Spin Calogero models associated with Riemannian symmetric spaces of negative curvature

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Abstract

The Hamiltonian symmetry reduction of the geodesics system on a symmetric space of negative curvature by the maximal compact subgroup of the isometry group is investigated at an *arbitrary* value of the momentum map. Restricting to regular elements in the configuration space, the reduction generically yields a spin Calogero model with hyperbolic interaction potentials defined by the root system of the symmetric space. These models come equipped with Lax pairs and many constants of motion, and can be integrated by the projection method. The special values of the momentum map leading to spinless Calogero models are classified under some conditions, explaining why the BC_n models with two independent coupling constants are associated with $SU(n + 1, n)/S(U(n + 1) \times U(n))$ as found by Olshanetsky and Perelomov. In the zero curvature limit our models reproduce rational spin Calogero models studied previously and similar models correspond to other (affine) symmetric spaces, too. The construction works at the quantized level as well.

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1 Introduction

The investigation of the structure and applications of 'Calogero type' models, pioneered in [1, 2, 3, 4, 5], is a fascinating subject receiving lots of attention. It is clear from the reviews (see e.g. [6, 7, 8]) that these models appear in extremely many contexts in physics as well as in mathematics. The present paper deals with their hyperbolic variants and extensions by internal ('spin') degrees of freedom [9], at the classical level. Among alternative approaches to generalized Calogero models, we are interested in their relationship to symmetric spaces, which was first realized in [10] and further studied in [11]-[21].

As introduced in [10], a hyperbolic Calogero type model is characterized by the Hamiltonian

$$H(q,p) = \frac{1}{2}\langle p, p\rangle + \sum_{\alpha\in\mathcal{R}_+} \frac{g_\alpha^2}{\sinh^2\alpha(q)}, \tag{1.1}$$

where $\mathcal{R}_+$ denotes the positive roots in a root system $\mathcal{R}$ and the coupling constants g_α can be different in principle for different orbits of the corresponding reflection group. Here, the crystallographic root systems are considered that occur in association with symmetric spaces and include, besides the root systems of the complex simple Lie algebras, the $BC_n = B_n \cup C_n$ systems [22, 23]. If $\mathcal{R}$ is of the classical A_n, B_n, C_n, D_n or BC_n type, and the coupling constants are subject to certain relations, then Olshanetsky and Perelomov were able to construct a Lax representation and a solution algorithm for the model by treating it as projection of geodesic motion on a symmetric space of negative curvature [10, 12, 16, 19]. Their method is equivalent to Hamiltonian symmetry reduction of the geodesic system by the maximal compact subgroup of the isometry group, $G_+ \subset G$, as was explained in the A_n case by Kazhdan, Kostant and Sternberg in [13]. (For general reviews of the theory of Hamitonian reduction, see e.g. [19, 24].)

The Hamiltonian reduction yields a Calogero type model (1.1) only if the value of the momentum map defining the reduction enjoys some very specific properties, which are known to occur only for particular symmetric spaces G/G_+, as described in [10, 13, 15, 16, 19, 20]. However, a classification of such 'good reductions' is not available. For reasons not very well understood, the classical mechanical models (1.1) based on the exceptional root systems, or on BC_n with three arbitrary coupling constants, are (up to now) not related to symmetric spaces. It is also not quite clear why it is the case [10] that for the BC_n models with two independent coupling constants the pertinent symmetric space is $SU(n+1,n)/S(U(n+1)\times U(n))$, although the root system of $SU(m,n)/S(U(m)\times U(n))$ is of BC_n type for any $m > n$. These problems motivated [25, 26] to set up new frameworks for studying Calogero models. Due to its universal applicability, the method developed in [26, 27, 28, 29] may be considered more natural than the traditional Olshanetsky-Perelomov approach to Calogero type models. Still, one would like to better understand the relation between these models and symmetric spaces.

In this paper we reformulate the question about the correspondence between symmetric spaces of negative curvature and Calogero type models by asking what is the reduced system that results from the geodesic system in general, at an *arbitrary value* μ_0 of the momentum map for the action of G_+ on $T^*(G/G_+)$. The answer turns out to be very simple. We demonstrate

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that the reduction generically yields a spin Calogero model with Hamiltonian of the form

$$H(q, p, \xi) = \frac{1}{2}\langle p, p \rangle + \frac{1}{2}\sum_{\alpha \in \mathcal{R}_+} \sum_{i=1}^{\nu_\alpha} \frac{(\xi_i^\alpha)^2}{\sinh^2 \alpha(q)}. \tag{1.2}$$

The phase space of this model is $T^*\check{\mathcal{A}} \times \mathcal{O}_{red} = \check{\mathcal{A}} \times \mathcal{A} \times \mathcal{O}_{red} = \{(q, p, \xi)\}$, where $\mathcal{O}_{red}$ is the reduction of the coadjoint orbit of G_+ through $-\mu_0$ by the action of a subgroup $M \subset G_+$ at the zero value of its momentum map. The ν_α are the multiplicities of the restricted roots [22, 23] with respect to the Cartan subalgebra $\mathcal{A}$ of the symmetric space, $\check{\mathcal{A}}$ is the interior of a Weyl chamber and M is the centralizer of $\mathcal{A}$ inside G_+.

After deriving the spin Calogero models (1.2), which seem to appear here for the first time, we show that the Hamiltonian reduction equips them naturally with many constants of motion and a spectral parameter dependent Lax pair. Their evolution equation belongs to a commuting family whose Hamiltonian flows can be constructed with the aid of the projection method. The model (1.2) simplifies to (1.1) if the space of spin degrees of freedom, $\mathcal{O}_{red}$, consists of a single point. There is only one mechanism known whereby this can be guaranteed. Namely, if G_+ contains a simple factor of $SU(k)$ type intersecting M in its maximal torus, then one can make use of the same orbit of $SU(k)$ (possibly 'dressed' by a contribution from the center of $\mathrm{Lie}(G_+)$), which was used in [13] in relation with the symmetric space $G/G_+ = SL(k, \mathbb{C})/SU(k)$. We shall classify the cases for which this 'KKS mechanism' is applicable, and thereby explain why the BC_n models with two independent coupling parameters are associated with $SU(n+1, n)/S(U(n+1) \times U(n))$ as found by Olshanetsky and Perelomov.

The rational analogues of the models (1.2) have been obtained recently in [30, 31] by reducing the geodesic motion on the symmetric spaces of zero curvature, as initiated in [11]. Our results concerning the list of spinless cases and Lax pairs, which are not addressed in [30, 31], can also be applied in the zero curvature limit. In [31] the rational spin Calogero models are presented as an illustration to the general theory of singular symplectic reduction of cotangent bundles advanced in this paper. In contrast, we here give a direct, simple derivation of the models (1.2). We shall proceed similarly to [32], where we obtained a different class of hyperbolic and trigonometric spin Calogero models by reducing the geodesic motion on a semisimple Lie group with the aid of the symmetry induced by twisted conjugations. Together with the above and several further results in the literature, the present work supports the following general statement. Heuristically formulated, the statement is that if one reduces geodesic motion on a space of matrices by the Hamiltonian action of a symmetry group whereby those matrices can be diagonalized, then the result is in general a spin Calogero type model, with coordinate variables parametrizing the diagonal matrices that arise. This heuristic statement can be promoted to a proper theorem under various more precise formulations of the conditions.

The organization of the paper and our results can be outlined as follows. Section 2 contains necessary background material and conventions. Our main result is given by Theorem 1 in Section 3 summarizing the outcome of the derivation of the reduced Hamiltonian system (1.2) from the geodesic motion. The subsections of Section 4 deal with the conserved quantities and the Lax representation of this system, with the results formulated in Theorem 4 and Proposition 5. Section 5 is devoted to explaining what is meant by the 'KKS mechanism' and

to presenting the list of cases in which this mechanism leads to spinless Calogero models of type (1.1). It is shown that in addition to the original $SL(k, \mathbb{C})/SU(k)$ case the KKS mechanism is applicable only to certain reductions of the symmetric spaces having $SU(m, n)$ as isometry group for some $m \geq n$, with the precise list of cases provided by Theorem 6. The corresponding Hamiltonians are collected in Proposition 7, recovering the classical examples [10, 16, 19] in our more systematic framework. Our conclusions are presented in Section 6. We here briefly discuss also the dynamical r-matrices and the quantization of the models (1.2), which will be elaborated in a future publication. Finally, Appendix A contains auxiliary material on $su(m, n)$.

2 Preliminaries on the system to be reduced

In this preparatory section we collect background material and conventions on Riemannian symmetric spaces of negative curvature. More details can be found, e.g., in [19, 22, 23] and the reader may also consult Section 5 with Appendix A for a concrete example. Our notations are adapted to matrix Lie groups for simplicity throughout the paper, but this does not mean any restriction of generality since all formulae can be rewritten in a more abstract manner as well.

2.1 Group theoretic preliminaries and conventions

Let G be a non-compact real simple Lie group with finite centre and $\mathcal{G}$ its Lie algebra. Up to conjugation, there is a unique Cartan involution θ of $\mathcal{G}$, which is characterized by the decomposition

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-, \qquad \theta(X_\pm) = \pm X_\pm \quad \forall X_\pm \in \mathcal{G}_\pm, \tag{2.1}$$

where the restriction of the Killing form $\langle \, , \, \rangle$ of $\mathcal{G}$ is negative (resp. positive) definite on $\mathcal{G}_+$ (resp. on $\mathcal{G}_-$). $\mathcal{G}_+$ is a maximal compact subalgebra of $\mathcal{G}$ and the elements of $\mathcal{G}_-$ are diagonalizable in the adjoint representation of $\mathcal{G}$. Any maximal Abelian subspace $\mathcal{A} \subset \mathcal{G}_-$ induces the decomposition

$$\mathcal{G} = \mathcal{A} \oplus \mathcal{M} \oplus \left(\oplus_{\alpha \in \mathcal{R}} \mathcal{G}_\alpha \right), \tag{2.2}$$

where

$$\mathcal{M} := \{ X \in \mathcal{G}_+ \, | \, [H, X] = 0 \ \forall H \in \mathcal{A} \}, \quad \mathcal{G}_\alpha := \{ X \in \mathcal{G} \, | \, [H, X] = \alpha(H)X \ \forall H \in \mathcal{A} \}. \tag{2.3}$$

The elements of $\mathcal{R} \subset \mathcal{A}^* \setminus \{0\}$ are called restricted roots. We fix a polarization $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ and choose weight vectors $E_\alpha^i \in \mathcal{G}_\alpha$ $(i = 1, \ldots, \nu_\alpha := \dim(\mathcal{G}_\alpha))$ so that

$$\theta(E_\alpha^i) = -E_{-\alpha}^i, \qquad \langle E_\alpha^i, E_\beta^j \rangle = \delta_{\alpha, -\beta} \delta^{i,j}. \tag{2.4}$$

The decomposition (2.1) can be refined as

$$\mathcal{G}_- = \mathcal{A} + \mathcal{A}^\perp, \qquad \mathcal{G}_+ = \mathcal{M} + \mathcal{M}^\perp, \tag{2.5}$$

where $\mathcal{M}^\perp$ and $\mathcal{A}^\perp$ are spanned by the basis vectors

$$E_\alpha^{+,i} = \frac{1}{\sqrt{2}} (E_\alpha^i + \theta(E_\alpha^i)) \in \mathcal{M}^\perp, \qquad E_\alpha^{-,i} = \frac{1}{\sqrt{2}} (E_\alpha^i - \theta(E_\alpha^i)) \in \mathcal{A}^\perp \qquad \forall \alpha \in \mathcal{R}_+. \tag{2.6}$$

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Lifting $\theta \in \mathrm{Aut}(\mathcal{G})$ to the Cartan involution Θ of G, let us introduce

$$G_+ = \{g_+ \in G \,|\, \Theta(g_+) = g_+\}, \qquad G_- = \{g_- \in G \,|\, \Theta(g_-) = g_-^{-1}\}. \tag{2.7}$$

G_+ is a maximal compact subgroup of G and the submanifold $G_- \subset G$ is diffeomorphic to $\mathcal{G}_-$ by the exponential map. The group G is diffeomorphic to $G_- \times G_+$ since any $g \in G$ admits a unique decomposition as

$$g = g_- g_+, \quad g_\pm \in G_\pm. \tag{2.8}$$

The symmetric spaces of negative curvature are the coset spaces G/G_+. A convenient model of such a coset space is provided by the identification

$$G/G_+ \simeq G_-, \tag{2.9}$$

where the corresponding projection $\pi : G \to G_-$ is by definition given by

$$\pi : g \mapsto \Lambda(g) := g\Theta(g^{-1}) = g_-^2 \quad \text{for} \quad g = g_- g_+. \tag{2.10}$$

The left translation on G by $\eta \in G$ descends to the action on the symmetric space G/G_+, which operates according to

$$G \ni \eta \mapsto \rho_\eta \in \mathrm{Diff}(G/G_+), \qquad \rho_\eta(\Lambda) = \eta \Lambda \Theta(\eta^{-1}). \tag{2.11}$$

2.2 Hamiltonian model of the geodesic motion on G/G_+

Later we shall reduce the Hamiltonian system of the geodesic motion on G/G_+ using the action of the symmetry group G_+ induced by (2.11). A very convenient model of this Hamiltonian system can be obtained by reducing the geodesic system on T^*G by the G_+ action defined by right translations, fixing the corresponding momentum map to zero. Indeed, as is easily verified, this leads to the model

$$(T^*(G/G_+), \Omega, \mathcal{H}), \tag{2.12}$$

where the various ingredients are identified as follows. First, the phase space is

$$T^*(G/G_+) \simeq T^*G_- \simeq G_- \times \mathcal{G}_- = \{(\Lambda, J_-) \,|\, \Lambda \in G_-, \ J_- \in \mathcal{G}_-\}. \tag{2.13}$$

To describe the symplectic form Ω and the Hamiltonian $\mathcal{H}$, let us introduce the $\mathcal{G}_+$ valued function J_+ by the formula

$$J_+(\Lambda, J_-) = (\tanh \mathrm{ad}_Q) J_- \quad \text{with} \quad Q := \frac{1}{2} \log \Lambda, \tag{2.14}$$

which is well-defined since ad_Q has real eigenvalues only. Then introduce $J : T^*(G/G_+) \to \mathcal{G}$ by

$$J(\Lambda, J_-) = J_- + J_+(\Lambda, J_-). \tag{2.15}$$

Note that the defining equation of J_+ can be rewritten as

$$\Lambda^{-1} J \Lambda = J_- - J_+. \tag{2.16}$$

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Now the symplectic form and the geodesic Hamiltonian are

$$\Omega = d\vartheta \quad \text{with} \quad \vartheta = \frac{1}{2}\langle J, d\Lambda\Lambda^{-1}\rangle, \qquad \mathcal{H} = \frac{1}{2}\langle J, J\rangle. \tag{2.17}$$

The Hamiltonian action of G operates on the phase space (2.13) by $\rho_\eta^* \in \mathrm{Diff}(T^*G_-)$,

$$\rho_\eta^* : (\Lambda, J_-) \mapsto (\eta\Lambda\Theta(\eta^{-1}), (\eta J(\Lambda, J_-)\eta^{-1})_-), \qquad \forall \eta \in G, \tag{2.18}$$

where we use the decomposition $X = X_+ + X_-$ for any $X \in \mathcal{G}$. In fact, $J : T^*(G/G_+) \to \mathcal{G}$ is nothing but the equivariant momentum map that generates this action. This means that if T_a is a basis of $\mathcal{G}$, and

$$J_a = \langle T_a, J\rangle, \qquad [T_a, T_b] = f_{ab}^c T_c, \tag{2.19}$$

then we have the Poisson brackets

$$\{\Lambda, J_a\} = T_a\Lambda - \Lambda\theta(T_a), \qquad \{J_a, J_b\} = f_{ab}^c J_c. \tag{2.20}$$

Naturally, $J_+ : T^*(G/G_+) \to \mathcal{G}_+$ is the momentum map for the restriction of the above action to G_+, which simplifies according to

$$\rho_\eta^* : (\Lambda, J_-) \mapsto (\eta\Lambda\eta^{-1}, \eta J_- \eta^{-1}) \qquad \forall \eta \in G_+. \tag{2.21}$$

The Hamiltonian equations of motion can be written as

$$\dot{\Lambda} = \{\Lambda, \mathcal{H}\} = J\Lambda - \Lambda\theta(J), \qquad \dot{J}_- = \{J_-, \mathcal{H}\} = 0. \tag{2.22}$$

By (2.16) the first formula is equivalent to

$$\dot{\Lambda}\Lambda^{-1} + \Lambda^{-1}\dot{\Lambda} = 4J_-. \tag{2.23}$$

The solution with initial value (Λ_0, J_-^0) is just the orbit of the one-parameter subgroup of G generated by $J_0 := J(\Lambda_0, J_-^0)$:

$$\Lambda(t) = e^{tJ_0}\Lambda_0 e^{-t\theta(J_0)}, \tag{2.24}$$

and the components of J are constants of motion.

3   Spin Calogero models from Hamiltonian reduction

This section contains our derivation of spin Calogero models from the geodesic motion on the symmetric space, with the result given by Theorem 1 and subsequent remarks.

We below use the subset of regular elements $\hat{\mathcal{A}} \subset \mathcal{A}$,

$$\hat{\mathcal{A}} = \{H \in \mathcal{A} \,|\, \alpha(H) \neq 0 \quad \forall \alpha \in \mathcal{R}\}, \tag{3.1}$$

and the open Weyl chamber

$$\check{\mathcal{A}} := \{H \in \mathcal{A} \,|\, \alpha(H) > 0 \quad \forall \alpha \in \mathcal{R}_+\}, \tag{3.2}$$

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which is a connected component of $\hat{\mathcal{A}}$. The G_+-conjugates of $\check{\mathcal{A}}$ form a dense open subset $\check{\mathcal{G}}_- \subset \mathcal{G}_-$, and we focus on the corresponding dense open submanifold of $T^*(G/G_+)$ furnished by

$$P := T^*\check{G}_-, \qquad \check{G}_- := \exp(\check{\mathcal{G}}_-). \tag{3.3}$$

We wish to reduce P under the Hamiltonian action of G_+ at an *arbitrary* value, μ_0, of the momentum map J_+. To characterize the Marsden-Weinstein reduction of the Hamiltonian system

$$(P, \Omega, \mathcal{H}), \tag{3.4}$$

we make use of the standard shifting trick of symplectic reduction (see, e.g., [24]). For this, we let

$$(\mathcal{O}, \omega^{\mathcal{O}}) \tag{3.5}$$

denote the coadjoint orbit of G_+ through $(-\mu_0)$ equipped with its natural symplectic form $\omega^{\mathcal{O}}$, where $\mathcal{G}_+^*$ is identified with $\mathcal{G}_+$ by the Killing form. The shifting trick states that the reduced system mentioned above is naturally isomorphic to the Marsden-Weinstein reduction of the 'extended system'

$$(P^{\mathcal{O}}, \Omega^{\mathcal{O}}, \mathcal{H}^{\mathcal{O}}) \tag{3.6}$$

at the *zero* value of the appropriate momentum map, Ψ. The extended system is

$$P^{\mathcal{O}} = P \times \mathcal{O}, \qquad \Omega^{\mathcal{O}} = \Omega + \omega^{\mathcal{O}}, \qquad \mathcal{H}^{\mathcal{O}}(\Lambda, J_-, \xi) = \mathcal{H}(\Lambda, J_-), \tag{3.7}$$

where $(\Lambda, J_-, \xi) \in P^{\mathcal{O}}$ is arbitrary. Using (2.15), $\mathcal{H}^{\mathcal{O}}$ can be written as

$$\mathcal{H}^{\mathcal{O}} = \frac{1}{2}\langle J^{\mathcal{O}}, J^{\mathcal{O}}\rangle \quad \text{with} \quad J^{\mathcal{O}}(\Lambda, J_-, \xi) := J(\Lambda, J_-). \tag{3.8}$$

The action of G_+ on $P^{\mathcal{O}}$ is the diagonal one, denoted as $\hat{\rho}$:

$$\hat{\rho}_\eta : (\Lambda, J_-, \xi) \mapsto (\eta\Lambda\eta^{-1}, \eta J_-\eta^{-1}, \eta\xi\eta^{-1}), \qquad \forall \eta \in G_+, \tag{3.9}$$

and this is generated by the momentum map

$$\Psi : P^{\mathcal{O}} \to \mathcal{G}_+, \qquad \Psi(\Lambda, J_-, \xi) = J_+(\Lambda, J_-) + \xi. \tag{3.10}$$

With $G_+(\mu_0)$ being the isotropy group of μ_0, the main point is the second equality in

$$P_{red} := P_{J_+=\mu_0}/G_+(\mu_0) = P^{\mathcal{O}}_{\Psi=0}/G_+. \tag{3.11}$$

After the foregoing preparations, we are now in the position to describe the reduced Hamiltonian system. The crucial step is to observe that all G_+ orbits in the constrained manifold $P^{\mathcal{O}}_{\Psi=0}$ intersect the following gauge slice:

$$S := \{(e^{2q}, J_-, \xi) \in P^{\mathcal{O}}_{\Psi=0} \mid q \in \check{\mathcal{A}}\}, \tag{3.12}$$

since every regular element of $\mathcal{G}_-$ can be conjugated into $\check{\mathcal{A}}$. This gauge slice is 'thick' in the sense that it represents only a partial gauge fixing of the 'gauge transformations' defined by the

G_+ action. In fact, the residual gauge transformations (that map an arbitrarily chosen point of S into S) are generated precisely by the centralizer subgroup M of $\mathcal{A}$ inside G_+:

$$M := \{m \in G_+ \mid mHm^{-1} = H \quad \forall H \in \mathcal{A}\}. \tag{3.13}$$

Therefore we obtain the identification

$$P_{red} = P^{\mathcal{O}}_{\Psi=0}/G_+ = S/M. \tag{3.14}$$

To proceed further, let us decompose $J_- \in \mathcal{G}_-$ and $\xi \in \mathcal{O} \subset \mathcal{G}_+$ according to (2.5) as

$$J_- = J_{\mathcal{A}} + J_{\mathcal{A}^\perp}, \qquad \xi = \xi_{\mathcal{M}} + \xi_{\mathcal{M}^\perp}. \tag{3.15}$$

Then one can check that the constraint $\Psi = 0$ on S is equivalent to the requirements

$$\xi_{\mathcal{M}} = 0 \qquad \text{and} \qquad J_{\mathcal{A}^\perp} = -(\coth \mathrm{ad}_q)\xi_{\mathcal{M}^\perp}. \tag{3.16}$$

This motivates to consider the smooth one-to-one map

$$I : (\check{\mathcal{A}} \times \mathcal{A}) \times (\mathcal{O} \cap \mathcal{M}^\perp) \to S, \qquad I(q, p, \xi_{\mathcal{M}^\perp}) := (e^{2q}, p - (\coth \mathrm{ad}_q)\xi_{\mathcal{M}^\perp}, \xi_{\mathcal{M}^\perp}). \tag{3.17}$$

The pull-back of $\Omega^{\mathcal{O}}|_S$ by I turns out to be

$$I^*(\Omega^{\mathcal{O}}|_S) = d\langle p, dq\rangle + \omega^{\mathcal{O}}|_{\mathcal{O} \cap \mathcal{M}^\perp}. \tag{3.18}$$

The first term is the canonical symplectic structure of

$$T^*\check{\mathcal{A}} = \check{\mathcal{A}} \times \mathcal{A} = \{(q, p)\}. \tag{3.19}$$

The second term in (3.18) is the restriction of $\omega^{\mathcal{O}}$ to the zero level set of the momentum map for the action of the subgroup $M \subset G_+$ on $\mathcal{O}$, which is provided by $\mathcal{O} \ni \xi \mapsto \xi_{\mathcal{M}} \in \mathcal{M} \simeq \mathcal{M}^*$. It is also important to note that I is an M equivariant map, where M acts trivially on $T^*\check{\mathcal{A}}$. As for the reduced Hamiltonian of the geodesic motion, we find from (3.8)

$$(\mathcal{H}^{\mathcal{O}} \circ I)(q, p, \xi_{\mathcal{M}^\perp}) = \frac{1}{2}\langle L(q, p, \xi_{\mathcal{M}^\perp}), L(q, p, \xi_{\mathcal{M}^\perp})\rangle \tag{3.20}$$

with the map $L := J^{\mathcal{O}} \circ I : T^*\check{\mathcal{A}} \times (\mathcal{O} \cap \mathcal{M}^\perp) \to \mathcal{G}$, which is equivariant under the natural actions of $M \subset G_+ \subset G$. By expanding $\xi_{\mathcal{M}^\perp}$ in the basis (2.6),

$$\xi_{\mathcal{M}^\perp} = \sum_{\alpha \in \mathcal{R}_+} \sum_{i=1}^{\nu_\alpha} \xi_i^\alpha E_\alpha^{+,i}, \tag{3.21}$$

L can be written explicitly as

$$L(q, p, \xi_{\mathcal{M}^\perp}) = p - (\coth \mathrm{ad}_q)\xi_{\mathcal{M}^\perp} - \xi_{\mathcal{M}^\perp} = p - \sum_{\alpha \in \mathcal{R}_+} \sum_{i=1}^{\nu_\alpha} \xi_i^\alpha \left(\coth \alpha(q) E_\alpha^{-,i} + E_\alpha^{+,i}\right). \tag{3.22}$$

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On account of its equivariance property, the map I (3.17) gives rise to the identification

$$S/M = T^*\check{\mathcal{A}} \times (\mathcal{O} \cap \mathcal{M}^\perp)/M. \tag{3.23}$$

Combining this with (3.14) proves the following theorem.

Theorem 1. *The reduction of the geodesic system on $\check{G}_- \subset G/G_+$ defined by (3.11) with (3.3) can be identified as $(P_{red}, \Omega_{red}, \mathcal{H}_{red})$ with*

$$P_{red} = T^*\check{\mathcal{A}} \times \mathcal{O}_{red}, \qquad \Omega_{red} = d\langle p, dq \rangle + \omega_{red}^{\mathcal{O}}, \tag{3.24}$$

where q, p are the natural variables on $T^*\check{\mathcal{A}}$ and $(\mathcal{O}_{red}, \omega_{red}^{\mathcal{O}})$ is the symplectic reduction of $(\mathcal{O}, \omega^{\mathcal{O}})$ by the subgroup $M \subset G_+$ (3.13) at the zero value of its momentum map,

$$\mathcal{O}_{red} = (\mathcal{O} \cap \mathcal{M}^\perp)/M. \tag{3.25}$$

The reduced Hamiltonian defines a hyperbolic spin Calogero type model in general, since as an M invariant function on $T^*\check{\mathcal{A}} \times \mathcal{O} \cap \mathcal{M}^\perp$ it has the form

$$\mathcal{H}_{red}(q, p, \xi_{\mathcal{M}^\perp}) = \frac{1}{2}\langle L(q, p, \xi_{\mathcal{M}^\perp}), L(q, p, \xi_{\mathcal{M}^\perp}) \rangle = \frac{1}{2}\langle p, p \rangle + \frac{1}{2}\sum_{\alpha \in \mathcal{R}_+}\sum_{i=1}^{\nu_\alpha}\frac{(\xi_i^\alpha)^2}{\sinh^2\alpha(q)}. \tag{3.26}$$

Remark 2. Instead of S (3.12), one could equally well use the slightly 'thicker' gauge slice

$$\hat{S} := \{(e^{2q}, J_-, \xi) \in P_{\Psi=0}^{\mathcal{O}} \mid q \in \hat{\mathcal{A}}\}, \tag{3.27}$$

where $\hat{\mathcal{A}}$ (3.1) is the union of all open Weyl chambers. The residual gauge transformations now belong to the normalizer

$$\hat{M} := \{n \in G_+ \mid nHn^{-1} \in \mathcal{A} \quad \forall H \in \mathcal{A}\}. \tag{3.28}$$

Recalling that $M \subset \hat{M}$ is a normal subgroup and

$$W := \hat{M}/M \tag{3.29}$$

is the Weyl group of the symmetric space, we obtain

$$P_{red} = \hat{S}/\hat{M} = (\hat{S}/M)/(\hat{M}/M) = \hat{P}_{red}/W \tag{3.30}$$

with

$$\hat{P}_{red} := \hat{S}/M = T^*\hat{\mathcal{A}} \times \mathcal{O}_{red}. \tag{3.31}$$

Here, $\hat{P}_{red}$ differs from P_{red} only in that q now varies in $\hat{\mathcal{A}}$. The geodesic system descends to a spin Calogero type system on $\hat{P}_{red}$, with Hamiltonian still of the form (3.26). This system enjoys Weyl symmetry, where W acts on all three components of $(q, p, [\xi_{\mathcal{M}^\perp}]) \in \hat{P}_{red}$ naturally. All our spin Calogero models possess a hidden Weyl group symmetry in this sense.

Remark 3. The reduced phase space P_{red} (3.24) is not a smooth manifold in general, since the space of 'spin' degrees of freedom has some singularities. For example, if $\mathcal{G}$ is a real split simple Lie algebra (like $sl(n, \mathbb{R})$), then $\mathcal{M} = \{0\}$, M is a finite group and $P_{red} = T^*\check{\mathcal{A}} \times (\mathcal{O}/M)$ is an orbifold. In general, $\mathcal{O}_{red}$ (3.25) is a stratified space, whose strata are smooth symplectic manifolds [24]. The restriction to the principal orbit type for the M-action on $\mathcal{O} \cap \mathcal{M}^\perp$ always leads to a dense open subset of $\mathcal{O}_{red}$, which is a smooth manifold. A detailed study of the non-principal strata appears as an interesting problem for the future. In certain special cases it so happens that $\mathcal{O}_{red}$ is a trivial manifold consisting of a single point, and then the reduced system is a Calogero type model (1.1) without spin. This is further discussed in Section 5.

4 Constants of motion and Lax pairs

The G_+ invariant constants of motion of the extended system (3.6) survive the Hamitonian reduction to P_{red} (3.11). By using this we exhibit a large family of conserved quantities for the spin Calogero model of Theorem 1, and prove that those of them that are associated (by equation (4.5)) with the G invariant functions on $\mathcal{G}$ are in involution. Then we show that these conserved quantities in involution admit the usual interpretation as G invariant functions of a suitable (spectral parameter dependent) Lax operator for the spin Calogero model. The commuting constants of motion include the G invariant functions of the symmetry generator J (2.15), for which the reduced Hamiltonian flows are easily obtained by the projection method.

4.1 Constants of motion

Observe from (3.7) that J_- and ξ are conserved quantities for the system (3.6). Therefore so is their linear combination $K(x) : P^{\mathcal{O}} \to \mathcal{G}$ given by

$$K(x) := J_- - x\,\xi, \tag{4.1}$$

where x is an arbitrary real number. In the definition of $K(x)$ we regard J_- and ξ as evaluation functions on the phase space, i.e., $K(x) : P^{\mathcal{O}} \ni (\Lambda, J_-, \xi) \mapsto J_- - x\xi \in \mathcal{G}$. Since $K(x)$ is equivariant with respect to the natural actions of the symmetry group G_+ on $P^{\mathcal{O}}$ and on $\mathcal{G}$, the composite $f \circ K(x)$ is a G_+ invariant function on $P^{\mathcal{O}}$ for any G_+ invariant (real) function on $\mathcal{G}$, $f \in C^\infty_{G_+}(\mathcal{G})$. Here and below we use the notations

$$C^\infty_{G_+}(\mathcal{G}) := \{f \in C^\infty(\mathcal{G}) \,|\, f(gXg^{-1}) = f(X) \quad \forall X \in \mathcal{G}, \forall g \in G_+\}, \tag{4.2}$$

$$C^\infty_G(\mathcal{G}) := \{f \in C^\infty(\mathcal{G}) \,|\, f(gXg^{-1}) = f(X) \quad \forall X \in \mathcal{G}, \forall g \in G\}. \tag{4.3}$$

Any G_+ invariant (smooth) function on $P^{\mathcal{O}}_{\Psi=0}$ can be regarded as a (smooth) function on the reduced phase space P_{red} defined by (3.11). In particular, if

$$\mathcal{E} : P^{\mathcal{O}}_{\Psi=0} \to P^{\mathcal{O}} \tag{4.4}$$

is the tautological embedding, then

$$f \circ K(x) \circ \mathcal{E} \in C^\infty(P_{red}) \qquad \forall f \in C^\infty_{G_+}(\mathcal{G}). \tag{4.5}$$

All functions of this form are constants of motion for the reduced system of Theorem 1. The Poisson brackets of these functions under the reduced Poisson structure on P_{red} are given by

$$\{f \circ K(x) \circ \mathcal{E}, h \circ K(y) \circ \mathcal{E}\}_{red} := \{f \circ K(x), h \circ K(y)\} \circ \mathcal{E}, \quad \forall f, h \in C^\infty_{G_+}(\mathcal{G}), \ \forall x, y \in \mathbb{R}. \quad (4.6)$$

On the right-hand-side the Poisson bracket of $P^{\mathcal{O}}$ is used, whose explicit form is determined by (2.20) together with the $\mathcal{G}_+$ Lie-Poisson brackets of the components of ξ.

For any real function $f \in C^\infty(\mathcal{G})$, its gradient $\nabla f \in C^\infty(\mathcal{G}, \mathcal{G})$ is defined by

$$\frac{d}{dt}\bigg|_{t=0} f(X + tY) = \langle Y, (\nabla f)(X)\rangle, \quad \forall X, Y \in \mathcal{G}, \quad (4.7)$$

and, using also (2.1), the infinitesimal versions of the invariance conditions (4.2), (4.3) read

$$[X, (\nabla f)(X)]_+ = 0, \quad \forall f \in C^\infty_{G_+}(\mathcal{G}), \ X \in \mathcal{G}, \quad (4.8)$$
$$[X, (\nabla f)(X)] = 0, \quad \forall f \in C^\infty_G(\mathcal{G}), \ X \in \mathcal{G}. \quad (4.9)$$

To formulate our next result, we again refer to (2.1) and introduce the decomposition

$$\nabla f = (\nabla f)_+ + (\nabla f)_-, \qquad (\nabla f)_\pm \in C^\infty(\mathcal{G}, \mathcal{G}_\pm). \quad (4.10)$$

Theorem 4. *The constants of motion of the spin Calogero model of Theorem 1 that are provided by equation (4.5) satisfy the Poisson bracket relation*

$$\{f \circ K(x), h \circ K(y)\} \circ \mathcal{E} = xy\langle \xi, [(\nabla f)_+ \circ K(x), (\nabla h)_+ \circ K(y)]\rangle \circ \mathcal{E}$$
$$- \langle \xi, [(\nabla f)_- \circ K(x), (\nabla h)_- \circ K(y)]\rangle \circ \mathcal{E} \quad (4.11)$$

$\forall \ f, h \in C^\infty_{G_+}(\mathcal{G})$ and $x, y, \in \mathbb{R}$, with ξ being the $\mathcal{O}$ valued evaluation function on $P^{\mathcal{O}}$ (3.7). This Poisson bracket vanishes identically for any x and y if both f and h belong to $C^\infty_G(\mathcal{G})$. It also vanishes identically $\forall f \in C^\infty_{G_+}(\mathcal{G})$, $x \in \mathbb{R}$ if $h \in C^\infty_G(\mathcal{G})$ and $y^2 = 1$.

Proof. Formula (4.11) itself is readily calculated by using the Poisson bracket on $P^{\mathcal{O}}$ and imposing the constraint $\Psi = J_+ + \xi = 0$ at the end of the calculation. To verify the claimed involution properties, we introduce the shorthand

$$A^f(x) := A^f_+(x) + A^f_-(x) := (\nabla f) \circ K(x) \quad (4.12)$$

with the subscripts referring to (2.1). Now, for any $f \in C^\infty_{G_+}(\mathcal{G})$ and $h \in C^\infty_G(\mathcal{G})$, notice that the identity

$$x\langle \xi, [A^f_+(x), A^h_+(y)]\rangle = y\langle \xi, [A^f_-(x), A^h_-(y)]\rangle \quad (4.13)$$

is valid for all $x, y \in \mathbb{R}$. Indeed, this comes from the following calculation

$$x\langle \xi, [A^f_+(x), A^h_+(y)]\rangle =$$
$$= -\langle K(x), [A^f_+(x), A^h_+(y)]\rangle = -\langle K(x), [A^f(x), A^h_+(y)]\rangle + \langle K(x), [A^f_-(x), A^h_+(y)]\rangle$$
$$= \langle K(y), [A^f_-(x), A^h_+(y)]\rangle = \langle K(y), [A^f_-(x), A^h(y)]\rangle - \langle K(y), [A^f_-(x), A^h_-(y)]\rangle$$
$$= y\langle \xi, [A^f_-(x), A^h_-(y)]\rangle. \quad (4.14)$$

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By applying this identity, (4.11) gives

$$\{f \circ K(x), h \circ K(y)\} \circ \mathcal{E} = (y^2 - 1)\langle \xi, [A_-^f(x), A_-^h(y)]\rangle \circ \mathcal{E}, \tag{4.15}$$

which implies the last sentence of the theorem. If both f and h belong to $C_G^\infty(\mathcal{G})$, then similarly to (4.13) we obtain

$$y\langle \xi, [A_+^f(x), A_+^h(y)]\rangle = x\langle \xi, [A_-^f(x), A_-^h(y)]\rangle. \tag{4.16}$$

By combining (4.13) and (4.16), it follows that

$$(x^2 - y^2)\langle \xi, [A_-^f(x), A_-^h(y)]\rangle \equiv 0, \tag{4.17}$$

and by introducing the open planar subset $\mathcal{D} := \mathbb{R}^2 \setminus \{(x,y) \mid x = \pm y\}$, we see that

$$\langle \xi, [A_-^f(x), A_-^h(y)]\rangle \equiv 0, \quad \forall (x,y) \in \mathcal{D}. \tag{4.18}$$

Thus (4.15) implies

$$\{f \circ K(x), h \circ K(y)\}(m) = 0 \quad \forall (x,y) \in \mathcal{D}, \forall m \in P_{\Psi=0}^{\mathcal{O}}. \tag{4.19}$$

Since the function $\mathbb{R}^2 \ni (x,y) \mapsto \{f \circ K(x), h \circ K(y)\}(m) \in \mathbb{R}$ is continuous, it is necessarily zero on the closure of $\mathcal{D}$. This proves that (4.11) vanishes indeed for all $x, y \in \mathbb{R}$ if f and h are G invariant functions on $\mathcal{G}$. Q.E.D.

Tracing the definitions, one sees that the spin Calogero Hamiltonian $\mathcal{H}_{red}$ (3.26) can be identified as

$$\mathcal{H}_{red} = h_2 \circ K(\pm 1) \circ \mathcal{E} \quad \text{with} \quad h_2(X) = \frac{1}{2}\langle X, X \rangle \quad \forall X \in \mathcal{G}, \tag{4.20}$$

and $h_2 \circ K(x) \circ \mathcal{E}$ for any x differs from $\mathcal{H}_{red}$ only by a multiple of the irrelevant Casimir function $\langle \xi, \xi \rangle$. Taking arbitrary $f \in C_G^\infty(\mathcal{G})$ and $x \in \mathbb{R}$, (4.5) yields a family of functions in involution that contain the spin Calogero Hamiltonian. This could be sufficient for the Liouville integrability of the reduced system on a generic (or any) symplectic leaf, but counting the number of independent invariants is tricky and we do not deal with it here.

Now we explain how the Hamiltonian flows of $\mathcal{H}_{red}$ and its constants of motion in involution considered below can be determined by the projection method. We start by observing that the functions $K(1)$ defined in (4.1) and $J^{\mathcal{O}}$ defined in (3.8) coincide on the constrained manifold $P_{\Psi=0}^{\mathcal{O}}$. Consequently, we have

$$f \circ K(1) \circ \mathcal{E} = f \circ J^{\mathcal{O}} \circ \mathcal{E} \quad \forall f \in C_G^\infty(\mathcal{G}). \tag{4.21}$$

This means that the functions $f \circ K(1)$ and $f \circ J^{\mathcal{O}}$ are the same from the point of view of the reduced system, whence their reduced Hamiltonian flows are also the same. The Hamiltonian flow of $f \circ J^{\mathcal{O}} \in C^\infty(P^{\mathcal{O}})$ with any initial value $(\Lambda_0, J_-^0, \xi^0) \in P^{\mathcal{O}}$ is given explicitly by

$$(\Lambda(t), J_-(t), \xi(t)) = (e^{t\nabla f(J_0)}\Lambda_0 e^{-t\theta(\nabla f(J_0))}, J_-^0, \xi^0), \quad J_0 := J_-^0 + J_+(\Lambda_0, J_-^0). \tag{4.22}$$

The flow (4.22) preserves $P^{\mathcal{O}}_{\Psi=0}$ and its projection to the reduced phase space integrates the Hamiltonian vector field of the function (4.21) regarded as an element of $C^\infty(P_{red})$. Developed in more detail, one can find the flows induced on P_{red} by the conserved quantities (4.21) as follows. The first step is to determine $\Lambda_0 = e^{2q_0}$ and $J^0_- = p_0 - (\coth \mathrm{ad}_{q_0})\xi^0$ from the initial value $(q_0, p_0, [\xi^0]) \in P_{red}$, where ξ^0 is any representative of $[\xi^0] \in \mathcal{O}_{red}$. The second step is to find the curve (4.22). Finally, one projects this curve to the reduced phase space by diagonalizing $\Lambda(t)$ as $\Lambda(t) = g_+(t)e^{2q(t)}g_+^{-1}(t)$ with $g_+(t) \in G_+$, whereby $q(t)$ gives the trajectory in $\check{\mathcal{A}}$, at least for small t. (It can in principle occur that $q(t)$ reaches the boundary of $\check{\mathcal{A}}$ at finite t, which corresponds to the incompleteness of the Hamiltonian vector field on P_{red}.) Incidentally, the set of functions (4.21) coincides with $\{f \circ K(-1) \circ \mathcal{E} \mid f \in C^\infty_G(\mathcal{G})\}$. For different conserved quantities, if exist, one must use a more complicated algorithm to find the flows.

In view of the involution properties given by Theorem 4, one may wonder if there exist any G_+ invariant functions for which (4.11) is non-vanishing, for some orbit $\mathcal{O}$ of some group G. We shall furnish examples of such functions at the end of Section 5.

4.2 Lax representation

In order to find a Lax pair for the system given by Theorem 1, let us start with a remark on how to obtain the Hamiltonian vector field of the reduced system in correspondence with an invariant Hamiltonian in general. Namely, suppose that V is the Hamiltonian vector field before reduction and σ is the gauge slice of a (partial or complete) gauge fixing in the constrained manifold defined by the momentum map constraint. Then the reduced evolution equation is generated by a vector field V^* on σ, which always has the form

$$V^* = V|_\sigma + Y, \tag{4.23}$$

where Y is the generator of certain infinitesimal gauge transformations. The 'correction term' Y is (partially or completely) determined by the condition that V^* must be tangent to σ. In the case of a complete gauge fixing, V^* is the Hamiltonian vector field with respect to the reduced Poisson bracket (alias the Dirac bracket) associated with the gauge slice σ.

We now take σ to be either the 'thick slice' S (3.12) or the cross section of a (local) complete gauge fixing inside S. We can parametrize the general element of σ as a triple

$$(e^{2q}, L_-, \xi_\sigma) \quad \text{with} \quad L_- = p - \coth(\mathrm{ad}_q)\xi_\sigma, \tag{4.24}$$

where $q \in \check{\mathcal{A}}$, $p \in \mathcal{A}$ and $\xi_\sigma \in \mathcal{O} \cap \mathcal{M}^\perp$ with further restrictions on the form of ξ_σ if σ is a complete gauge fixing. The derivatives, $\mathcal{L}_{V^*}$, of these variables along V^* are subject to

$$\mathcal{L}_{V^*}(q) = L_- - \frac{1}{2}\sinh(2\mathrm{ad}_q)\mathcal{Y}, \quad \mathcal{L}_{V^*}(L_-) = [\mathcal{Y}, L_-], \qquad \mathcal{L}_{V^*}(\xi_\sigma) = [\mathcal{Y}, \xi_\sigma], \tag{4.25}$$

where $\mathcal{Y}$ is a $\mathcal{G}_+$-valued function on σ realizing the term Y in (4.23). The formulae in (4.25) follow by combining the Hamiltonian vector field V of the system (3.6), which can be read off from (2.22) and $\mathcal{L}_V(\xi) = 0$, and the infinitesimal variant of the gauge transformations (3.9). By decomposing the gauge transformation parameter $\mathcal{Y}$ using (2.5),

$$\mathcal{Y} = \mathcal{Y}_{\mathcal{M}} + \mathcal{Y}_{\mathcal{M}^\perp}, \tag{4.26}$$

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the relation $\mathcal{L}_{V^*}(q) \in \mathcal{A}$ and the form of L_- (4.24) fix $\mathcal{Y}_{\mathcal{M}^\perp}$ uniquely as

$$\mathcal{Y}_{\mathcal{M}^\perp} = -w^2(\mathrm{ad}_q)\xi_\sigma \tag{4.27}$$

with the analytic function

$$w(z) = (\sinh z)^{-1}. \tag{4.28}$$

The component $\mathcal{Y}_{\mathcal{M}}$ is arbitrary if $\sigma = S$ (when it gives a residual gauge transformation), and in general it is subject to the requirement that the form of $\mathcal{L}_{V^*}(\xi_\sigma)$ must be consistent with the gauge fixing conditions imposed on ξ_σ. By (4.25) and (4.27), $\mathcal{Y}_{\mathcal{M}}$ can be taken to be independent of p, but depends on q and ξ_σ in general. Identifying $\mathcal{L}_{V^*}$ with the evolutional derivative, denoted by dot, we immediately obtain the following result.

Proposition 5. *Let us describe the spin Calogero system of Theorem 1 using a gauge slice $\sigma \subseteq S$ parametrized by (4.24). Then the evolution equation can be written as $\dot{q} = p$ and*

$$\dot{p} = [w^2(\mathrm{ad}_q)\xi_\sigma, \coth(\mathrm{ad}_q)\xi_\sigma]_{\mathcal{A}}, \tag{4.29}$$
$$\dot{\xi}_\sigma = [\mathcal{Y}_{\mathcal{M}} - w^2(\mathrm{ad}_q)\xi_\sigma, \xi_\sigma], \tag{4.30}$$

where $\mathcal{Y}_{\mathcal{M}} : \sigma \to \mathcal{M}$ yields an infinitesimal gauge transformation so that (4.30) is consistent with the form of ξ_σ. Defining the functions $L(x) : \sigma \to \mathcal{G}$ (for any $x \in \mathbb{R}$) and $\mathcal{Y} : \sigma \to \mathcal{G}_+$ by

$$L(x) := p - \coth(\mathrm{ad}_q)\xi_\sigma - x\,\xi_\sigma, \qquad \mathcal{Y} := \mathcal{Y}_{\mathcal{M}} - w^2(\mathrm{ad}_q)\xi_\sigma, \tag{4.31}$$

equations (4.29)-(4.30) are equivalent to

$$\dot{L}(x) = [\mathcal{Y}, L(x)]. \tag{4.32}$$

The conserved quantities associated with this Lax equation are the same as those exhibited in Theorem 4, since $L(x)$ is the restriction of the function $K(x)$ (4.1) to the gauge slice σ,

$$L(x) = K(x)|_\sigma. \tag{4.33}$$

To verify Proposition 5, it is sufficient to note that $\dot{q} = p$ and (4.29) follow, respectively, from the $\mathcal{A}$-components of the first and the second equations under (4.25), and (4.30) also follows directly from (4.25). (By construction, the evolution equation just obtained is generated by $\mathcal{H}_{red}$ (3.26) through the Dirac bracket if σ is a complete gauge fixing.) In view of (4.33), the conserved quantities in involution described in Theorem 4 receive the usual interpretation as the G invariant functions of the Lax matrix. This is valid since any point of $P_{\Psi=0}^{\mathcal{O}}$ can be transformed into the gauge slice σ by the action of G_+, and such a gauge transformation may be used to convert $K(x)$ into $L(x)$ since $K(x)$ is a G_+ equivariant function on $P^{\mathcal{O}}$.

If for some value of ξ_σ, say $\xi_\sigma = \mu$, with a suitable function $\mathcal{Y}_{\mathcal{M}}(q, \xi_\sigma) \in \mathcal{M}$, it so happens that

$$[\mathcal{Y}_{\mathcal{M}}(q, \mu) - w^2(\mathrm{ad}_q)\mu, \mu] = 0 \qquad \forall q \in \check{\mathcal{A}}, \tag{4.34}$$

then one can 'freeze' the spin variable to that value μ (see (4.30)). By using the identity

$$[w^2(\mathrm{ad}_q)Z, Z] = (\sinh \mathrm{ad}_q)[w(\mathrm{ad}_q)Z, w'(\mathrm{ad}_q)Z] \qquad \forall Z \in \mathcal{M}^\perp, \tag{4.35}$$

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one can check that (4.34) is equivalent to

$$[\mathcal{Y}_{\mathcal{M}}(q, \mu), w(\mathrm{ad}_q)\mu] = [w(\mathrm{ad}_q)\mu, w'(\mathrm{ad}_q)\mu]_{\mathcal{A}^\perp}, \qquad \forall q \in \check{\mathcal{A}}, \tag{4.36}$$

where the subscript means projection onto $\mathcal{A}^\perp$ according to (2.5). Equation (4.36) appears[1] in the work of Olshanetsky and Perelomov, too, as the key condition for obtaining Lax representations for Calogero type models (1.1). In the cases for which such a constant value μ exists, the specializations of our Lax operator $L(x)$ furnished by

$$L_- = L(0) = p - \coth(\mathrm{ad}_q)\xi_\sigma \quad \text{and} \quad \mathcal{L} := e^{-\mathrm{ad}_q} L(1) = p - w(\mathrm{ad}_q)\xi_\sigma, \tag{4.37}$$

reproduce precisely the alternative Lax operators of [10] upon setting $\xi_\sigma := \mu$. Notice that $L(1)$ is essentially the same as the function L introduced in (3.22), and the conjugation by e^{-q} is useful since it leads to a $\mathcal{G}_-$ valued Lax operator. By looking at the explicit form of $\mathcal{H}_{red}$ (3.26), it is reasonable to expect that (4.34) holds only if the reduced Poisson structure of $\mathcal{O}_{red}$ vanishes at $[\mu]$. This is the case automatically whenever $\mathcal{O}_{red}$ is a trivial space consisting of a single point, which is realized in the examples described in the subsequent section.

5 Spinless models obtainable by the KKS mechanism

Here we first recall that the spinless model (1.1) of A_{k-1} type (the 'hyperbolic Sutherland model') arises from the symmetric space $SL(k, \mathbb{C})/SU(k)$ by using a minimal coadjoint orbit of $SU(k)$. In fact, the reduced orbit (3.25) consists of a single point in this case [13]. Relying on the mechanism that works in this basic example, we then explain why the spinless BC_n model is associated with $SU(n+1, n)$, as presented in [10, 16, 19] without detailed explanation.

The standard Cartan involution of $G = SL(k, \mathbb{C})$ operates as $\Theta(g) = (g^\dagger)^{-1}$ and

$$sl(k, \mathbb{C}) = su(k) + \mathrm{i}\, su(k) \tag{5.1}$$

is the corresponding Cartan decomposition of the real simple Lie algebra $\mathcal{G} = sl(k, \mathbb{C})$. By using the natural embedding, we can take $\mathcal{A} = \mathrm{i}\mathcal{T}_{k-1}$, where $\mathcal{T}_{k-1}$ denotes the standard Cartan subalgebra of $su(k)$. Then $\mathcal{M} = \mathcal{T}_{k-1}$ and $M = \mathbf{T}_{k-1}$ is the maximal torus of $G_+ = SU(k)$. For any $u \in \mathbb{C}^k$, viewed as a column vector, we define

$$\eta(u) := \mathrm{i}\left(uu^\dagger - \frac{u^\dagger u}{k}\mathbf{1}_k\right) \in su(k), \tag{5.2}$$

with $\mathbf{1}_k$ denoting the unit matrix. The minimal coadjoint orbits of $SU(k)$ are provided by

$$\mathcal{O}^{k,\kappa} := \{\eta(u) \mid u \in \mathbb{C}^k, \quad u^\dagger u = k\kappa\}, \tag{5.3}$$

where $\kappa > 0$ is a constant. (Of course, $-\mathcal{O}^{k,\kappa}$ is also a minimal orbit, but it either coincides with $\mathcal{O}^{k,\kappa}$ or is obtained from it by an automorphism of $su(k)$. Since $\pm\mathcal{O}^{k,\kappa}$ always lead to similar systems, we may focus on $\mathcal{O}^{k,\kappa}$.) We need the constrained orbit

$$\mathcal{O}_0^{k,\kappa} := \{\eta(u) \in \mathcal{O}^{k,\kappa} \mid \eta(u)_{a,a} = 0 \quad \forall a = 1, \ldots, k\}. \tag{5.4}$$

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[1] Equation (4.36) corresponds to (2.22) in [10] by identifying $\mathcal{Y}_{\mathcal{M}}$ and $-w(\mathrm{ad}_q)\mu$ with the objects D and X used there. In effect, in [10] an ansatz was also adopted for μ, which is confirmed in the examples of Sect. 5.

Note that $\mathcal{O}_0^{k,\kappa} = \mathcal{O}^{k,\kappa} \cap \mathcal{M}^{\perp}$ and $\eta(u) \in \mathcal{O}_0^{k,\kappa}$ is associated with $u \in \mathbb{C}^k$ of the form

$$u_a = \sqrt{\kappa}e^{\mathrm{i}\beta_a}, \quad \beta_a \in \mathbb{R}, \quad \forall a = 1, \ldots, k. \tag{5.5}$$

This implies that any $\eta(u) \in \mathcal{O}_0^{k,\kappa}$ can be transformed by $\mathbf{T}_{k-1}$ into the representative $\mu^{k,\kappa}$ furnished by the matrix

$$(\mu^{k,\kappa})_{a,b} = \mathrm{i}\kappa(1 - \delta_{a,b}), \tag{5.6}$$

showing that

$$\mathcal{O}_{red}^{k,\kappa} = (\mathcal{O}^{k,\kappa} \cap \mathcal{M}^{\perp})/M = \mathcal{O}_0^{k,\kappa}/\mathbf{T}_{k-1} \tag{5.7}$$

consists of a single point indeed. One can readily calculate that the resulting Hamiltonian is given by (1.1) with $\mathcal{R}$ now being the root system of $sl(k, \mathbb{C})$ (and $g_\alpha^2 \sim \kappa^2$). The way whereby the orbital reduced space (5.7) is trivial is referred to below as the 'KKS mechanism', since the choice of the orbit (5.3) goes back to the classical work of Kazhdan, Kostant and Sternberg [13], where the Sutherland model was first derived by Hamiltonian reduction.

Now we make the following important observation: *A spinless Calogero model (1.1) arises from the symmetric space G/G_+ if the 'KKS mechanism' that works for $SL(k, \mathbb{C})$ as described above can be applied by embedding.* For this to be realized, G_+ must contain a simple factor of $SU(k)$ type and M must act on the minimal orbits of this $SU(k)$ factor as the maximal torus $\mathbf{T}_{k-1} \subset SU(k)$. By inspecting the properties of the real simple Lie algebras tabulated in [23], one sees that *these conditions single out the algebras $\mathcal{G} = su(m, n)$, for all $m \geq n$.* In fact, among the classical Lie algebras there are no other cases for which $\mathcal{G}_+$ contains $su(k)$ and at the same time $\mathcal{M}$ contains a non-zero Abelian factor.[2] The system of restricted roots of $su(m, n)$ is of C_n type if $m = n$, and BC_n type if $m > n$. Nevertheless, as we explain below, the spinless BC_n Calogero model can only be associated with $su(n + 1, n)$.

By using $I_{m,n} := \mathrm{diag}(\mathbf{1}_m, -\mathbf{1}_n)$ with $m \geq n$, the standard realizations of the Lie group $SU(m, n)$ and its Lie algebra $su(m, n)$ are

$$SU(m, n) = \{g \in SL(m + n, \mathbb{C}) \,|\, g^{\dagger}I_{m,n}g = I_{m,n}\}, \tag{5.8}$$

$$su(m, n) = \{X \in \ sl(m + n, \mathbb{C}) \,|\, X^{\dagger}I_{m,n} + I_{m,n}X = 0\}. \tag{5.9}$$

Written as a block matrix, $X \in \mathcal{G} = su(m, n)$ has the form

$$X = \begin{pmatrix} A & B \\ B^{\dagger} & D \end{pmatrix}, \tag{5.10}$$

where $B \in \mathbb{C}^{m \times n}$, $A \in u(m)$, $D \in u(n)$ and $\mathrm{tr}\,A + \mathrm{tr}\,D = 0$. The Cartan involution of $G = SU(m, n)$ is $\Theta : g \mapsto (g^{\dagger})^{-1}$, and thus

$$G_+ = S(U(m) \times U(n)), \tag{5.11}$$

$$\mathcal{G}_+ = su(m) \oplus su(n) \oplus \mathbb{R}C_{m,n} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + xC_{m,n} \,\middle|\, A \in su(m), D \in su(n), x \in \mathbb{R} \right\} \tag{5.12}$$

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[2]This is also true for the exceptional Lie algebras apart from E_6. There exists a real form of E_6 [23] for which $\mathcal{G}_+ = su(6) \oplus su(2)$ and $\mathcal{M}$ is Abelian of dimension 2. By studying the relative position of $\mathcal{M}$ and the $su(2)$ factor of $\mathcal{G}_+$, it would be interesting to investigate if the KKS mechanism is applicable in this case or not.

with the central element
$$C_{m,n} := \mathrm{diag}(in\mathbf{1}_m, -im\mathbf{1}_n). \tag{5.13}$$

A convenient choice for the maximal Abelian subspace of
$$\mathcal{G}_- = \left\{ \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} \,\middle|\, B \in \mathbb{C}^{m \times n} \right\} \tag{5.14}$$

is given by
$$\mathcal{A} := \left\{ \begin{pmatrix} \mathbf{0}_n & 0 & Q \\ 0 & \mathbf{0}_{m-n} & 0 \\ Q & 0 & \mathbf{0}_n \end{pmatrix} \in \mathcal{G}_- \,\middle|\, Q = \mathrm{diag}(q^1, \dots, q^n), \ q^j \in \mathbb{R} \right\}. \tag{5.15}$$

Taking $\chi := \mathrm{diag}(\chi_1, \dots, \chi_n)$ with any $\chi_j \in \mathbb{R}$, the centralizer of $\mathcal{A}$ in $\mathcal{G}_+$ is
$$\mathcal{M} = \{\mathrm{diag}(i\chi, \gamma, i\chi) \mid \gamma \in u(m-n), \ \mathrm{tr}\,\gamma + 2i\mathrm{tr}\,\chi = 0\}, \tag{5.16}$$

and the corresponding subgroup of G_+ is
$$M = \{\mathrm{diag}(e^{i\chi}, \Gamma, e^{i\chi}) \mid \Gamma \in U(m-n), \ (\det \Gamma)(\det e^{i2\chi}) = 1\}. \tag{5.17}$$

Let $\mathcal{O}^k$ be an arbitrary orbit of $SU(k)$. For $k \in \{m, n\}$, denote by $\tilde{\mathcal{O}}^k$ the natural embedding of $\mathcal{O}^k$ into an $su(k)$ factor of $\mathcal{G}_+$ (5.12). If $m \neq n$, then the most general coadjoint orbit of G_+ (5.11) has the form
$$\mathcal{O} = \tilde{\mathcal{O}}^m + \tilde{\mathcal{O}}^n + xC_{m,n} \qquad (x \in \mathbb{R}). \tag{5.18}$$

In the $SU(n,n)$ case we can similarly embed different orbits of $SU(n)$ into the two isomorphic factors. In order to get a trivial reduced space $\mathcal{O} \cap \mathcal{M}^\perp / M$ by the KKS mechanism, we must take the constituent orbits to be of the type $\mathcal{O}^{k,\kappa}$ (5.3). If $m \leq (n+1)$, then a simple dimension counting argument says that $\mathcal{O} \cap \mathcal{M}^\perp / M$ could possibly be a trivial space only if either $\tilde{\mathcal{O}}^m$ or $\tilde{\mathcal{O}}^n$ in (5.18) is taken to be zero. Furthermore, it follows from the structure of $\mathcal{M}$ (5.16) that if $m > (n+1)$, then the KKS mechanism could be applicable only if the non-trivial component of $\mathcal{O}$ is contained in the factor of size n. Detailed, simple inspection leads to the following result.

Theorem 6. *For any $m > n$, and similarly for $m = n$, consider the family (5.18) of non-zero coadjoint orbits of $SU(m,n)$ with minimal orbits (5.3) as constituents. In this family,*
$$\mathcal{O} \cap \mathcal{M}^\perp / M \tag{5.19}$$

consists of a single point precisely in the following cases:
$$\textit{for any } m \geq n, \textit{ the orbits of type } \quad \tilde{\mathcal{O}}^{n,\kappa} \quad \forall \kappa > 0, \tag{5.20}$$

$$\textit{for } m = n, \textit{ all non-zero orbits of the form } \quad \tilde{\mathcal{O}}^{n,\kappa} + xC_{n,n} \quad \forall \kappa \geq 0, \ x \in \mathbb{R}, \tag{5.21}$$

$$\textit{for } m = (n+1), \textit{ the orbits } \quad \tilde{\mathcal{O}}^{n+1,\kappa} + xC_{n+1,n} \quad \forall \kappa > 0, \ x \in \mathbb{R} \quad \textit{satisfying} \tag{5.22}$$

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in this case $(\kappa - nx) \geq 0$ and $(\kappa + x) \geq 0$.

Proof. Let us consider an orbit of $SU(n+1, n)$ of the form

$$\mathcal{O} := \tilde{\mathcal{O}}^{n+1,\kappa} + xC_{n+1,n}. \tag{5.23}$$

The general element $\xi \in \mathcal{O}$ can be written as

$$\xi = \tilde{\eta} + xC_{n+1,n}, \tag{5.24}$$

where $\tilde{\eta} \in \tilde{\mathcal{O}}^{n+1,\kappa}$ is the embedding of $\eta \in \mathcal{O}^{n+1,\kappa}$ into the factor $su(n+1)$ of $\mathcal{G}_+$ (5.12). By the mapping $\eta \mapsto \xi$ according to (5.24), the constraint

$$\mathrm{tr}\,(X\xi) = 0 \qquad \forall X \in \mathcal{M} \tag{5.25}$$

turns out to be equivalent to

$$\eta_{diag} = \mathrm{diag}(\mathrm{i}x\mathbf{1}_n, -\mathrm{i}xn), \tag{5.26}$$

where η_{diag} denotes the diagonal part of the matrix $\eta \in su(n+1)$. Thus we have a one-to-one correspondence between $\mathcal{O} \cap \mathcal{M}^\perp$ and the 'constrained KKS orbit' $\mathcal{O}_x^{n+1,\kappa}$ consisting of the elements $\eta \in \mathcal{O}^{n+1,\kappa}$ subject to (5.26). By this correspondence, the action of M (5.17) on $\mathcal{O} \cap \mathcal{M}^\perp$ can be represented as the action of $\mathbf{T}_n \subset SU(n+1)$ on $\mathcal{O}_x^{n+1,\kappa}$, which gives rise to a one-to-one map

$$\mathcal{O} \cap \mathcal{M}^\perp / M \longleftrightarrow \mathcal{O}_x^{n+1,\kappa} / \mathbf{T}_n. \tag{5.27}$$

Therefore we have to show that the latter space consists of a single point. Now write any $\eta \in \mathcal{O}^{n+1,\kappa}$ as $\eta(u)$ with

$$u = (u_1, \ldots, u_n, u_{n+1})^t, \tag{5.28}$$

using the notation (5.2). The constraint (5.26) requires that

$$|u_j|^2 = (\kappa + x) \quad \forall j = 1, \ldots, n, \quad \text{and} \quad |u_{n+1}|^2 = (\kappa - xn)\,. \tag{5.29}$$

Hence the constants κ and x has to be chosen so that the right hand sides above are non-negative. Now the point is that given the constraint (5.29), we can bring any u by a $\mathbf{T}_n$ transformation to the following normal form, say $\hat{u}$:

$$\hat{u}_j = e^{\mathrm{i}\alpha}\sqrt{\kappa + x} \quad \forall j = 1, \ldots, n, \qquad \hat{u}_{n+1} = e^{\mathrm{i}\alpha}\sqrt{\kappa - xn} \tag{5.30}$$

with some $\alpha \in \mathbb{R}$. Since u matters only up to phase, $\eta(e^{\mathrm{i}\alpha}u) = \eta(u)$, we see that every point of $\mathcal{O}_x^{n+1,\kappa}$ can be transformed into $\eta(\hat{u})$ by the action of $\mathbf{T}_n$. This means that $\mathcal{O}_x^{n+1,\kappa}$ consists of a single orbit of $\mathbf{T}_n$, and may be represented by $\eta(\hat{u})$. By the correspondence (5.27), it follows that the reduction of $\mathcal{O}$ (5.23) by M yields a trivial space, and as a representative of this space one may take the matrix

$$\xi_{red} := \tilde{\eta}(\hat{u}) + xC_{n+1,n} \in su(n+1, n). \tag{5.31}$$

It is quite similar but even simpler to verify that (5.19) consists of a single element also in the cases listed under (5.20) and (5.21). In all other cases when simple counting does not

exclude that $\mathcal{O} \cap \mathcal{M}^\perp$ contains a single orbit of M, the constraints, $\operatorname{tr}(X\xi) = 0 \ \forall X \in \mathcal{M}$, are found to be inconsistent with the form of the orbits considered. For example, one can check for $SU(n+1, n)$ that

$$(\tilde{\mathcal{O}}^{n,\kappa} + xC_{n+1,n}) \cap \mathcal{M}^\perp = \emptyset \qquad \forall x \neq 0, \ \kappa \geq 0. \tag{5.32}$$

Hence we may conclude that the list given in the theorem exhausts all cases for which (5.19) consists of a single point by the KKS mechanism. *Q.E.D.*

One may use the conventions collected in the appendix to obtain the Hamiltonians of the spinless Calogero models corresponding to the various cases listed under Theorem 6. In the most complicated case of equation (5.22), it is easily checked that the representative ξ_{red} (5.31) of the reduced orbit can be expanded in the form

$$\xi_{red} = 2g \sum_{1 \leq k < l \leq n} (E_{e_k - e_l}^{+,i} + E_{e_k + e_l}^{+,i}) + 2g_1 \sum_{k=1}^{n} E_{e_k}^{+,i} + 2g_2 \sum_{k=1}^{n} E_{2e_k}^{+,i}, \tag{5.33}$$

where we use the basis introduced in (A.6)-(A.13) together with the notation

$$g := \frac{\kappa + x}{2}, \qquad g_1 := \sqrt{\frac{(\kappa + x)(\kappa - nx)}{2}}, \qquad g_2 := \frac{(n+1)x}{\sqrt{2}}. \tag{5.34}$$

From (4.37) with (4.28), the corresponding Lax operator is

$$\mathcal{L}(q, p) = p - w(\mathrm{ad}_q)\xi_{red}, \tag{5.35}$$

where $q \in \check{\mathcal{A}}$ is parametrized by $\operatorname{diag}(q^1, \ldots, q^n)$ according to (5.15), now with $m = n+1$, and $p \in \mathcal{A}$ is similarly parametrized by $\operatorname{diag}(p_1, \ldots, p_n)$. This leads to the Hamiltonian

$$H_{BC_n}(q, p) := \frac{1}{4}\operatorname{tr}\left(\mathcal{L}(q, p)\right)^2 = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n} \frac{g_1^2}{\sinh^2(q^k)} + \sum_{k=1}^{n} \frac{g_2^2}{\sinh^2(2q^k)}$$

$$+ \sum_{1 \leq k < l \leq n} \frac{g^2}{\sinh^2(q^k - q^l)} + \sum_{1 \leq k < l \leq n} \frac{g^2}{\sinh^2(q^k + q^l)}. \tag{5.36}$$

On account of (5.34), the coupling constants satisfy the quadratic relation

$$g_1^2 - 2g^2 + \sqrt{2}gg_2 = 0. \tag{5.37}$$

One can similarly spell out the Hamiltonian in the other cases of Theorem 6. Although we do not obtain any spinless models that were not described before in the symmetric space framework, it is worth summarizing the list of the resulting models as a proposition.

Proposition 7. *The Calogero type Hamiltonian corresponding to case (5.22) of Theorem 6 is H_{BC_n} (5.36) with the relation (5.37). The Hamiltonian in the case (5.21) turns out to be*

$$H_{C_n}(q, p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{1 \leq k < l \leq n} \frac{\kappa^2/4}{\sinh^2(q^k - q^l)} + \sum_{1 \leq k < l \leq n} \frac{\kappa^2/4}{\sinh^2(q^k + q^l)} + \sum_{k=1}^{n} \frac{n^2 x^2/2}{\sinh^2(2q^k)}. \tag{5.38}$$

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The orbit (5.20) leads to

$$H_{D_n}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{1 \le k < l \le n} \frac{\kappa^2/4}{\sinh^2(q^k - q^l)} + \sum_{1 \le k < l \le n} \frac{\kappa^2/4}{\sinh^2(q^k + q^l)} . \tag{5.39}$$

The statement of Proposition 7 amounts to a systematization of known results. Indeed, the Lax matrix (5.35) of the BC_n model (5.36) reproduces[3] the original result of Olshanetsky and Perelomov. The C_n and D_n models obtained by the KKS mechanism are essentially degenerations of the BC_n model. The C_n model is treated in [20] by using Hamiltonian reduction (see also [21]). The constants of motion provided by the eigenvalues of the Lax matrix (5.35) guarantee Liouville integrability if (5.37) holds, but one needs a different approach for proving that the BC_n model (5.36) is also integrable with three arbitrary coupling constants [27].

Remark 8. For $\mathcal{G} := su(m,n)$ with any $m \ge n$, let us consider the functions

$$f_k : \mathcal{G} \to \mathbb{R}, \qquad f_k(X) := \text{tr}\,((ABDB^\dagger)^k) \qquad (k = 1, \ldots, n), \tag{5.40}$$

where $X \in \mathcal{G}$ is written in the form (5.10). These functions are $G_+ = S(U(m) \times U(n))$ invariant, and hence give rise to conserved quantities for all spin Calogero models based on $su(m,n)$ as follows from Theorem 4. For the models provided by Proposition 7 these constants of motion are not independent from the eigenvalues of the Lax matrices in (4.37). This can be seen by combining Theorem 4 with the fact [19] that the eigenvalues of any of the two Lax matrices (4.37) generate the same *maximal set of constants of motion in involution* for the above spinless models. However, for general spin Calogero models based on $su(m,n)$ the conserved quantities associated with the functions f_k are independent from the conserved quantities in involution furnished by Theorem 4. In fact, we have checked in several cases (even numerically) that the functions $f_k \circ K(x)$ Poisson commute neither with each other for different k nor with all of the invariants in involution $h \circ K(y)$ $(\forall h \in C_G^\infty(\mathcal{G}))$ described in Theorem 4. It could be an interesting problem for the future to clarify the various possible (Liouville, degenerate, super) integrability properties of the spin Calogero models that we obtained, and in particular to understand the role of the conserved quantities just exhibited.

6 Discussion

In this paper we investigated the symmetry reductions of the geodesic motion on a symmetric space of negative curvature G/G_+ based on the action of G_+ on G/G_+. Taking an *arbitrary* value of the momentum map and restricting to regular elements in the configuration space, the result turned out to be the hyperbolic spin Calogero model characterized by Theorem 1.

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[3]Our conventions are chosen so that (5.36) reproduces the BC_n Hamiltonian as given by Olshanetsky and Perelomov and (5.37) coincides with the correct relation (B.11) in [10]. (This quadratic relation is mistyped in (3.3) in [10] and also in [16, 19] where $g_1 g_2$ appears in place of gg_2.) Our Lax pair, defined by (5.35) with Proposition 5 and (4.37), reproduces their BC_n Lax pair after similarity transformation by a constant matrix.

We analyzed the integrability properties of this family of models in Section 4, describing many conserved quantities in Theorem 4 and a spectral parameter dependent Lax pair in Proposition 5. In Section 5 we classified the cases yielding spinless Calogero models of type (1.1) relying on the KKS mechanism. We conjecture that no other spinless models arise in the Hamiltonian reduction framework, even without assuming the applicability of the KKS mechanism.

Trigonometric spin Calogero models appear similarly in the positive curvature case, the corresponding rational models are related to the symmetric spaces of zero curvature [30, 31], and analogous elliptic models should also exist. Further generalizations can be obtained, for instance, by reducing the geodesic motion on affine symmetric spaces [33]. In fact, this yields spin extensions of the Calogero models attached to root systems with signature [14, 18]. All these examples fit in the theory of singular symplectic reduction of cotangent bundles with a single isotropy type in the configuration space ([24, 31, 34] and references therein).

Let us recall [35] that, in classical integrable systems that admit a diagonalizable Lax matrix with Poisson commuting eigenvalues, the Poisson brackets between the matrix elements of the Lax matrix are always encoded by some classical r-matrix that may depend on the dynamical variables. Starting from [36], a lot of effort went into finding the dynamical r-matrices of Calogero type models. In the Hamiltonian reduction setting the integrability properties can be analyzed directly, but we are nevertheless interested in the corresponding dynamical r-matrices, too. So far we computed the r-matrix belonging to the Lax matrix $\mathcal{L}$ (4.37) by using a complete (local) gauge fixing $\sigma \subset S$ of type (4.24). With the usual St Petersburg notation, we found that the Dirac brackets associated with the gauge fixing can be written as follows:

$$\{\mathcal{L}_1, \mathcal{L}_2\}^* = [r_{12} + d_{12}, \mathcal{L}_1] - [r_{21} + d_{21}, \mathcal{L}_2], \tag{6.1}$$

where $r_{12} \in \mathcal{M}^\perp \otimes \mathcal{A}^\perp$ depends on the variable q as

$$r_{12}(q) = \sum_{\alpha \in \mathcal{R}_+} \sum_{k=1}^{\nu_\alpha} \coth \alpha(q) E_\alpha^{+,k} \otimes E_\alpha^{-,k} \tag{6.2}$$

and $d_{12} \in \mathcal{M} \otimes \mathcal{A}^\perp$ depends in general also on the spin variable as

$$d_{12}(q, \xi_\sigma) = \sum_{\alpha, k, b} D_{k,b}^\alpha(\xi_\sigma)(\sinh \alpha(q))^{-1} M^b \otimes E_\alpha^{-,k} \tag{6.3}$$

with coefficients $D_{k,b}^\alpha(\xi_\sigma)$ determined by the constraints defining the gauge fixing. Here $\{M^b\}$ denotes a basis of $\mathcal{M}$, and ξ_σ becomes a constant if $\mathcal{O}_{red}$ consists of one point. In the spinless examples listed in Section 5 equation (6.1) reproduces and extends previous results of [20, 21]. Details will be presented elsewhere.

Finally, let us briefly discuss the quantization of the models (1.2). Clearly, the quantum mechanical analogue of the phase space (3.6) is $L^2(G/G_+, V_\Lambda)$, i.e., V_Λ valued square-integrable functions on G/G_+. Here V_Λ is an irreducible representation of G_+ corresponding to a quantizable coadjoint orbit $\mathcal{O}$. It is easy to see that quantum Hamiltonian reduction requires restriction to the G_+ equivariant wave functions in the Hilbert space, which can be represented by functions on the Weyl chamber $\check{\mathcal{A}}$ with values in $V_\Lambda[0]$, where $V_\Lambda[0]$ consists of the invariants in

V with respect to the action of the subgroup $M \subset G_+$. The Casimirs of $\mathcal{G}$ yield commuting self-adjoint operators on the reduced Hilbert space formed by the these $V_\Lambda[0]$ valued functions. Thus the quadratic Casimir gives rise to the Hamiltonian of the spin Calogero model, which is a spinless model at the quantum mechanical level if and only if $\dim(V_\Lambda[0]) = 1$. The analysis of quantum (spin) Calogero models translates in this way into problems in harmonic analysis and representation theory. Quite an analogous procedure can be applied starting with positive or zero curvature symmetric spaces, too. We plan to elaborate the quantization in the future building on the previous works dealing with special cases [17].

A   Restricted roots and convenient basis for $su(m,n)$

For reference in the main text, in this appendix we present the restricted roots and the basis elements $E_\alpha^{\pm,k}$ (2.6) explicitly for $\mathcal{G} := su(m,n)$ with any $m \geq n$, using the realization of this real simple Lie algebra and its Cartan involution specified in (5.8)-(5.15).

Now it proves convenient to present any matrix $X \in su(m,n)$ in a block-form corresponding to the partition $(m+n) = n + (m-n) + n$, i.e.,

$$X = \begin{pmatrix} a & v & b \\ -v^\dagger & e & w \\ b^\dagger & w^\dagger & d \end{pmatrix}, \qquad \mathrm{tr}\, a + \mathrm{tr}\, e + \mathrm{tr}\, d = 0, \tag{A.1}$$

where $a, d \in u(n)$, $e \in u(m-n)$ and $v \in \mathbb{C}^{n \times (m-n)}$ parametrize $\mathcal{G}_+$, and $b \in \mathbb{C}^{n \times n}$, $w \in \mathbb{C}^{(m-n) \times n}$ parametrize $\mathcal{G}_-$. Writing the general element of $\mathcal{A}$ (5.15) as

$$q := \begin{pmatrix} 0 & 0 & Q \\ 0 & 0 & 0 \\ Q & 0 & 0 \end{pmatrix} \quad \text{with} \quad Q = \mathrm{diag}(q^1, \ldots, q^n), \quad q^j \in \mathbb{R}, \tag{A.2}$$

one may introduce the functionals $e_k \in \mathcal{A}^*$ ($k = 1, \ldots, n$) by $e_k(q) := q^k$. The system of restricted roots, $\mathcal{R}$, is of BC_n type if $m > n$ and of C_n type if $m = n$. Indeed, $\mathcal{R}$ is given by $\mathcal{R} = \mathcal{R}_+ \cup (-\mathcal{R}_+)$ with

$$\mathcal{R}_+ := \{e_k \pm e_l \, (1 \leq k < l \leq n), \, 2e_k, e_k \, (1 \leq k \leq n)\} \quad \text{if} \quad m > n, \tag{A.3}$$

and

$$\mathcal{R}_+ := \{e_k \pm e_l \, (1 \leq k < l \leq n), \, 2e_k \, (1 \leq k \leq n)\} \quad \text{if} \quad m = n. \tag{A.4}$$

The corresponding multiplicities are

$$\nu_{e_k \pm e_l} = 2 \quad (1 \le k < l \le n), \quad \nu_{2e_k} = 1 \quad \text{and} \quad \nu_{e_k} = 2(m-n) \quad (1 \le k \le n). \tag{A.5}$$

Instead of the restricted root vectors E_α^j for which $[q, E_\alpha^j] = \alpha(q)E_\alpha^j$, we directly list their linear combinations (2.6) lying in $\mathcal{G}_\pm$. The two-dimensional subspaces of $\mathcal{M}^\perp \subset \mathcal{G}_+$ associated with $(e_k \pm e_l) \in \mathcal{R}_+$ for any $1 \le k < l \le n$ are spanned by the matrices

$$E_{e_k \pm e_l}^{+,\mathrm{r}} := \frac{1}{2} \begin{pmatrix} E_{kl} - E_{lk} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mp(E_{kl} - E_{lk}) \end{pmatrix}, \tag{A.6}$$

and

$$E_{e_k \pm e_l}^{+,\mathrm{i}} := \frac{\mathrm{i}}{2} \begin{pmatrix} E_{kl} + E_{lk} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mp(E_{kl} + E_{lk}) \end{pmatrix}, \tag{A.7}$$

whose real or imaginary character is indicated by the superscripts r or i, respectively. The generators corresponding to $2e_k \in \mathcal{R}_+$ are the imaginary matrices

$$E_{2e_k}^{+,\mathrm{i}} := \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} E_{kk} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -E_{kk} \end{pmatrix}. \tag{A.8}$$

If $m > n$, then the $2(m-n)$ basis vectors of the subspace of $\mathcal{G}_+$ belonging to $e_k \in \mathcal{R}_+$ are

$$E_{e_k}^{+,\mathrm{r},d} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & E_{kd} & 0 \\ -E_{dk} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_{e_k}^{+,\mathrm{i},d} := \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & E_{kd} & 0 \\ E_{dk} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.9}$$

for $1 \le d \le m-n$. Similarly, the basis of $\mathcal{A}^\perp \subset \mathcal{G}_-$ is given by the matrices

$$E_{e_k \pm e_l}^{-,\mathrm{r}} := \frac{1}{2} \begin{pmatrix} 0 & 0 & E_{lk} \mp E_{kl} \\ 0 & 0 & 0 \\ E_{kl} \mp E_{lk} & 0 & 0 \end{pmatrix} \tag{A.10}$$

and

$$E_{e_k \pm e_l}^{-,\mathrm{i}} := \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & 0 & -(E_{lk} \pm E_{kl}) \\ 0 & 0 & 0 \\ E_{kl} \pm E_{lk} & 0 & 0 \end{pmatrix}, \quad 1 \le k < l \le n, \tag{A.11}$$

together with

$$E_{2e_k}^{-,\mathrm{i}} := \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -E_{kk} \\ 0 & 0 & 0 \\ E_{kk} & 0 & 0 \end{pmatrix}, \quad 1 \le k \le n, \tag{A.12}$$

and

$$E_{e_k}^{-,\mathrm{r},d} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E_{dk} \\ 0 & E_{kd} & 0 \end{pmatrix}, \qquad E_{e_k}^{-,\mathrm{i},d} := \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E_{dk} \\ 0 & E_{kd} & 0 \end{pmatrix} \tag{A.13}$$

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for any $1 \leq k \leq n$, $1 \leq d \leq m - n$. Combined with bases of $\mathcal{A}$ and $\mathcal{M}$, the matrices listed under (A.6)-(A.13) span $su(m, n)$. Their normalization is fixed according (2.4), (2.6) with $\langle X, Y \rangle := \mathrm{tr}\,(XY)$. If desired, the restricted root vectors $E_{\pm\alpha}^{j}$ can be recovered easily since

$$[q, E_\alpha^{\pm,j}] = \alpha(q) E_\alpha^{\mp,j} \quad \forall \alpha \in \mathcal{R}_+, \, j = 1, \ldots, \nu_\alpha. \tag{A.14}$$

In the above formulae of the basis elements the E_{kl} and so on stand for the usual elementary matrices of suitable size given according to (A.1).

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