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Multidimensional Integrable Generalizations of the nonholonomic Chaplygin sphere problem

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- Preservation of Invariant Measure

$$\dot{x} = v(x), \qquad x = (x_1, \dots, x_n) \qquad\qquad (*)$$

The volume form $\mathcal{I} = \mu(x)\, dx_1 \wedge \dots \wedge dx_n$ is said to be an invariant measure of (*) if

$$\mathcal{L}_v \mathcal{I} = 0 \quad \Longleftrightarrow \quad \dot{\mu}(x) + \mu \operatorname{div}(v) = 0.$$

- Some Integrability Theorems:

Theorem 1. (Euler–Jacobi). *If the system (*) possesses an an invariant measure $\mathcal{I}$ and $n-2$ independent first integrals, then it is integrable by quadratures.*

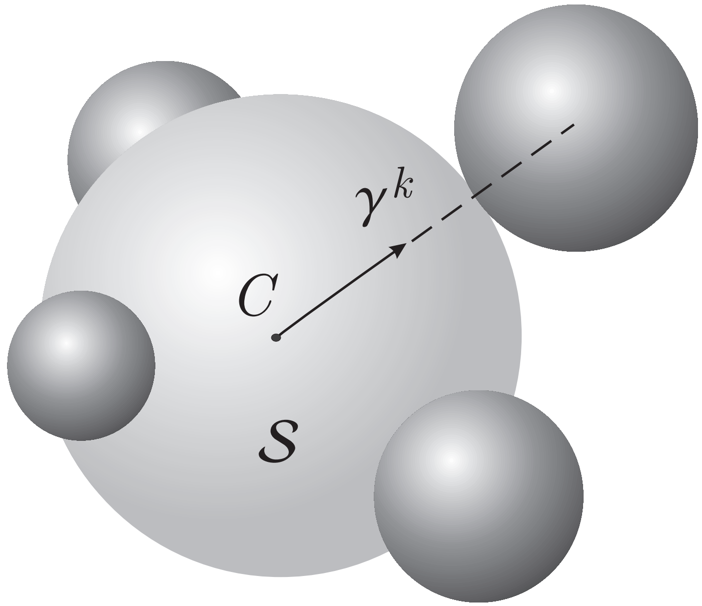
Theorem 2. (Kozlov). *Suppose that the system (*) has an invariant measure $\mathcal{I}(x)$, $k-1$ independent integrals $f_1(x), \dots, f_{k-1}(x)$, and there are vector fields $u_{k+1} = v$, $u_{k+2}, \dots, u_n$, independent at each point, commuting with each other, such that*

$$\mathcal{L}_{u_j} \mathcal{I} = 0, \quad \mathcal{L}_{u_j} f_i = 0, \quad i = 1, \dots, k, \quad j = k+1, \dots, n.$$

Then this system is integrable by quadratures.

- Example: *Spherical Supporter* (Yu. F., 1988)

A dynamically non-symmetric sphere surrounded by N symmetric spheres



$$J\dot{\omega} + \omega \times J\omega = \sum_{k=1}^{N} \gamma^k \times R^k, \quad D_k \dot{\omega}^k = \lambda_k \gamma^k \times R^k, \qquad k = 1, \ldots, N.$$

N nonholonomic constraints expressing absence of slipping at the contact points:

$$-\lambda_k \omega^k = \omega - (\omega, \gamma^k)\gamma^k - \lambda_k(\omega^k, \gamma^k)\gamma^k, \quad \lambda_k(\omega^k, \gamma^k) = c_k = \text{const}.$$

Taking into account the constraints, the first equation gives rise to

$$\Lambda\dot\omega + \omega\times\Lambda\omega = -\Gamma\dot\omega, \quad \Lambda = J + \sum_{k=1}^{N}(D_k/\lambda_k^2)\mathbf{I}, \quad \Gamma = \sum_{k=1}^{N} D_k\gamma^k\otimes\gamma^k/\lambda_k^2,$$

Setting $\Gamma = a\alpha\otimes\alpha + b\beta\otimes\beta + c\gamma\otimes\gamma$,
one obtains the *generalized Euler top* on $T\,SO(3)$:

$$\begin{gathered}\dot K = K\times\omega, \quad \dot\alpha = \alpha\times\omega, \quad \dot\beta = \beta\times\omega, \quad \dot\gamma = \gamma\times\omega,\\ K = (\Lambda+\Gamma)\omega = \Lambda\omega + a(\omega,\alpha)\alpha + b(\omega,\beta)\beta + c(\omega,\gamma)\gamma,\end{gathered}$$

• First integrals: (K,K),

$$\begin{gathered}(K,\alpha) = (\Lambda_a\omega,\alpha), \quad (K,\beta) = (\Lambda_b\omega,\beta), \quad (K,\gamma) = (\Lambda_c\omega,\gamma),\\ \Lambda_a = \Lambda + a\mathbf{I}, \quad \Lambda_b = \Lambda + b\mathbf{I}, \quad \Lambda_c = \Lambda + c\mathbf{I},\end{gathered}$$

of which any three integrals are independent. Besides, the system has the kinetic energy integral

$$\frac{1}{2}(\omega,(\Lambda+\Gamma)\omega) = \frac{1}{2}(\omega,\Lambda\omega) + \frac{a}{2}(\omega,\alpha)^2 + \frac{b}{2}(\omega,\beta)^2 + \frac{c}{2}(\omega,\gamma)^2.$$

It also possesses an invariant measure with density $\mu = \sqrt{\det(\Lambda+\Gamma)}$

• By the Euler–Jacobi theorem, the system is integrable by quadratures, and its generic invariant manifolds are two-dimensional tori.

• For $N=1$ we have $\Gamma = -D_1\gamma\otimes\gamma/\lambda_1^2, \quad \gamma = \gamma^1$,

$$\dot K = K\times\omega, \quad \dot\gamma = \gamma\times\omega, \quad K = \Lambda\omega - D(\omega,\gamma)\gamma,$$

and the spherical support becomes equivalent to the celebrated *Chaplygin sphere problem* (Chaplygin, 1903).

• **The Suslov problem** (A. Suslov, 1903)

Nonholonomic constraint: the projection of the angular velocity vector $\overrightarrow{\omega} \in \mathbb{R}^3$ to a certain *fixed in the body* unit vector $\overrightarrow{\gamma}$ equals zero:

$$(\overrightarrow{\omega}, \overrightarrow{\gamma}) = 0.$$

The Lagrangian $L = \frac{1}{2}(\overrightarrow{\omega}, \mathbb{I}\overrightarrow{\omega})$, the momentum $\overrightarrow{M} = (M_1, M_2, M_3)^T = \mathbb{I}\overrightarrow{\omega}$.

$$\frac{d}{dt}(\mathbb{I}\overrightarrow{\omega}) = \mathbb{I}\overrightarrow{\omega} \times \overrightarrow{\omega} + \lambda \overrightarrow{\gamma},$$

which is equivalent to

$$\frac{d}{dt}(\mathbb{I}\overrightarrow{\omega}) = (\mathbb{I}\overrightarrow{\omega}, \overrightarrow{\gamma})\, \overrightarrow{\omega} \times \mathbb{I}^{-1}\overrightarrow{\gamma}.$$

The Suslov system possesses the energy integral

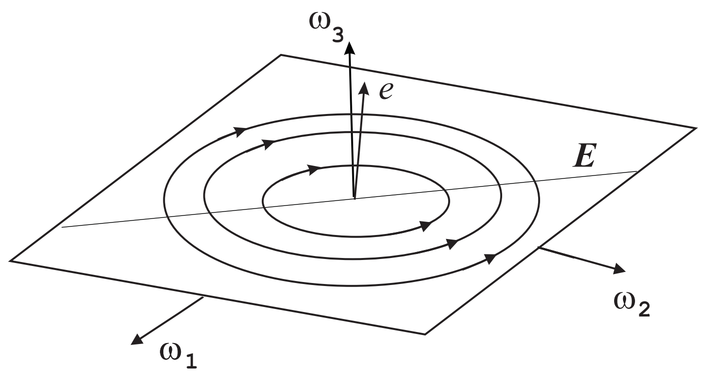
$$(\overrightarrow{\omega}, \mathbb{I}\overrightarrow{\omega}) \equiv (\overrightarrow{M}, \mathbb{I}^{-1}\overrightarrow{M}) = h, \qquad h = \text{const}$$

and a line of equilibria positions

$$E = \{(\overrightarrow{\omega}, \overrightarrow{\gamma}) = 0\} \cap \{(\mathbb{I}\overrightarrow{\omega}, \overrightarrow{\gamma}) = 0\}.$$

In the basis where $\overrightarrow{\gamma} = (0, 0, 1)^T$ the energy integral can be replaced by

$$\mathbb{I}_{22}M_1^2 - 2\mathbb{I}_{12}M_1M_2 + \mathbb{I}_{11}M_2^2.$$



ω3
e
E
ω2
ω1

• **Discrete nonholonomic mechanical system on** $Q \times Q$ (J.Cortés, S. Martínez, 1999)

(1) a *discrete Lagrangian* $\mathbb{L} : Q \times Q \to \mathbb{R}$;

(2) distribution $\mathcal{D}$ on TQ ($\mathcal{D} = \{\dot{q} \in TQ \mid \langle A^j(q), \dot{q} \rangle = 0,\ j = 1, \ldots, s\}$);

(3) a discrete constraint manifold $\mathcal{D}_d \subset Q \times Q$ (of the same dimension as $\mathcal{D}$ and $(q, q) \in \mathcal{D}_d$ for all $q \in Q$)
$\mathcal{F}_j(q_k, q_{k+1}) = 0, \qquad j = 1, \ldots, s$

Discrete Lagrange–d'Alembert equations with multipliers

$$\boxed{D_1\mathbb{L}(q_k, q_{k+1}) + D_2\mathbb{L}(q_{k-1}, q_k) = \sum_{j=1}^{s} \lambda_j^k A_j(q_k), \qquad \mathcal{F}_j(q_k, q_{k+1}) = 0}$$

This defines a *multi-valued* map $Q \times Q \mapsto Q \times Q$

• **Discrete Euler–Poincaré–Suslov Equations** (Yu. F., D. Zenkov, 2004)
- Assume $Q = G\{g\}$ and $\mathbb{L}(g\, g_k, g\, g_{k+1}) = \mathbb{L}(g_k, g_{k+1})$.
- Introduce *left displacement* $W_k = g_k^{-1} g_{k+1} \in G$.
There exists *reduced discrete Lagrangian* $l_d : (G \times G)/G \cong G \to \mathbb{R}$ such that $L_d(g_k, g_{k+1}) = l_d(W_k)$.
- The discrete body momentum $p_k : G \times G \mapsto \mathfrak{g}^*$

$$\langle p_k, \xi \rangle = -\frac{d}{ds}\Big|_{s=0} l_d\left(\exp(-s\xi) W_k\right), \quad p_k = R^*_{W_k} l'_d(W_k).$$

- Left-invariant distribution $\mathcal{D} \subset \mathcal{T}\,\mathcal{G}$, $\mathcal{D}_g = T L_g\, \mathfrak{d}$,

$$\mathfrak{d} = \{\xi \in \mathfrak{g} \mid \langle a^j, \xi \rangle = 0,\ j = 1, \ldots, s\}, \qquad a^j = \text{const}.$$

Then $\langle a^j, g^{-1}\dot{g} \rangle = 0, \qquad j = 1, \ldots, s.$

• **Discrete Euler–Poincaré–Suslov Equations** (continuation)
• Discrete left-invariant constraints $\mathcal{F}_j(g\, g_k, g\, g_{k+1}) = \mathcal{F}_j(g_k, g_{k+1})$
there exist functions $f_j : G \to \mathbb{R}$, $j = 1, \dots, s$, such that

$$\mathcal{F}_j(g_k, g_{k+1}) = f_j(W_k).$$

$\mathcal{D}_d \subset G \times G$ is completely defined by *the admissible displacement subvariety*

$$\mathcal{S} = \{f_1(W) = 0, \dots, f_s(W) = 0\} \subset G;$$

This implies that the discrete momentum p_k is restricted to *the admissible momentum subvariety*

$$\mathcal{U} = \{p \in \mathfrak{g}^* \mid p = L_W^* l_d'(W), W \in \mathcal{S}\} \subset g^*.$$

$$\boxed{p_{k+1} - \mathrm{Ad}^*_{W_k} p_k = \sum_{j=1}^{s} \lambda_{k+1}^j a^j, \quad \text{where} \quad W_k, W_{k+1} \in \mathcal{S}, \quad p_k, p_{k+1} \in \mathcal{U} \subset \mathfrak{g}^*}.$$

• Our choice of $\mathcal{S} \subset G$: $\mathcal{S} = \exp \mathfrak{d}$. *Usually* $\exp \mathfrak{d} = G$ *!*
Assume $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{d}$, $\mathfrak{h}$ being a subalgebra, such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{d}, \mathfrak{d}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{d}] \subset \mathfrak{d}.$$

In this case $\mathcal{S} = \exp \mathfrak{d}$ is a smooth submanifold of G homeomorphic to either the symmetric space G/H or to a quotient of G/H by a finite group action.

• **Discrete Suslov system** $G = SO(3)$ ($SO(n)$), $\{R_k\} \subset SO(3)$ ($SO(n)$)

Finite rotations $\Omega_k = R_k^{-1} R_{k+1}$ (discrete analogue of the body angular velocity $\omega = R^{-1}\dot{R}$)
Discrete Lagrangian is the same as for the discrete Euler top (A. Veselov, J.Moser, 1991)

$$\mathbb{L}(R_k, R_{k+1}) = \frac{1}{2}\,\mathrm{Tr}\,(R_k J R_{k+1}^T), \qquad l_d(\Omega_k) = \frac{1}{2}\,\mathrm{Tr}\,(\Omega_k J),$$

The discrete body angular momentum $M_k = \Omega_k J - J\Omega_k^T \in so(3)$

Continuous constraints are defined by the subspace

$$\mathfrak{d} = \begin{pmatrix} 0 & \dots & 0 & \omega_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \omega_{n-1,n} \\ -\omega_{1n} & \dots & -\omega_{n-1,n} & 0 \end{pmatrix} \subset so(n), \quad \mathfrak{d} = \begin{pmatrix} 0 & 0 & \omega_{13} \\ 0 & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \subset so(3)$$

Discrete constraints are defined by the admissible finite rotations

$$\mathcal{S} = \exp \mathfrak{d} = \{\Omega \in SO(n) \mid \Omega_{ij} = \Omega_{ji}, \quad \Omega_{in} = -\Omega_{ni}, \qquad 1 \le i, j \le n-1.\}$$

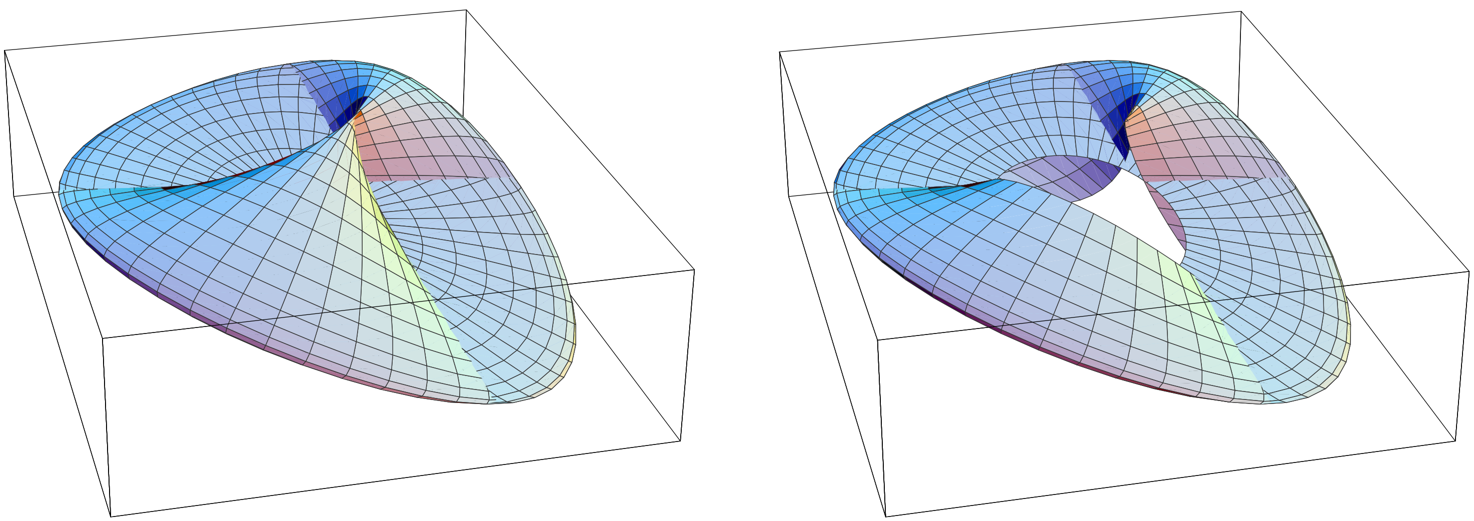
$\mathcal{S}$ is diffeomerphic to $\mathbb{RP}^{n-1} = S^{n-1}/\mathbb{Z}^2$.
For $n = 3$,

$$\Omega = \begin{pmatrix} 2(q_0^2 + q_1^2) - 1 & 2q_1q_2 & 2q_0q_2 \\ 2q_1q_2 & 2(q_0^2 + q_2^2) - 1 & -2q_0q_1 \\ -2q_0q_2 & 2q_0q_1 & 2q_0^2 - 1 \end{pmatrix}, \quad q_0^2 + q_1^2 + q_2^2 = 1.$$

• Discrete Euler–Poincaré–Suslov equations on $so^*(3)$.

$$\boxed{M_{k+1} = \Omega_k^T M_k \Omega_k + \lambda_k \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_k = \Omega_k J - J\Omega_k^T}$$

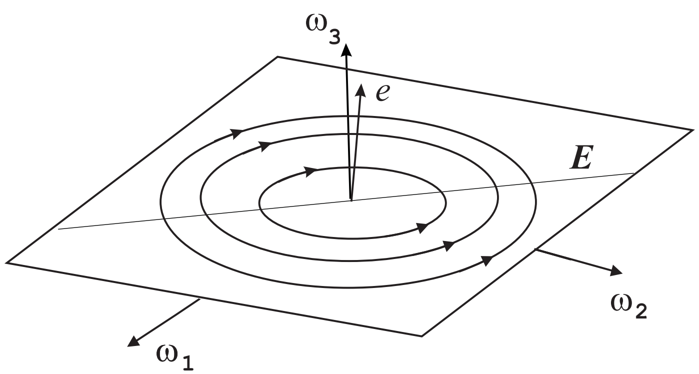
Admissible momentum locus in $so^*(3)$, $\mathcal{U} = \{\Omega_k J - J\Omega_k^T \mid \Omega_k \in \mathcal{S}\}$



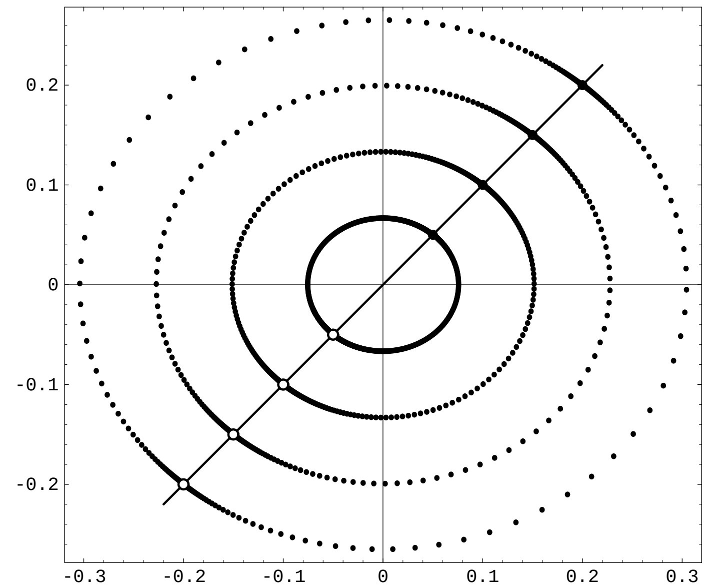
(a) (b)

Theorem 3. *Regardless to the branch of the map $M_k \mapsto M_{m+1}$, the discrete Suslov map preserves the reduced constrained energy*

$$E_c(M_{23}, \mathrm{M}_{13}) = (J_{11} + J_{33})\mathrm{M}_{23}^2 + 2J_{12}\mathrm{M}_{23}\mathrm{M}_{13} + (J_{22} + J_{33})\mathrm{M}_{13}^2.$$



(c)



(d)

• **A discretization of the Chaplygin sphere** $\{R_k, r_k\}$, $R_k \in SO(n), r_k \in \mathbb{R}^n$
Discrete Lagrangian $\mathbb{L} = \frac{1}{2}\mathrm{Tr}(R_k J R_{k+1}^T) + \frac{m}{2}\langle \Delta r_k, \Delta r_k \rangle, \qquad \Delta r_k = r_{k+1} - r_k.$
The discrete momentum of the sphere in the body frame

$$M_k = \Omega_k J - J\Omega_k^T, \qquad \Omega_k = R_k^T R_{k+1} \in SO(n),$$

• Continuous constraints expressing abcence of slipping at the contact point

$$\dot{r} + \rho\,\omega\vec{\gamma} = 0.$$

Discrete Euler–Poincaré equations with multipliers

$$M_k = \Omega_{k-1}^T M_{k-1}\Omega_{k-1} + \rho\,\vec{F}^k \wedge \gamma_k^T, \quad m(\Delta r_k - \Delta r_{k-1}) = \vec{f}^k\,.$$

where $\vec{F}^k = R_k^T \vec{f}^k, \quad \gamma_k = R_k^T \vec{\gamma}.$
• Our choice of discrete constraints that mimic the continuous constraints

$$\Delta r_k + \frac{\rho}{2}(\bar{\Omega}_k - \bar{\Omega}_k^T)\vec{\gamma} = 0, \qquad \bar{\Omega}_k = R_{k+1}^T R_k \in SO(n).$$

Proposition 4. *The map admits the following compact representation*

$$\mathcal{K}_k = \Omega_{k-1}^T \mathcal{K}_{k-1}\Omega_{k-1}, \quad \Gamma_k = \Omega_{k-1}^T \Gamma_{k-1}\Omega_{k-1},$$

where

$$\begin{aligned}
\mathcal{K}_k &= \Omega_k\left(J + \frac{D}{2}\Gamma_k\right) - \left(J + \frac{D}{2}\Gamma_k\right)\Omega_k^T + \frac{D}{2}(\Gamma_k\Omega_k - \Omega_k^T\Gamma_k) \\
&\equiv M_k + \frac{D}{2}(\Omega_k\Gamma_k - \Gamma_k\Omega_k^T) + \frac{D}{2}(\Gamma_k\Omega_k - \Omega_k^T\Gamma_k), \qquad D = m\rho^2, \\
&= \Omega_k\left(J + \frac{D}{2}(\Gamma_{k+1} + \Gamma_k)\right) - \left(J + \frac{D}{2}(\Gamma_{k+1} + \Gamma_k)\right)\Omega_k^T.
\end{aligned}$$

- The classical case $n=3$ Let

$$\vec{M}=(M_1,M_2,M_3)^T\equiv(M_{32},M_{13},M_{21})^T,\quad \vec{\mathcal{K}}=(\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3)^T\equiv(\mathcal{K}_{32},\mathcal{K}_{13},\mathcal{K}_{21})^T.$$

Then the map reads

$$\vec{\mathcal{K}}_k=\Omega_{k-1}^T\vec{\mathcal{K}}_{k-1},\quad \gamma_k=\Omega_{k-1}^T\gamma_{k-1}$$

and preserves 3 independent integrals

$$\langle\gamma,\gamma\rangle=1\quad\langle\mathcal{K},\gamma\rangle=h,\quad\langle\mathcal{K},\mathcal{K}\rangle=n.$$

The special case $\vec{\mathcal{K}}\parallel\gamma$. ($\vec{\mathcal{K}}_k=h\gamma_k$, h =const)

This defines map $\mathcal{G}_h:\ S^2\mapsto S^2$, $\mathcal{G}_h(\gamma_k)=\gamma_{k+1}$

Proposition 5. *Regardless to branch of the map $\mathcal{G}_h$, it has the quadratic integral*

$$\langle\gamma,\Lambda^{-1}\gamma\rangle=l\,.$$

Hence, γ admits the elliptic parameterization, e.g.,

$$\gamma_1=C_1\,\mathrm{cn}(u|k),\quad\gamma_2=C_2\,\mathrm{sn}(u|k),\quad\gamma_3=C_3\,\mathrm{dn}(u|k),$$

Therefore, for a fixed l, the map $\mathcal{G}_h$ is reduced to one-dimensional map

$$u_{k+1}=u_k+\Delta u_k(u_k,l)$$

Δu_k depens non-trivially on u_k !