodesic motion on $SU(k)$. For any $u \in \mathbb{C}^k$, viewed as a column vector, we define

$$\eta_\pm(u) := \pm \mathrm{i} \left(uu^\dagger - \frac{u^\dagger u}{k} \mathbf{1}_k \right) \in su(k). \tag{3.18}$$

The minimal coadjoint orbits of $SU(k)$ are provided by

$$\mathcal{O}_{k,\kappa,\pm} := \{ \xi \in su(k) \mid \exists u \in \mathbb{C}^k,\ u^\dagger u = k\kappa,\ \xi = \eta_\pm(u) \}, \tag{3.19}$$

where $\kappa > 0$ is a constant. For definiteness, we below take the plus sign. (Incidentally, $\eta_+(u) = \eta_+(e^{i\vartheta} u)$ for any $\vartheta \in \mathbb{R}$ and $\mathcal{O}_{k,\kappa,+}$ can be viewed as a reduction of the canonical phase space $\mathbb{C}^k \simeq \mathbb{R}^{2k}$ under the $U(1)$ action $u \mapsto e^{\mathrm{i}\vartheta} u$ for which $u \mapsto \mathrm{i} u^\dagger u$ is the momentum map.)

For $G = SU(n,n)$, we now consider the following coadjoint orbits of G_+:

$$\mathcal{O}^l := \mathcal{O}_{n,\kappa,+} + \{ x C_{n,n} \}, \qquad \mathcal{O}^r := \{ y C_{n,n} \}, \tag{3.20}$$

where x and y are real constants and $\mathcal{O}_{n,\kappa,+}$ is embedded say in the upper $su(n)$ block of $\mathcal{G}_+$. Since $C_{n,n} \in \mathcal{M}^\perp$, no restriction on x, y, κ arises from the constraint (2.20). One may confirm in the standard manner [26, 20] that the reduced orbit $\mathcal{O}_{\mathrm{red}}$ (2.27) consists of a single point, and as a representative one can take

$$\xi^l := \kappa \sum_{1 \le j < k \le n} \left(E^{+,\mathrm{i}}_{e_j + e_k} + E^{+,\mathrm{i}}_{e_j - e_k} \right) + \sqrt{2} x n \sum_{k=1}^n E^{+,\mathrm{i}}_{2e_k}, \quad \xi^r := y C_{n,n} = \sqrt{2} y n \sum_{k=1}^n E^{+,\mathrm{i}}_{2e_k}. \tag{3.21}$$

Upon substitution into (2.31) using the normalization (3.3), $\langle X, Y \rangle := \mathrm{tr}\,(XY)$, the reduced Hamiltonian (2.31) now gives

$$\frac{1}{2} \mathcal{H}^{SU(n,n)}_{\mathrm{red}}(q, p, \xi^l, \xi^r) = \mathcal{H}_{BC_n}(q,p) \quad \text{with} \quad g^2 = \frac{\kappa^2}{4}, \quad g_1^2 = \frac{xyn^2}{2}, \quad g_2^2 = \frac{(x-y)^2 n^2}{2}, \tag{3.22}$$

where we use the notation (1.1). The coupling constants g^2, g_1^2, g_2^2 can take arbitrary positive values, and we may even change the sign of g_1^2 by changing the sign of xy. This association of the classical BC_n Sutherland model with $SU(n,n)$ appears to be a new result. By setting $y = 0$ we reproduce the C_n type Hamiltonian previously known to arise from $SU(n,n)$ [2, 27] and $x = y \neq 0$ (resp. $x = y = 0$) yields the B_n (resp. D_n) type Sutherland Hamiltonian.

For $G = SU(n+1,n)$, we take $\mathcal{O}^l$ and $\mathcal{O}^r$ to be

$$\mathcal{O}^l := \mathcal{O}_{n+1,\kappa,+} + \{ x C_{n+1,n} \}, \qquad \mathcal{O}^r := \{ y C_{n+1,n} \}, \tag{3.23}$$

where $\mathcal{O}_{n+1,\kappa,+}$ is embedded into the $su(n+1)$ factor of $\mathcal{G}_+$. An analysis similar to [20] shows that the consistency of the constraint (2.20) requires

$$\kappa + x + y \ge 0 \quad \text{and} \quad \kappa - n(x+y) \ge 0. \tag{3.24}$$

The reduced orbit (2.27) again consists of a single point, and for a representative one can use

$$\xi^l := -\xi^r_{\mathcal{M}} + 2g \sum_{1\leq j<k\leq n} \left(E^{+,\mathrm{i}}_{e_j+e_k} + E^{+,\mathrm{i}}_{e_j-e_k} \right) + 2h_1 \sum_{k=1}^n E^{+,\mathrm{i},1}_{e_k} + 2h_2 \sum_{k=1}^n E^{+,\mathrm{i}}_{2e_k}, \tag{3.25}$$

$$g = \frac{\kappa + x + y}{2}, \quad h_1 = \frac{\sqrt{(\kappa+x+y)(\kappa - nx - ny)}}{\sqrt{2}}, \quad h_2 = \frac{2(n+1)x + y}{\sqrt{8}}, \tag{3.26}$$

$$\xi^r_{\mathcal{M}} = -\frac{\mathrm{i}y}{2}\mathrm{diag}(\mathbf{1}_n, -2n, \mathbf{1}_n), \quad \xi^r_{\mathcal{M}^\perp} = 2\tilde{h}_2 \sum_{k=1}^n E^{+,\mathrm{i}}_{2e_k}, \quad \tilde{h}_2 = \frac{y(2n+1)}{\sqrt{8}}. \tag{3.27}$$

Referring to $\mathcal{H}_{BC_n}$ in (1.1), in the present case we find

$$\frac{1}{2}\mathcal{H}^{SU(n+1,n)}_{\mathrm{red}}(q,p,\xi^l,\xi^r) = \mathcal{H}_{BC_n}(q,p) - \frac{y^2(2n^2+n)}{8} \quad \text{with} \quad g_1^2 = h_1^2 + h_2\tilde{h}_2, \quad g_2^2 = (h_2 - \tilde{h}_2)^2. \tag{3.28}$$

The coupling constants g^2, g_1^2, g_2^2 of $\mathcal{H}_{BC_n}$ depend on the three parameters x, y, κ subject to (3.24), and one recovers the result of [14, 20] upon setting $y = 0$.

In the above we have seen how the spinless BC_n Sutherland model with three arbitrary coupling constants arises from $SU(n,n)$ and from $SU(n+1,n)$. What happens if $m \geq (n+2)$? Briefly, in these cases we can obtain a one-point reduced orbit $\mathcal{O}_{\mathrm{red}}$ (2.27) if

$$\mathcal{O}^l = \mathcal{O}_{n,\kappa,+} + \{xC_{m,n}\}, \qquad \mathcal{O}^r = \{yC_{m,n}\}, \qquad x = -y. \tag{3.29}$$

The orbit $\mathcal{O}_{n,\kappa,+}$ is embedded in the $su(n)$ factor of $\mathcal{G}_+$ (3.9). The condition $(x+y) = 0$ is now enforced by the constraint (2.20). This leads again to the BC_n model, but with only two independent coupling parameters. Concretely, we find that

$$\frac{1}{2}\mathcal{H}^{SU(m,n)}_{\mathrm{red}} = \mathcal{H}_{BC_n} - \frac{y^2(m^2-n^2)n}{8} \quad \text{with} \quad g^2 = \frac{\kappa^2}{4}, \quad g_1^2 = -\frac{g_2^2}{4} = -\frac{y^2(m+n)^2}{8}. \tag{3.30}$$

In the $y=0$ case [20] the model $\mathcal{H}^{SU(m,n)}_{\mathrm{red}}$ becomes of type D_n. Finally, we note that the choice (3.29) is available for $m = n+1$ as well.

One can spell out the Lax matrices (2.34) for all the above cases and can also determine the explicit form of the 'compensating gauge transformation' $\mathcal{Y}_{\mathcal{M}}$ in (2.36). The Lax pair that results in the $m = n+1$ case is probably equivalent to the BC_n Lax pair of the same size found in [16]. In the other cases, including the most interesting $su(n,n)$ case, our Lax pairs appear to be new. It could be worthwhile to study in the future the dynamical r-matrix structures associated with these Lie algebraic Lax pairs of the BC_n model (1.1).

4 Discussion

The main results of this letter are the general description of the reduced geodesic system $(P_{\mathrm{red}}, \Omega_{\mathrm{red}}, \mathcal{H}_{\mathrm{red}})$ presented in Section 2 and the realization that this contains the spinless BC_n Sutherland models (1.1) with three independent coupling constants as explained in Section 3.

Next we discuss possible extensions and the quantum mechanical analogue of the construction that led to these results.

It is clear that the results can be extended to compact simple Lie groups without any difficulty. At the Lie algebra level, the compact real form corresponding to $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ is $\mathcal{G}_{\rm compact} = \mathcal{G}_+ + {\rm i}\mathcal{G}_-$ and reducing the cotangent bundle of the associated compact Lie group by $G_+ \times G_+$ leads to the trigonometric version of the hyperbolic spin Calogero model encoded by (2.31). It could be interesting to investigate also generalizations based on replacing the symmetry group $G_+ \times G_+$ by suitable groups $G'_+ \times G''_+$, where the factors are the fixed point sets of two commuting involutions of G. The construction works similarly to Section 2 whenever a 'good decomposition' analogous to (2.5) is available. This is closely related to the theory of affine symmetric spaces.

All spin Calogero models (2.31) and their generalizations just mentioned can be quantized by quantum Hamiltonian reduction as follows. One starts by replacing the coadjoint orbits $\mathcal{O}^\lambda$ in (2.8) by corresponding irreducible unitary representations ρ_λ of G_+ on vector spaces V_λ for $\lambda = l, r$ and considers also the associated representation ρ of $G_+ \times G_+$ on $V = V_l \otimes V_r$. The quantum analogue of the extended phase space $P^{\rm ext}$ (2.8) is the Hilbert space of V valued square integrable wave functions on $\check{G}$ and quantum Hamiltonian reduction amounts to allowing only those wave functions ψ that are equivariant in the sense that $\psi(g_+^l g (g_+^r)^{-1}) = \rho(g_+^l, g_+^r)\psi(g)$ holds. These functions are determined by their restrictions to the domain $\exp(\check{\mathcal{A}})$. Equivariance requires that the restricted wave functions take their values in the subspace V^M of V spanned by the vectors invariant under $M_{\rm diag}$ in the representation ρ. The allowed representations must therefore satisfy $\dim(V^M) > 0$, and spinless Calogero type models arise at the quantum mechanical level precisely if $\dim(V^M) = 1$. The reduced Hilbert space naturally comes equipped with a commuting family of self-adjoint operators induced by the center of the universal enveloping algebra of $\mathcal{G}$. In particular, the Laplace-Beltrami operator on the V valued wave functions on G induces the quantum mechanical Hamiltonian of the model. Thus all problems about the spectrum and the scattering theory for the resulting Hamiltonian are translated into problems in harmonic analysis. The wave functions of the reduced system are generalized spherical functions. This perspective on quantum Calogero type models originates from [28], where the trivial representation was taken for the ρ_λ above. Many interesting results obtained in this framework can be found in [3, 5, 11, 12, 29] and references therein.

Later we plan to elaborate the consequences of the quantum Hamiltonian reduction in detail for the general case paying particular attention to the spinless BC_n models. In this future work, which is in progress, we shall also describe the precise relationship between the reduction procedure proposed in Section 3 and the interpretation of BC_n type Jacobi polynomials as generalized spherical functions on $GL(m+n,\mathbb{C})/(GL(m,\mathbb{C}) \times GL(n,\mathbb{C}))$ put forward by Oblomkov [21]. It is known [5] that these polynomials are essentially the eigenstates of the BC_n type trigonometric Sutherland model. However, the natural compact analogue of Oblomkov's construction, obtained through replacing $GL(m+n,\mathbb{C})$ by $SU(m+n)$, does not seem to coincide with the quantized version of our classical Hamiltonian reduction in general. In fact, the finite dimensional representations (the V_λ for $\lambda = l, r$ mentioned above) that he uses do not correspond to the geometric quantization of the coadjoint orbits $\mathcal{O}^\lambda$ that we utilized in Section 3, except in the $m = n$ case. In general, his construction and ours may produce the eigenstates of the BC_n Hamiltonian for different discrete sets of the coupling constants, but it requires further work to fully clarify the connection.

Acknowledgements. The work of L.F. was supported in part by the Hungarian Scientific Research Fund (OTKA) under the grants T043159, T049495 and by the EU networks 'EUCLID' (contract number HPRN-CT-2002-00325) and 'ENIGMA' (contract number MRTN-CT-2004-5652). He is indebted to M.A. Olshanetsky for a discussion and for encouragement. B.G.P. wishes to thank J. Harnad for hospitality in Montreal.

References

[1] F. Calogero, *Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials,* J. Math. Phys. 12 (1971) 419-436.

[2] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Rept. 71 (1981) 313-400.

[3] M.A. Olshanetsky and A.M. Perelomov, *Quantum integrable systems related to Lie algebras*, Phys. Rept. 94 (1983) 313-404.

[4] A.M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkhäuser, 1990.

[5] G. Heckman, *Hypergeometric and spherical functions*, pp. 1-89 in: G. Heckman and H. Schlicktkrull, Harmonic Analysis and Special Functions on Symmetric Spaces, Perspectives in Mathematics 16, Academic Press, 1994.

[6] N. Nekrasov, *Infinite-dimensional algebras, many-body systems and gauge theories*, pp. 263-299 in: Moscow Seminar in Mathematical Physics, AMS Transl. Ser. 2, A.Yu. Morozov and M.A. Olshanetsky (Editors), Amer. Math. Soc., 1999.

[7] E. D'Hoker and D.H. Phong, *Seiberg-Witten theory and Calogero-Moser systems*, Prog. Theor. Phys. Suppl. 135 (1999) 75-93, hep-th/9906027.

[8] J.F. van Diejen and L. Vinet (Editors), Calogero-Moser-Sutherland Models, Spinger, 2000.

[9] B. Sutherland, Beautiful Models, World Scientific, 2004.

[10] F. Finkel, D. Gómez-Ullate, A. González-López, M.A. Rodríguez and R. Zhdanov, *A survey of quasi-exactly solvable systems and spin Calogero-Sutherland models*, pp. 173-186 in: Superintegrability in Classical and Quantum Systems, P. Tempesta et. al. (Editors), Amer. Math. Soc., 2004.

[11] P. Etingof, *Lectures on Calogero-Moser systems*, math.QA/0606233.

[12] A.P. Polychronakos, *Physics and mathematics of Calogero particles*, J. Phys. A: Math. Gen. 39 (2006) 12793-12845, hep-th/0607033.

[13] I. Aniceto and A. Jevicki, *Notes on collective field theory of matrix and spin Calogero models*, J. Phys. A: Math. Gen. 39 (2006) 12765-12791, hep-th/0607152.

[14] M.A. Olshanetsky and A.M. Perelomov, *Completely integrable Hamiltonian systems connected with semisimple Lie algebras,* Invent. Math. 37 (1976) 93-108.

[15] M.A. Olshanetsky and A.M. Perelomov, *Explicit solutions of some completely integrable systems,* Lett. Nuovo Cim. 17 (1976) 97-101.

[16] V.I. Inozemtsev and D.V. Meshcheryakov, *Extension of the class of integrable dynamical systems connected with semisimple Lie algebras*, Lett. Math. Phys. 9 (1985) 13-18.

[17] E.M. Opdam, *Root systems and hypergeometric functions IV*, Compositio Math. 67 (1988) 191-209.

[18] T. Oshima and H. Sekiguchi, *Commuting families of differential operators invariant under the action of a Weyl group*, J. Math. Sci. Univ. Tokyo 2 (1995) 1-75.

[19] A.J. Bordner, R. Sasaki and K. Takasaki, *Calogero-Moser models II: symmetries and foldings*, Prog. Theor. Phys. 101 (1999) 487-518, hep-th/9809068.

[20] L. Fehér and B.G. Pusztai, *Spin Calogero models associated with Riemannian symmetric spaces of negative curvature*, Nucl. Phys. B 751 (2006) 436-458, math-ph/0604073.

[21] A. Oblomkov, *Heckman-Opdam's Jacobi polynomials for the BC_n root system and generalized spherical functions*, Adv. Math. 186 (2004) 153-180, math.RT/0202076.

[22] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, 1978.

[23] A.W. Knapp, Lie Groups Beyond an Introduction, Progress in Mathematics 140, Birkhäuser, 2002.

[24] J.-P. Ortega and T.S. Ratiu, Momentum Maps and Hamiltonian Reduction, Progress in Mathematics 222, Birkhäuser, 2004.

[25] S. Hochgerner, *Singular cotangent bundle reduction and spin Calogero-Moser systems,* math.SG/0411068.

[26] D. Kazhdan, B. Kostant and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type,* Comm. Pure Appl. Math. XXXI (1978) 481-507.

[27] J. Avan, O. Babelon and M. Talon, *Construction of the classical R-matrices for the Toda and Calogero models,* Alg. and Anal. 6 (1994) 67-89, hep-th/9306102.

[28] M.A. Olshanetsky and A.M. Perelomov, *Quantum systems related to root systems, and radial parts of Laplace operators*, Funct. Anal. Appl. 12 (1978) 121-128, math-ph/0203031.

[29] P.I. Etingof, I.B. Frenkel and A.A. Kirillov Jr., *Spherical functions on affine Lie groups*, Duke Math. J. 80 (1995) 59-90, hep-th/9407047.