Real algebraic geometry

March, 10th

Exercice 1. Let R be a real closed field, and $V \subset R^n$ be an irreducible algebraic set. Let $f \in \mathcal{P}(V)$.

(1) Assume f is a sum of squares of rational functions on V. Prove that f is non-negative on $\mathrm{Reg}(V)$.

(2) Assume f is non-negative on $\mathrm{Reg}(V)$.

  (a) Prove there exists $h \in \mathcal{P}(V)$ such that $\mathrm{Sing}(V) = h^{-1}(0)$.

  (b) Prove that f is a sum of squares of rational functions on V.

(3) Let $f \in \mathcal{K}(V)$. Prove that f is a sum of squares of rational functions on V if and only if f is non-negative where f is a well-defined function on $\mathrm{Reg}(V)$.

Exercice 2.
Prove that $\mathbb{P}^1(\mathbb{R})$ is biregularly isomorphic to the circle $\mathcal{S}^1$.

Exercice 3. Consider the plane curve C defined by $x(x^3 - y^5)(x^3 + y^5) = 0$.

(1) What is the singular locus of C ? Draw the allure of C.

(2) Resolve the singularities of C. Make a picture showing the exceptional divisors and the strict transforms.

Exercice 4.

(1) Let $f : \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}^2$ be given by $f([x : y : z]) = (\frac{x^2}{x^2+y^2+z^2}, \frac{y^2}{x^2+y^2+z^2})$.

  (a) Prove that f is a proper regular map. What is the image of f ?

  (b) Compute the pushforward $f_*(\mathbf{1}_{\mathbb{P}^2(\mathbb{R})})$.

  (c) Compute $\Lambda f_*(\mathbf{1}_{\mathbb{P}^2(\mathbb{R})})$.

(2) Consider the semialgebraic set $A \subset \mathbb{R}^2$ given by

$$A = \{(x, y) : x^2 + y^2 = 1\} \cup \{(x, y) : -1 \leq x \leq 1, \quad y = 0\}.$$

Is A semialgebraically homeomorphic to a real algebraic set ?