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Hamiltonian reductions of the free particle on a simple Lie group to (spin) Calogero type integrable models

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Integrable systems of 'Calogero type' are beautiful and are connected with important areas of physics and mathematics.

Landmark contributions: Calogero (71, 75), Sutherland (72), Moser (75), Olshanetsky-Perelomov (76, 78), Kazhdan-Kostant-Sternberg (78), Gibbons-Hermsen (84), Heckman-Opdam (87-88), Oshima-Sekiguchi (95), Etingof-Frenkel-Kirillov (95), Sasaki et al (98-99) . . . Many aspects, surveyed in over 20 reviews ...

In our study we follow the (OP, KKS) '**projection method**' to explore hyperbolic/trigonometric (spin) Calogero models.

Sutherland models associated with root systems [OP, 76]

$\mathcal{R}$: crystallographic root system

$$\mathcal{H}_{\mathcal{R}}(q,p) := \frac{1}{2}\langle p,p\rangle + \sum_{\alpha\in\mathcal{R}_+} \frac{g_\alpha^2}{\sinh^2 \alpha(q)}$$

Coupling constants g_α^2 may arbitrarily **depend on orbits** of the corresponding reflection group. An important case is $\mathcal{R} = BC_n$:

$$\mathcal{H}_{BC_n} = \frac{1}{2}\sum_{k=1}^n p_k^2 + \sum_{1\leq j<k\leq n} \left(\frac{g^2}{\sinh^2(q^j - q^k)} + \frac{g^2}{\sinh^2(q^j + q^k)}\right)$$
$$+ \sum_{k=1}^n \left(\frac{g_1^2}{\sinh^2(q^k)} + \frac{g_2^2}{\sinh^2(2q^k)}\right)$$

[OP, 76]: BC_n model is 'projection' of geodesics on symmetric space $SU(n+1,n)/(S(U(n+1)\times U(n))$ **if** $g_1^2 - 2g^2 + \sqrt{2}gg_2 = 0$. **Why** this symmetric space? Can one get rid of the restriction?

Take a non-compact real simple Lie group G and denote by G_+ its maximal compact subgroup. The Olshanetsy-Perelomov treatment of the BC_n model amounts to (very special) reduction of geodesic motion on coset space G/G_+, based on the G_+ symmetry generated by left-multiplications, for $G = SU(n+1, n)$.

Geodesic system of G/G_+ is reduction of geodesics on G, at zero momentum for the right-multiplications defined by G_+.

We describe the reduction of the geodesics on G at **any** value of the momentum map for the 'left $\times$ right' action of $G_+ \times G_+$.

Not surprisingly, this leads to (hyperbolic) Calogero type models decorated with internal ('spin') degrees of freedom in general.
The detailed description of this class of models is new.
We characterize the cases without spin, and using a 'character trick' obtain the BC_n model with **3 arbitrary** coupling constants.

Some notations and group theoretic facts

$\mathcal{G} \equiv \mathrm{Lie}(G)$, $\mathcal{G}_+ \equiv \mathrm{Lie}(G_+)$, Cartan decomposition: $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$
$\mathcal{G}_\pm$ are eigesubspaces of Cartan involution $\theta \in \mathrm{Aut}(\mathcal{G})$

Choose maximal Abelian subspace $\mathcal{A} \subset \mathcal{G}_-$. Centralizer

$$\mathcal{M} := \{X \in \mathcal{G}_+ \,|\, [X,Y] = 0 \ \ \forall Y \in \mathcal{A}\} = \mathrm{Lie}(M) \quad \text{with}$$

$M := \{m \in G_+ \,|\, mYm^{-1} = Y \quad \forall Y \in \mathcal{A}\}$ using matrix notations

The Killing form $\langle \ , \ \rangle$ of $\mathcal{G}$ induces

$$\mathcal{G}_- = \mathcal{A} + \mathcal{A}^\perp, \quad \mathcal{G}_+ = \mathcal{M} + \mathcal{M}^\perp$$

$X \in \mathcal{A}$ is called **regular** if $\mathrm{Ker}(\mathrm{ad}_X) = \mathcal{A} + \mathcal{M}$ (and not larger).
The regular elements form dense open subset, $\hat{\mathcal{A}} \subset \mathcal{A}$.
Open Weyl chamber, $\check{\mathcal{A}}$, is connected component of $\hat{\mathcal{A}}$.

$\check{G} := \{g = g_+ e^q h_+ \,|\, g_+, h_+ \in G_+,\ q \in \check{\mathcal{A}}\}$ dense, open subset of G
(g_+, h_+, q) unique up to $(g_+, h_+) \to (g_+ m^{-1}, m h_+)$ $\forall m \in M$.

/Can also write $\check{G} = \{g = g_+ e^q h_+ \,|\, g_+, h_+ \in G_+,\ q \in \hat{\mathcal{A}}\}$, with the ambiguity $(g_+, h_+, q) \to (g_+ n^{-1}, n h_+, n q n^{-1})$ $\forall n$ from <u>normalizer</u> $N := \{n \in G_+ \,|\, n Y n^{-1} \in \mathcal{A} \ \ \forall Y \in \mathcal{A}\}$./

ad_Y diagonable $\forall Y \in \mathcal{A}$, $\mathcal{G}_\alpha := \{X \in \mathcal{G} \,|\, [Y, X] = \alpha(Y) X \ \ \forall Y \in \mathcal{A}\}$.
The <u>restricted roots</u>, $\alpha \subset \mathcal{A}^* \setminus \{0\}$ with $\nu_\alpha := \dim(\mathcal{G}_\alpha) \neq 0$, <u>form a crystallographic root system</u>, denoted by $\mathcal{R}$.

<u>Weyl group:</u> $W = N/M$ is reflection group generated by $\mathcal{R}$.
If $\{E_\alpha^a\}_{a=1}^{\nu_\alpha}$ basis of $\mathcal{G}_\alpha$ for $\alpha \in \mathcal{R}_+$, then $\mathcal{M}^\perp$ and $\mathcal{A}^\perp$ spanned by

$$E_\alpha^{+,a} = \frac{1}{\sqrt{2}}(E_\alpha^a + \theta(E_\alpha^a)) \in \mathcal{M}^\perp, \qquad E_\alpha^{-,a} = \frac{1}{\sqrt{2}}(E_\alpha^a - \theta(E_\alpha^a)) \in \mathcal{A}^\perp$$

Some details on $G = SU(m,n)$, $m \geq n$

$$SU(m,n) = \{g \in SL(m+n,\mathbb{C}) \,|\, g^\dagger I_{m,n} g = I_{m,n}\}$$
$$su(m,n) = \{X \in \ sl(m+n,\mathbb{C}) \,|\, X^\dagger I_{m,n} + I_{m,n} X = 0\}$$

where $I_{m,n} := \mathrm{diag}(\mathbf{1}_m, -\mathbf{1}_n)$. Any $X \in \mathcal{G} = su(m,n)$ has the form

$$X = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}$$

with $B \in \mathbb{C}^{m\times n}$, $A \in u(m)$, $D \in u(n)$ and $\mathrm{tr}\, A + \mathrm{tr}\, D = 0$. With Cartan involution $\Theta : g \mapsto (g^\dagger)^{-1}$, $\theta : X \mapsto -X^\dagger$, one obtains $G_+ = S(U(m) \times U(n))$ and $\mathcal{G}_+ = su(m) \oplus su(n) \oplus \mathbb{R}C_{m,n}$. Then $\mathcal{G}_-$ consists of block off-diagonal, hermitian matrices. Next we fix maximal Abelian subspace $\mathcal{A} \subset \mathcal{G}_-$ and describe its centralizer.

$$\mathcal{A} := \left\{ q := \begin{pmatrix} \mathbf{0}_n & 0 & Q \\ 0 & \mathbf{0}_{m-n} & 0 \\ Q & 0 & \mathbf{0}_n \end{pmatrix} \,\Big|\, Q = \mathrm{diag}(q^1, \ldots, q^n),\ q^j \in \mathbb{R} \right\}$$

Using $\chi := \mathrm{diag}(\chi_1, \ldots, \chi_n)\ \forall \chi_j \in \mathbb{R}$, centralizer of $\mathcal{A}$ reads

$$\mathcal{M} = \{\mathrm{diag}(\mathrm{i}\chi, \gamma, \mathrm{i}\chi) \,|\, \gamma \in u(m-n),\ \mathrm{tr}\,\gamma + 2\mathrm{i}\mathrm{tr}\,\chi = 0\} \subset \mathcal{G}_+$$

$$M = \{\mathrm{diag}(e^{\mathrm{i}\chi}, \Gamma, e^{\mathrm{i}\chi}) \,|\, \Gamma \in U(m-n),\ (\det \Gamma)(\det e^{\mathrm{i}2\chi}) = 1\} \subset G_+.$$

Define $e_k \in \mathcal{A}^*$ $(k = 1, \ldots, n)$ by $e_k(q) := q^k$. **Restricted roots:**

$\underline{BC_n:}$ $\quad \mathcal{R}_+ = \{e_j \pm e_k (1 \le j < k \le n),\ 2e_k, e_k (1 \le k \le n)\}$ $\underline{\text{if } m > n}$

$\underline{C_n:}$ $\quad \mathcal{R}_+ = \{e_j \pm e_k\, (1 \le j < k \le n),\ 2e_k\, (1 \le k \le n)\}$ $\underline{\text{if } m = n}$

<u>multiplicities:</u> $\quad \nu_{e_j \pm e_k} = 2, \quad \nu_{2e_k} = 1, \quad \nu_{e_k} = 2(m-n)$

Basis vectors of $\mathcal{M}^\perp$ for $su(m,n)$

$E^{+,\mathrm{r}}_{e_j \pm e_k}$, $\quad E^{+,\mathrm{i}}_{e_j \pm e_k}$, $\quad E^{+,\mathrm{i}}_{2e_k}$, $\quad E^{+,\mathrm{r},d}_{e_k}$, $\quad E^{+,\mathrm{i},d}_{e_k}$ for $1 \le d \le (m-n)$

The superscripts r or i refer to purely real or imaginary matrices.
Block notation associated to partition $(m+n) = n + (m-n) + n$:

$$X = \begin{pmatrix} a & v & b \\ -v^\dagger & e & w \\ b^\dagger & w^\dagger & d \end{pmatrix}, \qquad \mathrm{tr}\, a + \mathrm{tr}\, e + \mathrm{tr}\, d = 0,$$

$a, d \in u(n)$, $e \in u(m-n)$, $v \in \mathbb{C}^{n\times(m-n)}$; $b \in \mathbb{C}^{n\times n}$, $w \in \mathbb{C}^{(m-n)\times n}$.
With elementary matrices E_{kl} of suitable size, we define

$$E^{+,\mathrm{r}}_{e_k \pm e_l} := \frac{1}{2} \begin{pmatrix} E_{kl} - E_{lk} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mp(E_{kl} - E_{lk}) \end{pmatrix}$$

$$E^{+,\mathrm{i}}_{e_k \pm e_l} := \frac{\mathrm{i}}{2} \begin{pmatrix} E_{kl} + E_{lk} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mp(E_{kl} + E_{lk}) \end{pmatrix}$$

$$E^{+,\mathrm{i}}_{2e_k} := \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} E_{kk} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -E_{kk} \end{pmatrix}$$

If $m > n$, then for $1 \leq d \leq m - n$ we also have

$$E^{+,\mathrm{r},d}_{e_k} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & E_{kd} & 0 \\ -E_{dk} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E^{+,\mathrm{i},d}_{e_k} := \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & E_{kd} & 0 \\ E_{dk} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis of $\mathcal{A}^{\perp} \subset \mathcal{G}_{-}$ can be given similarly. Basis is 'orthonormal' with respect to $\langle X, Y \rangle := \mathrm{tr}\,(XY)$. Useful relation, $\forall q \in \mathcal{A}$,

$$[q, E^{\pm,j}_{\alpha}] = \alpha(q) E^{\mp,j}_{\alpha} \quad \forall \alpha \in \mathcal{R}_{+} \quad \text{and multiplicity index} \quad j.$$

Hamiltonian system to be reduced: $(P, \Omega, \mathcal{H})$

$$P := T^*\check{G} \simeq \check{G} \times \mathcal{G} = \{(g, J^l) \,|\, g \in \check{G},\ \ J^l \in \mathcal{G}\,\} \quad (\mathcal{G} \simeq \mathcal{G}^*)$$

$$\Omega = d\langle J^l, (dg)g^{-1}\rangle, \qquad \mathcal{H}(g, J^l) = \frac{1}{2}\langle J^l, J^l\rangle$$

Reduce by $G_+ \times G_+ \subset G \times G$ symmetry.

To use 'shifting trick' consider extended system $(P^{\text{ext}}, \Omega^{\text{ext}}, \mathcal{H}^{\text{ext}})$:

$$P^{\text{ext}} := P \times \mathcal{O}^l \times \mathcal{O}^r = \{(g, J^l, \xi^l, \xi^r) \,|\, g \in \check{G},\ J^l \in \mathcal{G},\ \xi^l \in \mathcal{O}^l,\ \xi^r \in \mathcal{O}^r\,\}$$

$$\Omega^{\text{ext}} := \Omega + \omega^l + \omega^r, \quad \mathcal{H}^{\text{ext}}(g, J^l, \xi^l, \xi^r) := \mathcal{H}(g, J^l),$$

with G_+ coadjoint orbits $(\mathcal{O}^\lambda, \omega^\lambda)$ $(\lambda = l, r)$. Extended dynamics

$$\dot{g} = \{g, \mathcal{H}^{\text{ext}}\} = J^l g, \quad \dot{J}^l = \{J^l, \mathcal{H}^{\text{ext}}\} = 0, \quad \dot{\xi}^\lambda = \{\xi^\lambda, \mathcal{H}^{\text{ext}}\} = 0$$

is integrated obviously by the geodesics $g(t) = e^{tJ^l} g(0)$.

Definition of reduced system

Symmetry transformations: $\forall (g_+^l, g_+^r) \in G_+ \times G_+$ acts on P^{ext}

$$(g, J^l, \xi^l, \xi^r) \mapsto (g_+^l g (g_+^r)^{-1}, g_+^l J^l (g_+^l)^{-1}, g_+^l \xi^l (g_+^l)^{-1}, g_+^r \xi^r (g_+^r)^{-1})$$

Equivariant momentum map: $\Psi = (\Psi^l, \Psi^r) : P^{\text{ext}} \to \mathcal{G}_+^* \oplus \mathcal{G}_+^*$

$$\Psi(g, J^l, \xi^l, \xi^r) = (J_+^l + \xi^l, -(g^{-1} J^l g)_+ + \xi^r), \quad (\mathcal{G}_+^* \simeq \mathcal{G}_+).$$

Reduced geodesic system $(P_{\text{red}}, \Omega_{\text{red}}, \mathcal{H}_{\text{red}})$ carried by

$$P_{\text{red}} := P^{\text{ext}}_{\Psi=0} / (G_+ \times G_+).$$

This is equivalent to (singular) Marsden-Weinstein reduction of original system $(P, \Omega, \mathcal{H})$ at any $(-\mu^l, -\mu^r) \in \mathcal{O}^l \oplus \mathcal{O}^r$.

The reduced system: $(P_{\rm red}, \Omega_{\rm red}, \mathcal{H}_{\rm red})$

$$P_{\rm red} = T^*\check{\mathcal{A}} \times \mathcal{O}_{\rm red}, \quad \Omega_{\rm red} = d\langle p, dq\rangle + \omega_{\rm red}$$

Here q, p natural variables on $T^*\check{\mathcal{A}} \simeq \check{\mathcal{A}} \times \mathcal{A}$ and

$$\mathcal{O}_{\rm red} := (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\rm diag}/M_{\rm diag}, \quad (\mathcal{M}_{\rm diag} = \mathrm{Lie}(M_{\rm diag}) \simeq \mathcal{M})$$

reduction of orbit $\mathcal{O}^l \oplus \mathcal{O}^r$ at zero momentum of $M_{\rm diag} \subset G_+ \times G_+$.

As $M_{\rm diag}$-invariant function on $(\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\rm diag}$

$$2\,\mathcal{H}_{\rm red}(q,p,\xi^l,\xi^r) = \langle p,p\rangle + \langle \xi^l_{\mathcal{M}}, \xi^l_{\mathcal{M}}\rangle - \langle \xi^l_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle$$
$$-\langle \xi^r_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp}\rangle + \langle \xi^r_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle - \langle \xi^r_{\mathcal{M}^\perp}, w^2(\frac{1}{2}\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle$$

where $w(z) = \frac{1}{\sinh z}$ and $\xi^l_{\mathcal{M}} + \xi^r_{\mathcal{M}} = 0$.
Spin Calogero model in general. *When is $\mathcal{O}_{\rm red}$ 1-point space?*

To characterize the reduced system, consider the gauge slice

$$S := \{(e^q, J^l, \xi^l, \xi^r) \in P^{\mathrm{ext}}_{\Psi=0} \,|\, q \in \check{\mathcal{A}}\}.$$

The 'residual gauge transformations' are generated by subgroup $M_{\mathrm{diag}} := \{(m,m) \in G_+ \times G_+ \,|\, m \in M\}$, and thus

$$P_{\mathrm{red}} = P^{\mathrm{ext}}_{\Psi=0}/(G_+ \times G_+) = S/M_{\mathrm{diag}}.$$

Using $J^l = J^l_{\mathcal{A}} + J^l_{\mathcal{A}^\perp} + J^l_{\mathcal{M}} + J^l_{\mathcal{M}^\perp}$, $\xi^\lambda = \xi^\lambda_{\mathcal{M}} + \xi^\lambda_{\mathcal{M}^\perp}$, $\Psi = 0$ requires

$$J^l = J^l_{\mathcal{A}} - F(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp} - w(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp} - \xi^l \quad \text{and} \qquad \xi^l_{\mathcal{M}} + \xi^r_{\mathcal{M}} = 0$$

with $F(z) = \coth z$, $w(z) = \frac{1}{\sinh z}$. In terms of $(q, p := J^l_{\mathcal{A}}) \in \check{\mathcal{A}} \times \mathcal{A}$ Ω^{ext} yields

$$\Omega^{\mathrm{ext}}|_S = d\langle p, dq\rangle + \left(\omega^l + \omega^r\right)|_{(\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\mathrm{diag}}}$$

and $\frac{1}{2}\langle J^l, J^l\rangle$ becomes the reduced Hamiltonian.

To understand **'hidden Weyl group symmetry'** of the reduced system, consider slightly 'thicker' gauge slice

$$\widehat{S} := \{(e^q, J^l, \xi^l, \xi^r) \in P^{\text{ext}}_{\Psi=0} \,|\, q \in \hat{\mathcal{A}}\}.$$

The corresponding residual gauge transformations belong to the **normalizer** $N_{\text{diag}} := \{(n,n) \in G_+ \times G_+ \,|\, n \in N\}$. It follows that

$$P_{\text{red}} = S/M_{\text{diag}} = \widehat{S}/N_{\text{diag}} = (\widehat{S}/M_{\text{diag}})/(N_{\text{diag}}/M_{\text{diag}}) = \widehat{P}_{\text{red}}/W$$

with

$$\widehat{P}_{\text{red}} = \widehat{S}/M_{\text{diag}}, \qquad W = N/M \simeq N_{\text{diag}}/M_{\text{diag}}.$$

The (spin) Calogero type model defined on

$$\widehat{P}_{\text{red}} := \widehat{S}/M_{\text{diag}} = T^*\hat{\mathcal{A}} \times \mathcal{O}_{\text{red}}$$

enjoys W-symmetry. /In the 'spinless cases' $\mathcal{O}_{\text{red}}$ is trivial, and W acts in effect only on $T^*\hat{\mathcal{A}}$. Thus one recovers the usual Weyl group symmetry of the scalar Sutherland models that arise./

Constants of motion and integrability

Any $G_+ \times G_+$ invariant function depending on J^λ and ξ^λ ($\lambda = l, r$) induces a constant of motion for the reduced dynamics.
For example, if $h \in C^\infty(\mathcal{G})^G$ and $v \in \mathbb{R}$, then the function on P^{ext}

$$h(K^\lambda(v)) \quad \text{with} \quad K^\lambda(v) := J^\lambda_- - v\xi^\lambda$$

yields conserved quantities, which are in involution $\forall h$, v, λ.
/Correspond to 'Casimirs' if $v = 1$, since $K^\lambda(1) = J^\lambda$ on $P^{\text{ext}}_{\Psi=0}$./

The Liouville integrability of the reduced system probably follows (in all cases on all symplectic leaves).

Reduced dynamics can be integrated **algebraically** by *projection method*. It may be interesting to present the solutions explicitly.

Lax pairs from Hamiltonian reduction

Consider a gauge slice $\sigma \subseteq S \subset P^{\text{ext}}_{\Psi=0}$, where $g = e^q$ with $q \in \check{\mathcal{A}}$.
Before reduction, $K^\lambda(v) = J^\lambda_- - v\xi^\lambda$ satisfies $\dot{K}^\lambda(v) = 0$.
By projecting Hamiltonian vector field to σ, we get Lax equation

$$\dot{L}^\lambda(v) = [\mathcal{Y}^\lambda, L^\lambda(v)] \quad \text{for} \quad L^\lambda(v) \equiv K^\lambda(v)|_\sigma \quad (\lambda = l, r)$$

where the projection is implemented by the 'compensating gauge transformation' belonging to $\mathcal{Y} : \sigma \to \mathcal{G}_+$. Explicitly:

$$\mathcal{Y}^l = \mathcal{Y}_{\mathcal{M}} + \frac{1}{2}\xi^l_{\mathcal{M}} - w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp} - (wF)(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp}$$
$$\mathcal{Y}^r = \mathcal{Y}_{\mathcal{M}} + \frac{1}{2}\xi^r_{\mathcal{M}} - w^2(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp} - (wF)(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}$$

with $w(z) = 1/(\sinh z)$, $F(z) = \coth z$ and some $\mathcal{Y}_{\mathcal{M}} : \sigma \to \mathcal{M}$.
/`Olshanetsky-Perelomov (1976)` BC_n Lax pairs can be recovered./

The basic example and the 'KKS mechanism'

Consider $G := SL(n,\mathbb{C})$ with Cartan involution $\Theta : g \mapsto (g^\dagger)^{-1}$.
$\mathcal{T}_{n-1}$: Lie algebra of maximal torus $\mathbf{T}_{n-1} \subset SU(n) = G_+$.
Now $sl(n,\mathbb{C}) = su(n) + \mathrm{i}\, su(n)$ and $\mathcal{A} = \mathrm{i}\mathcal{T}_{n-1}$, $M = \mathbf{T}_{n-1}$.

$$\text{If } \mathcal{O}^r = \{0\}\text{, then } \mathcal{O}_{\text{red}} \simeq (\mathcal{O}^l \cap \mathcal{T}_{n-1}^\perp)/\mathbf{T}_{n-1}.$$

This is 1-point space iff $\mathcal{O}^l$ is minimal orbit of $SU(n)$.

The minimal orbits of $SU(n)$ are $\mathcal{O}_{n,\kappa,\pm}$ for $\kappa > 0$, consisting of the elements $\xi = \pm\mathrm{i}(uu^\dagger - \frac{u^\dagger u}{n}\mathbf{1}_n)$ for some $u \in \mathbb{C}^n$, $u^\dagger u = n\kappa$.
Imposing $\xi_{a,a} = 0$ requires $u_a = \sqrt{\kappa}e^{\mathrm{i}\beta_a}$, leading to representative with $\xi_{a,b} = \pm\mathrm{i}\kappa(1 - \delta_{a,b})$. Reproduces original Sutherland model.

One gets 1-point space if G_+ has an $SU(k)$ factor and above arguments are applicable to $\mathcal{O}_{\text{red}} = (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}_{\text{diag}}^\perp / M_{\text{diag}}$.

Deformation of (spin) Sutherland models using a character

Suppose $C \in \mathcal{G}_+ \simeq \mathcal{G}_+^*$ forms a 1-point coadjoint orbit of G_+.
/Such characters exist iff G/G_+ is Hermitian symmetric space./
Then $(\mathcal{O}^r + yC)$ and $(\mathcal{O}^l - yC)$ 1-parameter families of G_+ orbits,

$$\mathcal{O}^y_{\text{red}} := \Big(\big(\mathcal{O}^l - yC\big) \oplus (\mathcal{O}^r + yC)\Big) \cap \mathcal{M}^\perp_{\text{diag}}/M_{\text{diag}}, \quad \forall y \in \mathbb{R}$$

yields deformation of system associated with $y = 0$.

If $\mathcal{O}^{y=0}_{\text{red}}$ is a 1-point space, then this holds $\forall y \in \mathbb{R}$.

Besides $G = SL(n, \mathbb{C})$, the KKS mechanism works **iff** $G = SU(m, n)$.
In this case $G_+ = S(U(m) \times U(n)) = SU(m) \times SU(n) \times U(1)$ and a 1-parameter family of characters exists.

For $G = SU(m,n)$, the system of restricted roots is of BC_n type if $m > n$ and of C_n type if $m = n$. The 1-parameter family of characters is spanned by $C_{m,n} := \mathrm{diag}(\mathrm{i}n\mathbf{1}_m, -\mathrm{i}m\mathbf{1}_n)$.

Spinless BC_n Sutherland models result in the following cases.

<u>If $m = n$:</u> $\quad \mathcal{O}^l := \mathcal{O}_{n,\kappa,+} + \{xC_{n,n}\}, \quad \mathcal{O}^r := \{yC_{n,n}\}, \quad \forall x, y, \kappa.$
One gets 3 couplings $g^2 = \kappa^2/4$, $g_1^2 = xyn^2/2$, $g_2^2 = (x-y)^2 n^2/2$.

<u>If $m = n+1$:</u> one obtains the BC_n model by taking
$\mathcal{O}^l := \mathcal{O}_{n+1,\kappa,+} + \{xC_{n+1,n}\}, \quad \mathcal{O}^r := \{yC_{n+1,n}\}$ with
3 parameters subject to $\kappa + x + y \geq 0$ and $\kappa - n(x+y) \geq 0$.

<u>If $m \geq n+1$:</u> model with 2 independent couplings comes from
$\mathcal{O}^l = \mathcal{O}_{n,\kappa,+} + \{xC_{m,n}\}$ and $\mathcal{O}^r = \{yC_{m,n}\}$ with $x = -y$.

`A. Oblomkov (math.RT/0202076)` considered **quantum** Hamiltonian reduction for holomorphic analogue of the above $SU(n,n)$ case.

Some remarks on the derivation of spinless BC_n models

For $SU(n,n)$, $\mathcal{O}_{\mathrm{red}}$ can be represented by the point (ξ^l, ξ^r) with

$$\xi^l := \kappa \sum_{1\leq j<k\leq n} \left(E^{+,\mathrm{i}}_{e_j+e_k} + E^{+,\mathrm{i}}_{e_j-e_k} \right) + \sqrt{2}xn \sum_{k=1}^{n} E^{+,\mathrm{i}}_{2e_k}$$

$$\xi^r = yC_{n,n} = \sqrt{2}yn \sum_{k=1}^{n} E^{+,\mathrm{i}}_{2e_k}.$$

The Lax pair that results in this case is apparently new. It could be interesting to study associated classical dynamical r-matrices.

Using $SU(n+1,1)$, one recovers the `Olshanetsky-Perelomov (1976)` and probably also the `Inozemtsev-Meshcheryakov (1985)` Lax pairs of the spinless BC_n Sutherland model.

Concluding remarks on further developments

Construction can be applied also to compact simple Lie groups. This amounts to replacing $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ by $\mathcal{G}_{\text{compact}} = \mathcal{G}_+ + \mathrm{i}\mathcal{G}_-$, and leads to trigonometric version of (spin) Sutherland models.

May replace symmetry group $G_+ \times G_+$ by other groups $G'_+ \times G''_+$. Results survive if generalized KAK decomposition of G exists, i.e., if a dense subset of G admits parametrization as $g = g'_+ e^q g''_+$.

Quantized analogue of $P^{\text{ext}} = T^*\check{G} \times \mathcal{O}^l \times \mathcal{O}^r$ is $L^2(G, V_l \otimes V_r)$, where $V_l \otimes V_r$ is irrep. of $G_+ \times G_+$. **Quantum Hamiltonian reduction** $\Leftrightarrow$ restrict to $G_+ \times G_+$ equivariant wave functions. *Laplacian on 'generalized spherical functions' yields Hamiltonian of quantum mechanical (spin) Sutherland model.* Originally, `Olshanetsky-Perelomov (1978)` studied the $\dim(V_l \otimes V_r) = 1$ case.