Motivic invariants of Arc-Symmetric sets and Blow-Nash Equivalence

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Abstract

We define invariants of the blow-Nash equivalence of Nash function germs, in a similar way that the motivic zeta functions of Denef & Loeser. As a key ingredient, we extend the virtual Betti numbers, which were known for real algebraic sets, as a generalized Euler characteristic for projective constructible arc-symmetric sets. Actually we prove more: the virtual Betti numbers are not only algebraic invariants, but also Nash invariants of arc-symmetric sets. Our zeta functions enable to distinguish the blow-Nash equivalence classes of Brieskorn polynomials of two variables. We prove moreover that there is no moduli for blow-Nash equivalence in the case of an algebraic family with isolated singularities.

In the study of real analytic function germs, the choice of a good equivalence relation between germs is a crucial topic. Whereas the topological equivalence is too coarse and the C^1-equivalence too fine, the blow-analytic equivalence, a notion introduced by T.-C. Kuo in 1985 (see [Kuo85], and [FKK98] for a survey) seems to behave better, especially with respect to finiteness properties. In this paper, we will focus on a particular case of blow-analytic equivalence, called blow-Nash equivalence, for which we add algebraicity assumptions.

Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be Nash function germs. Then f and g are said to be blow-Nash equivalent if there exist two algebraic modifications

$$\pi_f : \left(M_f, \pi_f^{-1}(0)\right) \longrightarrow (\mathbb{R}^d, 0) \quad \text{and} \quad \pi_g : \left(M_g, \pi_g^{-1}(0)\right) \longrightarrow (\mathbb{R}^d, 0),$$

and a Nash-isomorphism $\phi : \left(M_f, \pi_f^{-1}(0)\right) \longrightarrow \left(M_g, \pi_g^{-1}(0)\right)$, that is an analytic isomorphism with semi-algebraic graph, which respects the multiplicity of the jacobian determinants of π_f and π_g and which induces a homeomorphism h between neighbourhoods of 0 in $\mathbb{R}^d$ such that $f = g \circ h$. Here, by a modification π of f, we mean a proper birational map which is an isomorphism over the complement of the zero locus of f and such that $f \circ \pi$ is in normal crossing. One can define such a relation on Nash sets, and S. Koike ([Koi97, Koi00]) proved that finiteness properties hold in this case. In the case of germs of functions, the question of moduli is still open in general. However, in part 4, we prove that there is no moduli for a Nash family with isolated singularities under some algebraic assumptions on modifications. In particular, an algebraic family of isolated singularities does not admit moduli for the blow-Nash equivalence.

A common issue for blow-analytic equivalence and blow-Nash equivalence is to prove that, when it is the case, two given germs of real analytic functions are not equivalent. The difficulty rises in the lack of invariants known for these relations. Up to now, just two kind of invariants have been known: the Fukui invariants and the zeta functions of S. Koike and A. Parusiński [KP]. With an analytic function germ f, the Fukui invariants associate the set of possible orders n of series $f \circ \gamma(t) = a_n t^n + \cdots, a_n \neq 0$, for $\gamma : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^d, 0)$ an analytic arc ([Fu97, IKK02]). There exists also a version of the Fukui invariants related to the sign of f. Using motivic integration combined

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with the construction of a computable motivic invariant for arc-symmetrics sets, the virtual Betti numbers, we introduce in this paper zeta functions $Z(T), Z^{\pm}(T)$ of a real analytic function germ that belong to $\mathbb{Z}[u,u^{-1}][[T]]$, and take into account not only the orders of the series $f \circ \gamma(t)$ but also the geometry of the sets $\mathcal{X}_n(f)$ of arcs γ that realize a given order n (for precise definitions, see section 3.1). These zeta functions are similar to the motivic zeta functions of Denef & Loeser [DL01], and also to those of S. Koike and A. Parusiński, defined using the Euler characteristic with compact supports.

We prove that our zeta functions are invariants of the blow-Nash equivalence. The proof is directly inspired by the work of Denef & Loeser via their formulae for the zeta functions in terms of a modification of the zero locus of the given function germ (propositions 3.2, 3.5). It uses the powerfull machinery of motivic integration, a theory introduced by M. Kontsevitch in 1995 [Kont] and developped further by J. Denef and F. Loeser [DL01, DL99, DL02, DL98], in particular the fundamental change of variables formula (proposition 3.10).

In order to dispose of computable invariants, motivic integration requires computable measures, or in other words generalized Euler characteristics. A generalized Euler characteristic is an additive and multiplicative invariant defined on the level of the Grothendieck group of varieties. In our setting of the blow-Nash equivalence, we need invariants of the Zariski constructible sets over real algebraic varieties $\mathcal{X}_n(f)$ (real algebraic variety is the sense of [BCR]), and we ask it to be respected by Nash isomorphisms. It leads naturally to the category of Nash varieties, and more generally of arc-symmetric sets, a category introduced by K. Kurdyka [Kur88].

In part 2, we give conditions, inspired by a result of F. Bittner [Bit], on an invariant defined on connected components of compact nonsingular real algebraic varieties such that it extends to an additive invariant on the constructible category of arc-symmetric sets. Additive means that $\chi(A) = \chi(B) + \chi(A \setminus B)$ for a closed inclusion $B \subset A$ of arc-symmetric sets. As a fundamental example, we prove that the Betti numbers with $\mathbb{Z}_2$-coefficient defined on connected components A of compact nonsingular real algebraic sets by $b_k(A) = \dim H_k(A, \mathbb{Z}_2)$, give such an additive invariant β_k on arc-symmetric sets for each $k \in \mathbb{N}$ (corollary 2.5), called k-Virtual Betti number. We make them multiplivative by putting $\beta(A) = \sum_{k=0}^{\dim A} \beta_k(A) u^k \in \mathbb{Z}[u]$, called virtual Poincaré polynomial of A.

Let us stress the fact that the unique such additive invariant known up to now in the real case is the classical Euler characteristic with compact supports, and as a matter of fact it is the unique generalized Euler characteristic for semi-algebraic sets up to homeomorphism [Qua01].

The virtual Poincaré polynomial is not a topological invariant. It respects the dimension (see remark 2.11) whereas the Euler characteristic with compact supports may identify the dimension. Note that the virtual Betti numbers have been proven recently to be additive invariant of real algebraic varieties by C. McCrory and A. Parusiński in [MCP03] and independently by B. Totaro [Tot02]; in this paper, we extend these numbers to the more general context of arc-symmetric sets, and we prove the invariance not only under algebraic isomorphisms but also under Nash isomorphisms (see 2.15).

1. Arc-symmetric sets

The category of arc-symmetric sets contains the real algebraic varieties and, in some sense, this category has a better behaviour that the one of real algebraic varieties, similar to complex algebraic varieties. As an example, a closed and irreducible arc-symmetric set is connected whereas a closed and irreducible real algebraic variety may have as connected components as one wants!

In this part, we introduce arc-symmetric sets, with a slightly different definition that the one of K. Kurdyka, and we state several basic properties of them in relation with closure, dimension and irreducibility. Then, we study in more details the nonsingular arc-symmetric sets and especially the resolution of singularities for arc-symmetric sets.

1.1 Arc-symmetric sets and closure

We fix a compactification of $\mathbb{R}^n$, for instance $\mathbb{R}^n \subset \mathbb{P}^n$.

Definition 1.1. Let $A \subset \mathbb{P}^n$ be a semi-algebraic set. We say that A is arc-symmetric if, for every real analytic arc $\gamma :]-1,1[\longrightarrow \mathbb{P}^n$ such that $\gamma(]-1,0[) \subset A$, there exists $\epsilon > 0$ such that $\gamma(]0,\epsilon[) \subset A$.

This definition is the one of A. Parusiński [Pa]. Note that a closed arc-symmetric set is necessarily compact. This definition differs from the one of K. Kurdyka [Kur88] who only considers closed arc-symmetric sets in $\mathbb{R}^n$. One can think about our arc-symmetric sets as the projective constructible arc-symmetric sets of K. Kurdyka. Remark moreover that an arc-symmetric set needs not to be an analytic variety (cf. [Kur88], example 1.2).

Example 1.2.

i) A real algebraic variety is an arc-symmetric set. A connected component of a compact real algebraic variety is also an arc-symmetric set.
ii) The 2-dimensional sheet of the Whitney umbrella of equation $zx^2 = y^2$ is arc-symmetric.

Remark that the arc-symmetric sets form a constructible category of semi-algebraic sets in the sense of [Pa], denoted $\mathcal{AS}$, that is:

- $\mathcal{AS}$ contains the algebraic sets,
- $\mathcal{AS}$ is stable under set-theoretic operations $\cup$, $\cap$, $\setminus$,
- $\mathcal{AS}$ is stable by inverse images of $\mathcal{AS}$-map (i.e. whose graph is in $\mathcal{AS}$) and by images of injective $\mathcal{AS}$-map,
- each $A \in \mathcal{AS}$ has a well-defined fundamental class with coefficients in $\mathbb{Z}_2$.

In particular there is a notion of closure in $\mathcal{AS}$ (we refer to [Pa] for a proof).

Proposition 1.3. *Every $A \in \mathcal{AS}$ admits a smallest arc-symmetric set, denoted by $\overline{A}^{\mathcal{AS}}$, containing A and closed in $\mathbb{P}^n$.*

Remark 1.4. Arc-symmetric sets are not stable under the euclidean closure. Consider for example the regular part A of the Whitney umbrella $zx^2 = y^2$. The closure of A in $\mathcal{AS}$ is the entire Whitney umbrella.

We can define irreducible arc-symmetric sets in the usual way: $A \in \mathcal{AS}$ is irreducible if the existence of a decomposition $A = B \cup C$, with B and C closed in A and arc-symmetric, implies that either $B \subset C$ or $C \subset B$. Remark that an irreducible arc-symmetric set is not necessarily connected, with our definition of arc-symmetric sets (as an example consider a hyperbola in the plane). Nevertheless, as proved in [Kur88], an arc-symmetric set A admits a unique decomposition as a finite union of irreducible arc-symmetric sets closed in A.

Proposition 1.5. *If $A \in \mathcal{AS}$ is irreducible, then so is $\overline{A}^{\mathcal{AS}}$.*

Proof. Assume that $\overline{A}^{\mathcal{AS}}$ can be decomposed into $\overline{A}^{\mathcal{AS}} = B \cup C$ with B and C arc-symmetric and closed in $\overline{A}^{\mathcal{AS}}$. Then B and C are closed and A splits in

$$A = A \cap \overline{A}^{\mathcal{AS}} = (A \cap B) \cup (A \cap C),$$

with $A \cap B$ and $A \cap C$ arc-symmetric and closed in A. But A is irreducible so either $A \cap B \subset A \cap C$ or the reverse inclusion holds. By symmetry, one can assume that $A \cap B \subset A \cap C$. Then A equals $A \cap C$, so A is included in C and finally $\overline{A}^{\mathcal{AS}}$ is equal to C because C is arc-symmetric and closed. ☐

Define the dimension of an arc-symmetric set to be its dimension as a semi-algebraic set. Then, it equals to the dimension of its Zariski closure in the projective space [Kur88] (recall that by real algebraic variety, we mean in the sense of [BCR]) and therefore

$$\dim A = \dim \overline{A}^{\mathcal{AS}} = \dim \overline{A}^{Z}$$

for $A \in \mathcal{AS}$. The following result relies the dimension of an arc-symmetric set and the dimension of its arc-symmetric closure. It will be useful in the sequel when dealing with proofs by induction, e.g. for theorem 2.3 and 2.16.

Proposition 1.6. *Let $A \in \mathcal{AS}$. Then $\overline{A}^{\mathcal{AS}} = A \cup \overline{\overline{A} \setminus A}^{\mathcal{AS}}$. In particular $\dim \overline{A}^{\mathcal{AS}} \setminus A < \dim A$.*

Proof. Note that, as a union of arc-symmetric sets, $F = A \cup \overline{\overline{A} \setminus A}^{\mathcal{AS}}$ is arc-symmetric. Moreover F can be decomposed into

$$F = A \cup (\overline{A} \setminus A) \cup (\overline{\overline{A} \setminus A}^{\mathcal{AS}}) = \overline{A} \cup \overline{\overline{A} \setminus A}^{\mathcal{AS}},$$

thus F is closed. So the inclusion $\overline{A}^{\mathcal{AS}} \subset F$ holds.

Moreover $\overline{A}$ is included in $\overline{A}^{\mathcal{AS}}$ because $\overline{A}^{\mathcal{AS}}$ is closed, thus $\overline{\overline{A} \setminus A}^{\mathcal{AS}} \subset \overline{A}^{\mathcal{AS}}$, and so $F \subset \overline{A}^{\mathcal{AS}}$. Consequently $F = \overline{A}^{\mathcal{AS}}$. □

We can adapt proposition 1.5 of [Kur99] to our definition of arc-symmetric sets. It is another example of the good behaviour of irreducible arc-symmetric sets.

Proposition 1.7. *Let $A \in \mathcal{AS}$ be irreducible, and $B \subset A$ be a closed arc-symmetric subset of A of the same dimension. Then $B = A$.*

Proof. A can be decomposed into the union of two arc-symmetric sets closed in A as follows: $A = B \cup (\overline{A \setminus B}^{\mathcal{AS}} \cap A)$. Then, by irreducibility of A, either $B \subset \overline{A \setminus B}^{\mathcal{AS}} \cap A$ or $\overline{A \setminus B}^{\mathcal{AS}} \cap A \subset B$.

In the second case B is equal to A, and in the first one B is included in $(\overline{A \setminus B}^{\mathcal{AS}}) \setminus (A \setminus B)$. But this can not happen for the dimension of this arc-symmetric set is stricly less than $\dim B$ by proposition 1.6. □

1.2 Nonsingular arc-symmetric sets

Let us define a nonsingular arc-symmetric set with relation to its Zariski closure in the projective space.

Definition 1.8. An arc-symmetric set A is nonsingular if $A \cap \mathrm{Sing}(\overline{A}^{Z}) = \emptyset$.

Lemma 1.9. *([Kur99]) A nonsingular and connected arc-symmetric set is irreducible.*

Let us state a definition of an isomorphism between arc-symmetric sets.

Definition 1.10. Let $A, B \in \mathcal{AS}$. Then A is isomorphic to B if and only if there exist Zariski open subsets U and V in $\overline{A}^{\mathcal{Z}}$ and $\overline{B}^{\mathcal{Z}}$ containing A and B respectively, and an algebraic isomorphism $\phi : U \longrightarrow V$ such that $\phi(A) = B$.

Remark 1.11. At this point, we are only interested in the algebraic point of view, because we have in mind to study algebraic singularities of arc-symmetric sets, and to use Hironaka's Desingularisation theorem. In section 2.2 we will give another definition of isomorphism between arc-symmetric sets (see definition 2.15).

Proposition 1.12. *Let $A \in \mathcal{AS}$ be compact and nonsingular. Then A is isomorphic to a union of connected components of some compact nonsingular real algebraic variety.*

Proof. Let $X = \overline{A}^{\mathcal{Z}}$ be the Zariski closure of A in the projective space, and let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of the singularities of X. Remark that the three spaces A, X and $\widetilde{X}$ have the same dimension and that A is isomorphic to the subset $\pi^{-1}(A) = \widetilde{A}$ of $\widetilde{X}$ because $A \subset \operatorname{Reg}(X)$ and $\operatorname{Reg}(X)$ is a Zariski open subset of X isomorphic to $\pi^{-1}\big(\operatorname{Reg}(X)\big) \subset \widetilde{X}$.

Now, denote by $\widetilde{X} = \bigcup_{i\in I} C_i$ the decomposition of $\widetilde{X}$ into connected components. Each $C_i, i \in I$, is a closed and nonsingular arc-symmetric set, hence irreducible by proposition 1.9. Therefore $\widetilde{A}\cap C_i$ is either equal to C_i or empty because of proposition 1.7, and so $\widetilde{A}$ is a union of connected components of $\widetilde{X}$ as claimed. $\square$

1.3 Arc-symmetric sets and resolution of singularities

The following proposition is an adaptation of Theorem 2.6 of [Kur88] to our definition of arc-symmetric sets. It asserts that, up to desingularization, we can think about an irreducible arc-symmetric set as a connected component of a real algebraic variety.

PROPOSITION 1.13. *Let $A \in \mathcal{AS}$ be irreducible. Let X be a compact real algebraic variety containing A with $\dim X = \dim A$, and $\pi : \widetilde{X} \longrightarrow X$ a resolution of singularities for X (cf. [Hir64]). Then, there exists a unique connected component $\widetilde{A}$ of $\widetilde{X}$ such that $\pi(\widetilde{A}) = \overline{\operatorname{Reg}(A)}$.*

Proof. Let $\widetilde{A_0}$ be an irreducible arc-symmetric component of dimension $\dim A$ of $\pi^{-1}(A)$. Such an $\widetilde{A_0}$ exists because the dimension of $\pi^{-1}(A)$ and A coincides. Then $\widetilde{A_0}$ is contained in some connected component $\widetilde{A}$ of $\widetilde{X}$. Actually $\overline{\widetilde{A_0}}^{\mathcal{AS}}$ is irreducible (because so is $\widetilde{A_0}$, cf. proposition 1.5) and closed, therefore is connected by proposition 1.12. Now $\overline{\widetilde{A_0}}^{\mathcal{AS}}$ is included in some connected component of $\widetilde{X}$, and is equal to this component by proposition 1.7. We can put $\widetilde{A} = \overline{\widetilde{A_0}}^{\mathcal{AS}}$.

Let us prove that the equality $\pi(\widetilde{A}) = \overline{\operatorname{Reg}(A)}$ announced holds. In fact, it suffices to prove that

$$\dim \overline{\pi(\widetilde{A})}^{\mathcal{AS}} \setminus \pi(\widetilde{A}) < \dim A,$$

what will be done in the next lemma.

This is sufficient for the following reasons. On one hand $\overline{\pi(\widetilde{A})}^{\mathcal{AS}}$ is equal to $\overline{A}^{\mathcal{AS}}$ by proposition 1.7, so $\dim \overline{A}^{\mathcal{AS}} \setminus \pi(\widetilde{A}) < \dim A$. Now $\operatorname{Reg}(A) \cap \big(\overline{A}^{\mathcal{AS}} \setminus \pi(\widetilde{A})\big)$ is an open subset of $\overline{A}^{\mathcal{AS}}$ of dimension stricly less than $\dim A$, so $\operatorname{Reg}(A) \cap \big(\overline{A}^{\mathcal{AS}} \setminus \pi(\widetilde{A})\big) = \emptyset$. This implies the inclusion $\operatorname{Reg}(A) \subset \pi(\widetilde{A})$.

On the other hand, if E denotes the exceptional divisor of the resolution, then $\pi(\widetilde{A} \setminus E)$ is included in $\operatorname{Reg}(\overline{A}^{\mathcal{AS}})$. However $\operatorname{Reg}(\overline{A}^{\mathcal{AS}})$ is included in $\overline{\operatorname{Reg}(A)}$ because $\dim \overline{A}^{\mathcal{AS}} \setminus A < \dim A$ by proposition 1.6.

Thus the following inclusions $\pi(\widetilde{A} \setminus E) \subset \overline{\operatorname{Reg}(A)} \subset \pi(\widetilde{A})$ holds and gives the conclusion by taking the closure. $\square$

LEMMA 1.14. *Let A and $\widetilde{A}$ be as in the proof of proposition 1.13. Then*

$$\dim \overline{\pi(\widetilde{A})}^{\mathcal{AS}} \setminus \pi(\widetilde{A}) < \dim A.$$

Proof. Let us prove that the inclusion $\overline{\pi(\widetilde{A})}^{\mathcal{AS}} \subset \pi(\widetilde{A}) \cup \overline{\pi(E)}^{\mathcal{AS}}$ holds. Denote by F the set $\pi(\widetilde{A}) \cup \overline{\pi(E)}^{\mathcal{AS}}$. Remark that if F is closed and arc-symmetric, the lemma is proved.

As π is proper, $\pi(\widetilde{A})$ is closed and so is F. Now, let $\gamma :]-\epsilon,\epsilon[\longrightarrow \mathbb{P}^n$ be a real analytic arc such that $\operatorname{int} \gamma^{-1}(F) \neq \emptyset$. Then

- either $\operatorname{int} \gamma^{-1}(\overline{\pi(E)}^{\mathcal{AS}}) \neq \emptyset$ and $\gamma(]-\epsilon,\epsilon[) \subset \overline{\pi(E)}^{\mathcal{AS}}$
- or $\operatorname{int} \gamma^{-1}(\pi(\widetilde{A}) \setminus \overline{\pi(E)}^{\mathcal{AS}}) \neq \emptyset$.

In the latter case, there exists a unique analytic arc $\widetilde{\gamma} :]-\epsilon,\epsilon[\longrightarrow \mathbb{P}^m$ such that $\pi\circ\widetilde{\gamma}=\gamma$. One has $\operatorname{int}\widetilde{\gamma}^{-1}(\widetilde{A})\neq\emptyset$, therefore $\widetilde{\gamma}(]-\epsilon,\epsilon[)\subset\widetilde{A}$ because $\widetilde{A}$ is arc-symmetric. Finally $\gamma(]-\epsilon,\epsilon[)\subset\pi(\widetilde{A})\subset F$, and thus F is arc-symmetric. □

Remark 1.15. Denote by D the singular locus of X and by E the exceptional divisor of the resolution of proposition 1.13. Then $\pi:\widetilde{A}\setminus E\longrightarrow \overline{A}^{\mathcal{AS}}\setminus D$ is an isomorphism of arc-symmetric sets (restriction of an algebraic ismorphism). If we add the assumption that A is nonsingular, then the conclusion of proposition 1.13 becomes simply $\pi(\widetilde{A})=\overline{A}$. Moreover $\pi:\widetilde{A}\setminus\big(\widetilde{A}\setminus\pi^{-1}(A)\big)\longrightarrow A$ is an isomorphism of arc-symmetric sets, and $\widetilde{A}$ is close to A is the sense that $\dim\widetilde{A}\setminus\pi^{-1}(A)<\dim A$.

Actually, it is easy to prove that the symmetric difference of A and $\pi(\widetilde{A})$ consists of a semialgebraic set of dimension stricly less than $\dim A$. More precisely:

Proposition 1.16. *Let A and $\widetilde{A}$ be as in proposition 1.13. Then*

$$A\setminus(\pi(\widetilde{A})\cap A)=\{x\in\operatorname{Sing}(A);\ \dim_x A<\dim A\},$$

$$\pi(\widetilde{A})\setminus\big(A\cap\pi(\widetilde{A})\big)=\{x\in\overline{A}^{\mathcal{AS}}\setminus A;\ \dim_x\overline{A}^{\mathcal{AS}}=\dim A\}.$$

Let us finish this section by stating the particular case of the "blowing-up" of a real algebraic variety along a closed nonsingular arc-symmetric set. In virtue of proposition 1.12, it is just the blowing-up of a nonsingular real algebraic variety along a nonsingular center.

Proposition 1.17. *Let $Y\subset X$ be compact nonsingular algebraic varieties such that $\dim Y<\dim X$, and let $A\subset X$ be a connected component of X. Denote by $\pi:\widetilde{X}\longrightarrow X$ the blowing-up of X along Y. Then π is surjective and $\pi^{-1}(A)$ is a connected component of $\widetilde{X}$.*

2. Virtual Betti numbers

In the theory of motivic integration, generalized Euler characteristics play the role of a measure for certain subsets of the arc space of a variety. In the first section of this part, we give a new example of such a generalized Euler characteristic of arc-symmetric sets, constructed from the Betti numbers of compact nonsingular arc-symmetric sets.

This example, called the virtual Poincaré polynomial, was already known for real algebraic varieties; it has been introduced independently by C. McCrory and A. Parusiński [MCP03] and by B. Totaro [Tot02]. Here we define the virtual Poincaré polynomial for the larger category of arc-symmetric sets. The way to perform this is, following an idea of F. Bittner [Bit], to extend an invariant of the compact nonsingular arc-symmetric sets to the whole category of arc-symmetric sets. The key ingredients are the resolution of singularities, which enables to define the invariant for all arc-symmetric sets, and the weak factorisation theorem which simplifies the proof of the independance from the choices we have to make.

Moreover we prove, in the second section of this part, that the virtual Betti numbers are invariants of the arc-symmetric sets under not only algebraic isomorphisms, but also Nash isomorphisms. This result will be useful when studying the blow-Nash equivalence of germs of real analytic functions in the next part.

2.1 Generalized Euler characteristics of arc-symmetric sets

Definition 2.1. An additive map on $\mathcal{AS}$ with values in an abelian group is a map χ defined on $\mathcal{AS}$ such that

i) for arc-symmetric sets A and B which are isomorphic, $\chi(A)=\chi(B)$,

ii) for a closed arc-symmetric subset B of A, $\chi(A) = \chi(B) + \chi(A \setminus B)$.

If moreover χ takes values in a commutative ring and satisfies $\chi(A \times B) = \chi(A) \cdot \chi(B)$ for arc-symmetric sets A and B, then we say that χ is a generalized Euler characteristic on $\mathcal{AS}$.

Remark 2.2.

i) One can construct a universal generalized Euler characteristic with values in the Grothendieck ring of arc-symmetric sets [DL01]. But this ring is rather complicated, and we are interested in more computable invariants.

ii) The Euler characteristic with compact supports is a generalized Euler characteristic on $\mathcal{AS}$, and maybe the simplest one. Actually, it is unique if we just consider semi-algebraic sets, with isomorphisms replaced by homeomorphisms [Qua01].
However, for complex algebraic varieties, a lot of such generalized Euler characteristics exist, for example deduced from mixed Hodge structures [DL01, Loo00].

The following theorem gives sufficiently good conditions, on an invariant χ over the closed (i.e. compact) and nonsingular arc-symmetric sets, such that χ extends to an additive map on $\mathcal{AS}$. We state the theorem in terms of connected components of real algebraic varieties thanks to proposition 1.12. The method is inspired by the one of F. Bittner [Bit], who proves the result for algebraic varieties over a field of characteristic zero.

Theorem 2.3. *Let χ be a map defined on connected components of compact nonsingular real algebraic varieties with values in an abelian group and such that*

P1 $\chi(\emptyset) = 0$,

P2 *if A and B are connected components of compact nonsingular real algebraic varieties which are isomorphic as arc-symmetric sets, then $\chi(A) = \chi(B)$,*

P3 *with the notations and assumptions of proposition 1.17,*

$$\chi\big(\pi^{-1}(A)\big) - \chi\big(\pi^{-1}(A) \cap \pi^{-1}(A \cap Y)\big) = \chi(A) - \chi(A \cap Y).$$

Then χ extends uniquely to an additive map defined on $\mathcal{AS}$.

Remark 2.4. Property **P3** of theorem 2.3 is a kind of additivity property for nonsingular arcs-symmetric sets.

Before giving the proof of the theorem, let us state some consequences. First, this result enables us to give another example of such an additive map by considering the homology with coefficients in $\mathbb{Z}_2$. For $i \in \mathbb{N}$, denote by b_i the i-th Betti number with coefficients in $\mathbb{Z}_2$, defined by $b_i(\cdot) = \dim H_i(\cdot, \mathbb{Z}_2)$.

Corollary 2.5. *There exist additive maps on $\mathcal{AS}$ with values in $\mathbb{Z}$, noted β_i and called virtual Betti numbers, such that β_i coincides with the classical Betti numbers b_i on the connected component of compact nonsingular real algebraic varieties.*

Remark 2.6. C. McCrory and A. Parusiński [MCP03] have proven the same result for real algebraic varieties. In particular, they have defined the virtual Betti numbers of real algebraic varieties. The proof below is inspired by their argument.

Proof of Corollary 2.5. We adopt the notations of property P3 of theorem 2.3. One checks in a similar way to [MCP03] that there exist exact sequences of the form:

$$0 \longrightarrow H_i(\pi^{-1}(A) \cap \pi^{-1}(A \cap Y)) \longrightarrow H_i(A \cap Y) \oplus H_i(\pi^{-1}(A)) \longrightarrow H_i(A) \longrightarrow 0,$$

where the homology is considered with coefficients in $\mathbb{Z}_2$. Therefore

$$b_i(\pi^{-1}(A)) - b_i\big(\pi^{-1}(A) \cap \pi^{-1}(A \cap Y)\big) = b_i(A) - b_i(A \cap Y),$$

and we can apply theorem 2.3. □

It turns out to be easy to adapt theorem 2.3 in order to obtain not only additive maps but also generalized Euler characteristics.

Theorem 2.7. *Let χ be as in theorem 2.3. Assume moreover that χ takes values in a commutative ring, and that for connected components of compact nonsingular real algebraic varieties A and B, the relation $\chi(A \times B) = \chi(A)\chi(B)$ holds. Then the unique extension of χ on $\mathcal{AS}$ of theorem 2.3 is a generalized Euler characteristic.*

The following corollary is an immediate consequence of the Künneth formula.

Corollary 2.8. *Let β be defined on $A \in \mathcal{AS}$ by $\beta(A) = \sum_{i=0}^{\dim A} \beta_i(A)u^i$. Then β is a generalized Euler characteristic on $\mathcal{AS}$, called virtual Poincaré polynomial.*

Remark 2.9.

i) The name of "virtual Poincaré polynomial" is inspired by [Ful], where W. Fulton studies such a virtual Poincaré polynomial for complex algebraic varieties. It is related to the weighted characteristic associated with mixed Hodge structures.
ii) By unicity in theorem 2.7, one recovers the Euler characteristic with compact supports by evaluating β at $u = -1$.
iii) The virtual Poincaré polynomial is not a topological invariant [MCP03].

Example 2.10.

i) If $\mathbb{P}^k$ denotes the real projective space of dimension k, which is nonsingular and compact, then $\beta(\mathbb{P}^k) = 1 + u + \cdots + u^k$. Now, compactify the affine line $\mathbb{A}^1_{\mathbb{R}}$ in $\mathbb{P}^1$ by adding one point at the infinity. By additivity $\beta(\mathbb{A}^1_{\mathbb{R}}) = \beta(\mathbb{P}^1) - \beta(point) = u$, and so $\beta(\mathbb{A}^k_{\mathbb{R}}) = u^k$.
ii) Let W be the Whitney umbrella, and L be the line included in W. Then the additivity property implies $\beta(W) = \beta(W \setminus L) + \beta(L)$. Moreover $W \setminus L$ is isomorphic, via the blowing-up of W along L, to the strict transform of W minus a parabola P, therefore $\beta(W \setminus L) = \beta(\mathbb{A}^1_{\mathbb{R}} \times P) - \beta(P) = \big(\beta(\mathbb{A}^1_{\mathbb{R}}) - 1\big)\beta(P) = (u-1)u$. Finally $\beta(W) = u^2$.

Remark 2.11. The virtual Poincaré polynomial satisfies $\dim(A) = \deg\big(\beta(A)\big)$ for an arc-symmetric set A. In particular it respects the dimension.

Proof of theorem 2.3. We prove theorem 2.3 by induction on the dimension; the rank n inductive hypothesis claims that χ is defined on arc-symmetric sets of dimension less than or equal to n, is invariant under isomorphisms of arc-symmetric sets, and is additive.

For $n = 0$ the arc-symmetric sets are just finite unions of points and the result is clearly true. Assume that the inductive hypothesis is true at rank $n-1$. We prove the result at rank n in two steps:

i) if χ is an additive map on the nonsingular elements of $\mathcal{AS}$ of dimension less than or equal to n, then χ extends to an additive map on all arc-symmetric sets of dimension less than or equal to n,
ii) if χ satisfies property P1,P2 and P3, then χ extends to an additive map on the nonsingular elements of $\mathcal{AS}$ of dimension less than or equal to n.

Step 1. Let $A \in \mathcal{AS}$ of dimension n. There exists a stratification $\overline{A}^{\mathcal{Z}} = \bigcup_{S \in \mathcal{S}} S$ of $\overline{A}^{\mathcal{Z}}$ with nonsingular algebraic strata, i.e. $\overline{A}^{\mathcal{Z}}$ is a disjoint union of locally closed algebraic varieties (note in particular that we do not ask the strata to be connected). Then $S \cap A$, for each $S \in \mathcal{S}$, is a nonsingular arc-symmetric set, and thus $\chi(S \cap A)$ is defined. Put $\chi(A) = \sum_{S \in \mathcal{S}} \chi(S \cap A)$. One has to check that $\chi(A)$ is well-defined and satisfies the additivity property.

We show firstly that $\chi(A)$ is equal to $\sum_{S\in\mathcal{S}}\chi(S\cap A)$ in the case where A is nonsingular, by induction on the number of elements in $\mathcal{S}$. Indeed, take $N_0\in\mathcal{S}$; then

$$\chi(A)=\chi\big(A\setminus(A\cap N_0)\big)+\chi(A\cap N_0),\quad\text{and}\quad \chi\big(A\setminus(A\cap N_0)\big)=\sum_{S\in\mathcal{S}\setminus\{N_0\}}\chi(S\cap A)$$

by induction, so the result follows.

Now, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two stratifications of $\overline{A}^{\mathcal{Z}}$, one can find a common refinement $\mathcal{S}$ of $\mathcal{S}_1$ and $\mathcal{S}_2$. The independence in the nonsingular case induces that

$$\sum_{S\in\mathcal{S}_1}\chi(S\cap A)=\sum_{S\in\mathcal{S}}\chi(S\cap A)=\sum_{S\in\mathcal{S}_2}\chi(S\cap A),$$

thus χ does not depend on the choice of the stratification.

Let us show finally that χ is additive. Take A and B in $\in\mathcal{AS}$, with $B\subset A$, of dimension less than or equal to n. One can choose a stratification $\bigcup_{S\in\mathcal{S}}S$ of $\overline{A}^{\mathcal{Z}}$ such that $\overline{B}^{\mathcal{Z}}$ and $\overline{A\setminus B}^{\mathcal{Z}}$ are unions of strata. Then

$$\sum_{S\in\mathcal{S}}\chi(S\cap B)+\sum_{S\in\mathcal{S}}\chi\big(S\cap(A\setminus B)\big)=\sum_{S\in\mathcal{S}}\Big(\chi(S\cap B)+\chi\big(S\cap(A\setminus B)\big)\Big),$$

and $\chi\big(S\cap(A\setminus B)\big)+\chi(S\cap B)=\chi(S\cap A)$ because the strata are nonsingular, so χ is additive.

Step 2. The second step constitutes the heart of the work. Define χ over the nonsingular arc-symmetric sets of dimension n in the following way:

- **D1** if $A=\bigcup_{i\in I}A_i$ denotes the decomposition of A into irreducible components, put $\chi(A)=\sum_{i\in I}\chi(A_i)$,
- **D2** if $A\in\mathcal{AS}$ is nonsingular and irreducible, then define $\chi(A)$ by $\chi(A)=\chi(\widetilde{A})-\chi\big(\widetilde{A}\setminus\pi^{-1}(A)\big)$, where $\widetilde{A}$ is the connected component of a resolution of singularities π of $\overline{A}^{\mathcal{Z}}$ given by remark 1.15.

We have to prove that χ is well-defined, invariant under isomorphisms and additive over the nonsingular elements of $\mathcal{AS}$.

The following lemma will be useful in the sequel.

Lemma 2.12. *Let A and B in $\mathcal{AS}$ be nonsingular, irreducible and isomorphic. Suppose that $\overline{A}^{\mathcal{Z}}$ and $\overline{B}^{\mathcal{Z}}$ are nonsingular, and denote by $\widetilde{A}\subset\overline{A}^{\mathcal{Z}}$ and $\widetilde{B}\subset\overline{B}^{\mathcal{Z}}$ the connected components containing A and B respectively. Then*

$$\chi(\widetilde{A})-\chi(\widetilde{A}\setminus A)=\chi(\widetilde{B})-\chi(\widetilde{B}\setminus B).$$

Proof. By definition of an isomorphism between arc-symmetric sets, we know that $\overline{A}^{\mathcal{Z}}$ and $\overline{B}^{\mathcal{Z}}$ are birationally equivalent, and the weak factorization theorem [AKMW02, Wlo] factors this birational isomorphism in a succession of blowings-up and blowings-down. In particular, we can assume that the birational isomorphism between $\overline{A}^{\mathcal{Z}}$ and $\overline{B}^{\mathcal{Z}}$ is just a blowing-up $\pi:\overline{A}^{\mathcal{Z}}\longrightarrow\overline{B}^{\mathcal{Z}}$ along a nonsingular variety C such that $C\cap B=\emptyset$. Note that $\pi^{-1}(\widetilde{B})=\widetilde{A}$ by proposition 1.17. Now $\chi(\widetilde{B}\setminus B)=\chi(\widetilde{B}\cap C)+\chi\big(\widetilde{B}\setminus(B\cup C)\big)$ by the additivity inductive hypothesis because $\dim\widetilde{B}\setminus B$ is stricly less than $\dim B$ by proposition 1.6. Moreover

$$\chi(\widetilde{B})-\chi(\widetilde{B}\cap C)=\chi(\widetilde{A})-\chi\big(\widetilde{A}\cap\pi^{-1}(C)\big)\quad\text{and}\quad\chi\big(\widetilde{B}\setminus(B\cup C)\big)=\chi\Big(\widetilde{A}\setminus\big(A\cup\pi^{-1}(C)\big)\Big)$$

by property P3, and the inductive hypothesis on invariance under isomorphisms respectively. Therefore

$$\chi(\widetilde{B})-\chi(\widetilde{B}\setminus B)=\chi(\widetilde{A})-\chi\big(\widetilde{A}\cap\pi^{-1}(C)\big)-\chi\Big(\widetilde{A}\setminus\big(A\cup\pi^{-1}(C)\big)\Big),$$

which is equal to $\chi(\widetilde{A})-\chi(\widetilde{A}\setminus A)$ by the additivity inductive hypothesis. $\square$

Let us check that the definition of χ, for the nonsingular and irreducible arc-symmetric sets of dimension n, does not depend on the choice of the resolution of singularities of remark 1.15.

Let $A\in\mathcal{AS}$ be nonsingular and irreducible, and let $\pi_i:\widetilde{X_i}\longrightarrow\overline{A}^{\mathcal{Z}}$, for $i\in\{1,2\}$, be resolutions of singularities of $\overline{A}^{\mathcal{Z}}$. Let $\widetilde{A_i}$ be the connected components of $\widetilde{X_i}$ given by proposition 1.13. One has to show that: $\chi(\widetilde{A_1})-\chi\big(\widetilde{A_1}\setminus\pi_1^{-1}(A)\big)=\chi(\widetilde{A_2})-\chi\big(\widetilde{A_2}\setminus\pi_2^{-1}(A)\big)$.

But $\pi_1^{-1}(A)$ and $\pi_2^{-1}(A)$ are isomorphic irreducible nonsingular arc-symmetric sets because π_i is an isomorphism on a Zariski open subsets of $\widetilde{X_i}$ containing $\pi_i^{-1}(A)$, for $i\in\{1,2\}$. Therefore lemma 2.12 applies and χ is well-defined.

Now let us show that χ is invariant under isomorphisms of arc-symmetric sets. The proof is very similar to the last one. Let A and B in $\mathcal{AS}$ be nonsingular, irreducible and isomorphic. Then there exists Zariski open subsets U and V in $\overline{A}^{\mathcal{Z}}$ and $\overline{B}^{\mathcal{Z}}$ respectively, and an algebraic isomorphism $\phi:U\longrightarrow V$ such that $\phi(A)=B$. Choose resolutions of singularities

$$\pi_A:\widetilde{X}\longrightarrow\overline{A}^{\mathcal{Z}}\quad\text{and}\quad\pi_B:\widetilde{Y}\longrightarrow\overline{B}^{\mathcal{Z}}$$

for $\overline{A}^{\mathcal{Z}}$ and $\overline{B}^{\mathcal{Z}}$ respectively. Then $\pi_A^{-1}(A)$ and $\pi_B^{-1}(B)$ are isomorphic as arc-symmetric sets, and then by lemma 2.12: $\chi\big(\pi_A^{-1}(A)\big)=\chi\big(\pi_B^{-1}(B)\big)$.

Moreover $\chi\big(\pi_A^{-1}(A)\big)$ equals $\chi(A)$ because both are equal to $\chi(\widetilde{A})-\chi\big(\widetilde{A}\setminus\pi_A^{-1}(A)\big)$, where $\widetilde{A}$ is the connected component of $\widetilde{X}$ given by proposition 1.13. In the same way $\chi\big(\pi_B^{-1}(B)\big)$ is equal to $\chi(B)$, hence the equalities $\chi(A)=\chi\big(\pi_A^{-1}(A)\big)=\chi\big(\pi_B^{-1}(B)\big)=\chi(B)$ hold.

In the case where A and B are not irreducible, it suffices to decompose A and B in irreducible components, and to apply the property D1 because an isomorphism between arc-symmetric sets respects the irreducible components.

Finally, let us check that χ is additive. Let $B\subset A$ be an inclusion of nonsingular arc-symmetric sets. Note that, by definition of χ, we need to prove the result only in the case where A is irreducible.

If A and B have the same dimension, then $\overline{B}^{\mathcal{AS}}=\overline{A}^{\mathcal{AS}}$ by proposition 1.7, and so $\overline{B}^{\mathcal{Z}}=\overline{A}^{\mathcal{Z}}$. Now choose a resolution of singularities $\pi:\widetilde{X}\longrightarrow\overline{A}^{\mathcal{Z}}$ for $\overline{A}^{\mathcal{Z}}$. If $\widetilde{A}$ denotes the connected component of $\widetilde{X}$ given by proposition 1.13 for A, then it is also the component associated to B, and therefore $\chi(B)=\chi(\widetilde{A})-\chi\big(\widetilde{A}\setminus\pi^{-1}(B)\big)$.

Now $\chi\big(\widetilde{A}\setminus\pi^{-1}(B)\big)=\chi\big(\widetilde{A}\setminus\pi^{-1}(A)\big)+\chi\big(\pi^{-1}(A)\setminus\pi^{-1}(B)\big)$ by the inductive hypothesis on additivity. As a consequence $\chi(B)$ equals $\chi(A)-\chi(A\setminus B)$ because $\chi\big(\pi^{-1}(A)\setminus\pi^{-1}(B)\big)$ is equal to $\chi(A\setminus B)$ by the invariance under isomorphisms in dimensions smaller that n.

If $\dim B<\dim A$, choose a resolution of singularities $\pi:\widetilde{X}\longrightarrow\overline{A}^{\mathcal{Z}}$ for $\overline{A}^{\mathcal{Z}}$. It is also a resolution of singularities of $\overline{A\setminus B}^{\mathcal{Z}}=\overline{A}^{\mathcal{Z}}$ and so $\chi(A\setminus B)$ equals the difference $\chi(\widetilde{A})-\chi\big(\widetilde{A}\setminus\pi^{-1}(A\setminus B)\big)$. Now

$$\chi\big(\widetilde{A}\setminus\pi^{-1}(A\setminus B)\big)=\chi\Big(\big(\widetilde{A}\setminus\pi^{-1}(A)\big)\cup\pi^{-1}(B)\Big)=\chi\big(\widetilde{A}\setminus\pi^{-1}(A)\big)+\chi\big(\pi^{-1}(B)\big)$$

by the inductive assumption, and once more by the inductive assumption $\chi\big(\pi^{-1}(B)\big)$ is equal to $\chi(B)$. Finally $\chi(A\setminus B)=\chi(A)-\chi(B)$, which achieves the proof of step 2, and thus the proof of theorem 2.3. $\square$

As it was the case for the previous proof, we are going to prove theorem 2.7 by induction on the dimension. The following relations will be useful:

(1) $\chi(\sqcup_{i=1}^k A_i) = \sum_{i=1}^k \chi(A_i)$, where the union of the arc-symmetric sets $A_i, i = 1, \dots, k$ is disjoint, and

(2) $\chi(A) = \chi(\widetilde{A}) - \chi(\widetilde{A} \setminus A)$, where A is a nonsingular arc-symmetric set whose arc-symmetric closure $\widetilde{A}$ is nonsingular.

Proof of theorem 2.7. Put, as an inductive hypothesis at rank n, that χ is multiplicative for all arc-symmetric sets of dimension stricly less than or equal to n.

Remark that we can restrict our attention to the nonsingular case because, by considering stratifications of arc-symmetric sets with nonsingular strata, we prove the multiplicativity directly with formula (2.1).

Assume therefore that A and B are nonsingular arc-symmetric sets of dimension less than or equal to n; suppose that $\dim A = n$ for instance.

In the case where A is compact, the result follows from another induction, finite this one, on the dimension of B: indeed, resolving the singularities of $\overline{B}^{\mathcal{Z}}$, one can assume that $B \subset \widetilde{B}$, where $\widetilde{B}$ is the nonsingular arc-symmetric closure of B. Then, by (2.1), $\chi(A \times B)$ is equal to $\chi(A \times \widetilde{B}) - \chi\big(A \times (\widetilde{B} \setminus B)\big)$. However $\chi(A \times \widetilde{B}) = \chi(A)\chi(\widetilde{B})$ for they are compact and nonsingular, and $\chi\big(A \times (\widetilde{B} \setminus B)\big) = \chi(A)\chi\big(\widetilde{B} \setminus B)\big)$ as we can see by stratifying $\widetilde{B} \setminus B$ with nonsingular strata and using the inductive assumption of the second induction, because $\dim \widetilde{B} \setminus B < \dim B$ by lemma 1.6. Consequently

$$\chi(A \times B) = \chi(A)\big(\chi(\widetilde{B}) - \chi(\widetilde{B} \setminus B)\big) = \chi(A)\chi(B).$$

If A is no longer compact, then compactify A and B in $\widetilde{A}$ and $\widetilde{B}$ respectively, and assume that $\widetilde{A}, \widetilde{B}$ are nonsingular, even if it means resolving singularities, as before.

Then, by additivity, $\chi(A \times B) = \chi(\widetilde{A} \times B) - \chi\big((\widetilde{A} \setminus A) \times \widetilde{B}\big) + \chi\big((\widetilde{A} \setminus A) \times (\widetilde{B} \setminus B)\big)$.

The multiplicativity of the first two terms comes from the preceding case (in the second one, stratify the possibly singular set $\widetilde{A} \setminus A$), and the multiplicativity of the third is obtained by the inductive assumption for $\max(\dim \widetilde{A} \setminus A, \dim \widetilde{B} \setminus B) < n$ by lemma 1.6. Therefore $\chi(A \times B) = \chi(\widetilde{A})\chi(B) - \chi(\widetilde{A} \setminus A)\chi(\widetilde{B}) + \chi(\widetilde{A} \setminus A)\chi(\widetilde{B} \setminus B) = \chi(A)\chi(B)$, and theorem 2.7 is proven. ☐

2.2 Virtual Betti numbers and Nash isomorphisms

The definition of an isomorphism between arc-symmetric sets, we gave in part 1, is algebraic, via birational morphisms. But arc-symmetric sets are also closely related to analytic objects. As an example, the following proposition emphasizes the good behaviour of the virtual Poincaré polynomial β with respect to the compact algebraic varieties which are nonsingular as analytic varieties. Recall that by $b_i(X)$ we denote the i-th Betti number of X with coefficients in $\mathbb{Z}_2$, and let us put $b(X) = \sum_{i=0}^{\dim X} b_i(X)u^i$.

Proposition 2.13. *Let X be a compact algebraic variety which is nonsingular as an analytic space. Then the virtual Poincaré polynomial β evaluated on X is equal to $b(X)$.*

Proof. One can desingularize the algebraic singularities of X by a sequence of blowings-up with smooth centers ([BM97, Hir64]). At each step of the desingularization, one has the following relation, where $Bl_C X$ designs the blowing-up of X along the nonsingular subvariety C, and E is the exceptional divisor: $\beta(Bl_C X) - \beta(E) = \beta(X) - \beta(C)$, because the blowing-up is birational, and $b(Bl_C X) - b(E) = b(X) - b(C)$, because X and C are smooth and the blowing-up is a degree one morphism (cf. corollary 2.5).

Remark that $\beta(E)$ and $\beta(C)$ are equal to $b(E)$ and $b(C)$ respectively by definition of β, because E and C are nonsingular and compact arc-symmetric sets. The same is true for $\widetilde{X}$, the desingularization of X. Then $\beta(X)$ and $b(X)$ can be expressed by the same formulae in terms of β for the former, and b for the latter, where the spaces involved are nonsingular and compact. Therefore, for each one of these spaces, β and b coincide, and then $\beta(X)$ is equal to $b(X)$. □

Remark 2.14.

i) Note that a real algebraic variety which is nonsingular as an analytic space is not necessarily nonsingular as an algebraic variety (cf. [BCR], example 3.3.12.b).
ii) We will see in the proof of theorem 2.16 that the assumption "X is an algebraic variety" can be replaced by the weaker "X is a semi-algebraic set".

In order to relate the analytic aspect of arc-symmetric sets to the behaviour of the virtual Poincaré polynomial β, we propose the following definition of a Nash isomorphism between arc-symmetric sets.

Definition 2.15. Let $A, B \in \mathcal{AS}$. Assume that there exist compact analytic varieties V_1, V_2 containing A, B respectively, and also an analytic isomorphism ϕ from V_1 to V_2 such that $\phi(A) = B$. If moreover one can choose V_1, V_2 to be semi-algebraic sets and ϕ to be a semi-algebraic map, then we say that A and B are Nash isomorphic.

Theorem 2.16. *Nash isomorphic arc-symmetric sets have the same value under the virtual Poincaré polynomial.*

Proof of theorem 2.15. Once more, we are going to prove the result by an induction on the dimension. As a first step, let us generalize the result of proposition 2.13.

Step 1. Let A be a compact arc-symmetric set which is also a nonsingular analytic subspace of the Zariski closure X of A. Then $\beta(A)$ is equal to $b(A)$.

In order to prove this claim, one wants to apply the same method as in the proof of proposition 2.13. But, if C is a smooth center of blowing-up for X, it is not true in general that $C \cap A$ is still nonsingular, so the equality $\beta(C \cap A) = b(C \cap A)$ does no longer hold. In order to solve this problem, consider the algebraic normalization $\widetilde{X}$ of X. There exists $\widetilde{A} \subset \widetilde{X}$ the analytic normalization of A ([Loj]), which is analytically isomorphic to A because A is nonsingular as an analytic space. Then $b(A)$ is equal to $b(\widetilde{A})$ because b is invariant under homeomorphisms.

Moreover $\beta(A)$ is equal to $\beta(\widetilde{A})$; actually the algebraic normalization is a birational map, hence it is an algebraic isomorphism outside compact subvarieties E and D of $\widetilde{X}$ and X respectively, of dimension stricly less than $\dim X = \dim A$. Thus $\beta(\widetilde{A} \setminus E)$ is equal to $\beta(A \setminus D)$ by corollary 2.5, and the algebraic normalization, restricted to $\widetilde{A} \cap E$, is an analytic isomorphism onto $A \cap D$, so $\beta(\widetilde{A} \cap E)$ is equal to $\beta(A \cap D)$ by the inductive assumption.

Note that $\widetilde{X}$ is locally analytically irreducible as a normal space, therefore $\widetilde{A}$ is a union of connected components of $\widetilde{X}$. Now it is true that $C \cap \widetilde{A}$ is nonsingular when C is nonsingular, and the method of the proof of proposition 2.13 applies, therefore $\beta(\widetilde{A})$ is equal to $b(\widetilde{A})$. It follows that $\beta(A) = \beta(\widetilde{A}) = b(\widetilde{A}) = b(A)$, and then Step 1 is acheived.

Step 2. Let A_1 and A_2 be Nash isomorphic arc-symmetric sets. Let us prove the theorem in the particular case where A_1 and A_2 are nonsingular arc-symmetric sets and moreover, with the assumptions of the definition of a Nash isomorphism, the compact analytic variety V_1 and V_2 are supposed to be smooth as analytic spaces.

- First we show that $\beta(\overline{A_2}^{\mathcal{AS}}) = \beta(\overline{A_1}^{\mathcal{AS}})$.

Remark that $\overline{A_2}^{\mathcal{AS}}$ is a union of connected components of V_2 by proposition 1.7. Thus $\overline{A_2}^{\mathcal{AS}}$ is also nonsingular as an analytic variety and $\beta(\overline{A_2}^{\mathcal{AS}})$ is equal to $b(\overline{A_2}^{\mathcal{AS}})$ by step 1.

Moreover $\overline{A_2}^{\mathcal{AS}}$ is isomorphic to $\overline{A_1}^{\mathcal{AS}}$ by ϕ. Indeed, $\phi^{-1}(\overline{A_2}^{\mathcal{AS}})$ is a closed arc-symmetric set because ϕ have an arc-symmetric graph and ϕ is continuous, and it contains A_1, so $\overline{A_1}^{\mathcal{AS}} \subset \phi^{-1}(\overline{A_2}^{\mathcal{AS}})$. The reverse inclusion comes from the fact that the image by an injective map with arc-symmetric graph of an arc-symmetric set is still an arc-symmetric set (recall that $\mathcal{AS}$ form a constructible category, cf. part 1). Consequently, $\overline{A_1}^{\mathcal{AS}}$ is nonsingular as an analytic variety because so is $\overline{A_2}^{\mathcal{AS}}$ and ϕ is an analytic isomorphism, hence $\beta(\overline{A_1}^{\mathcal{AS}})$ equals $b(\overline{A_1}^{\mathcal{AS}})$ by the first step.

Remark also that $b(\overline{A_2}^{\mathcal{AS}})$ is equal to $b(\overline{A_1}^{\mathcal{AS}})$ because ϕ is a homeomorphism between these two smooth compact topological varieties.

These equalities imply that $\beta(\overline{A_2}^{\mathcal{AS}})$ equals $\beta(\overline{A_1}^{\mathcal{AS}})$.

• Then, remark that $\beta(\overline{A_1}^{\mathcal{AS}} \setminus A_1)$ is equal to $\beta(\overline{A_2}^{\mathcal{AS}} \setminus A_2)$. Indeed this follows from the inductive hypothesis, for $\overline{A_1}^{\mathcal{AS}} \setminus A_1$ and $\overline{A_2}^{\mathcal{AS}} \setminus A_2$ are Nash isomorphic arc-symmetric sets of dimension strictly less than $\dim A_2$.

• Finally $\beta(A_1)$ is equal to $\beta(A_2)$. Actually $\beta(A_1) = \beta(\overline{A_1}^{\mathcal{AS}}) - \beta(\overline{A_1}^{\mathcal{AS}} \setminus A_1)$ and $\beta(A_2) = \beta(\overline{A_2}^{\mathcal{AS}}) - \beta(\overline{A_2}^{\mathcal{AS}} \setminus A_2)$, and we have proved that the second members are equal, so $\beta(A_1)$ equals $\beta(A_2)$ by additivity of the virtual Poincaré polynomial β.

Step 3. Reduction of the problem to Step 2.

Let A_1 and A_2 be Nash isomorphic arc-symmetric sets. By definition of a Nash isomorphism, there exist compact analytic varieties V_1 and V_2 containing A and B respectively, and an analytic isomorphism $\phi : V_1 \longrightarrow V_2$ such that $\phi(A_1) = A_2$, and moreover V_1 and V_2 are semi-algebraic sets and ϕ is a semi-algebraic map.

Denote by X_1 and X_2 the Zariski closures of V_1 and V_2 respectively.

As a first step, we are going to obtain a regular morphism rather than a semi-algebraic map between V_1 and V_2. Denote by Γ the graph of ϕ. This graph is semi-algebraic and analytic, thus arc-symmetric. Then the projection p_i from $Z = \overline{\Gamma}^{\mathcal{Z}}$ onto X_i, for $i \in \{1, 2\}$, is a regular morphism whose restriction to Γ is an analytic isomorphism onto V_i. Moreover, the preimages by these restrictions of A_1 and A_2 coincide, so one can put $B = p^{-1}(A)$, where $A = A_i \subset V_i = V \subset X_i = X$ for $i \in \{1, 2\}$ and $p : Z \longrightarrow X$ denotes the natural projection. Therefore B is an arc-symmetric set which is Nash isomorphic to A, and the issue is now to prove that $\beta(B)$ equals $\beta(A)$.

In order to do this, we want to come down to nonsingular objects.

Desingularize X by a sequence of blowings-up with respect to coherent algebraic sheaves of ideals (this is possible by [BM97, Hir64]). By blowing-up Z with respect to the corresponding inverse image ideal sheaves with respect to p, one has at each step a regular morphism which lifts the projection $p : Z \longrightarrow X$ to the corresponding blowing-up by the universal property of algebraic blowing-up. Let $\pi_X : \widetilde{X} \longrightarrow X$ denotes the resolution of singularities of X and $\pi_Z : \widetilde{Z} \longrightarrow Z$ the corresponding composition of blowings-up of Z. If $\widetilde{p}$ denotes the morphism obtained between $\widetilde{Z}$ and $\widetilde{X}$ by the universal property, one has the following diagram:

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\widetilde{p}"] \arrow[d, "\pi_Z"'] & \widetilde{X} \arrow[d, "\pi_X"] \\ Z \arrow[r, "p"'] & X \end{tikzcd}$$

which is commutative. Moreover $\widetilde{p}$ restricted to the analytic strict transform $\widetilde{\Gamma}$ of Γ is an analytic isomorphism onto the strict transform $\widetilde{V}$ of V because so is p between Γ and V (here we consider the blowing-up as an analytic one).

Now we reduce the problem to the case where A and B are nonsingular by the inductive hypothesis. Actually the singular part of A and B are not necessarily exchanged by $p_{|\Gamma}$, but

$$\operatorname{Sing}(A) \cup p_{|\Gamma}^{-1}\big(\operatorname{Sing}(B)\big) \quad \text{and} \quad \operatorname{Sing}(B) \cup p_{|\Gamma}\big(\operatorname{Sing}(A)\big)$$

are Nash isomorphic by the restriction of $p_{|\Gamma}$. Moreover the dimension of these arc-symmetric sets is stricly less than $\dim A = \dim B$, so they have the same image by β thanks to the inductive hypothesis. Let us denote by A' and B' the respective complements of these sets in A and B. Now A' and B' are nonsingular.

As A' is nonsingular, it is isomorphic to its preimage in the desingularization $\widetilde{X}$ of X, in the sense of definition 1.10. Consequently B' is also isomorphic to its preimage in $\widetilde{\Gamma}$ by commutativity of the first diagram. As a consequence $\beta(A')$ and $\beta(B')$ are equal to $\beta\big(\pi^{-1}(A')\big)$ and $\beta\big(\pi^{-1}(B')\big)$ respectively, and we have reduced the problem to step 2. $\square$

3. Zeta functions

The zeta functions of a real analytic function germ we consider in this part are directly inspired by the work of J. Denef & F. Loeser [DL01] on their motivic zeta functions. In particular, our zeta functions are defined by considering the image, under the virtual Poincaré polynomial, of certain constructible real algebraic subsets of the arc space of an affine space.

In the case the function germ is Nash (that is semi-algebraic and analytic), we connect these zeta functions with the blow-analytic equivalence in part 4.

In this part, after the definition and some examples of zeta functions, we focus on the Denef & Loeser formulae. These formulae enable to compute the zeta functions in terms of a modification of the Nash function germ we consider. It gives some possibilities to compute more easily these zeta functions, and it is also a key ingredient for the application to blow-Nash equivalence in part 4.

Note that we state also a Thom-Sebastiani formula for analytic germs of the same sign, but we do not obtain such a formula in the general case.

3.1 Zeta functions and Denef & Loeser Formula

We first define the zeta functions for a germ of real analytic functions. Then we give a formula to compute these zeta functions in terms of a modification in the case the germ is Nash.

Denote by $\mathcal{L}$ the space of arcs at the origin $0 \in \mathbb{R}^d$, defined by:

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d, 0) = \{\gamma : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^d, 0) : \gamma \text{ formal}\},$$

and by $\mathcal{L}_n$ the space of truncated arcs at the order $n+1$:

$$\mathcal{L}_n = \mathcal{L}_n(\mathbb{R}^d, 0) = \{\gamma \in \mathcal{L} : \gamma(t) = a_1 t + a_2 t^2 + \cdots a_n t^n,\ a_i \in \mathbb{R}^d\},$$

for $n \geqslant 0$ an integer . Let $\pi_n : \mathcal{L} \longrightarrow \mathcal{L}_n$ and $\pi_{n,i} : \mathcal{L}_n \longrightarrow \mathcal{L}_i$, with $n \geqslant i$, be the truncation morphisms.

Consider $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ a real analytic function germ. We define the naive zeta function $Z_f(T)$ of f as the following element of $\mathbb{Z}[u, u^{-1}][[T]]$:

$$Z_f(T) = \sum_{n \geqslant 1} \beta(\mathcal{X}_n) u^{-nd} T^n,$$

where

$$\mathcal{X}_n = \{\gamma \in \mathcal{L}_n : ord(f\circ\gamma) = n\} = \{\gamma \in \mathcal{L}_n : f\circ\gamma(t) = bt^n + \cdots, b \neq 0\}.$$

Similarly, we define zeta functions with sign, by

$$Z_f^+(T) = \sum_{n\geqslant 1} \beta(\mathcal{X}_n^+)u^{-nd}T^n \quad \text{and} \quad Z_f^-(T) = \sum_{n\geqslant 1} \beta(\mathcal{X}_n^-)u^{-nd}T^n,$$

where

$$\mathcal{X}_n^+ = \{\gamma \in \mathcal{L}_n : f\circ\gamma(t) = +t^n + \cdots\} \quad \text{and} \quad \mathcal{X}_n^- = \{\gamma \in \mathcal{L}_n : f\circ\gamma(t) = -t^n + \cdots\}.$$

Remark that $\mathcal{X}_n$ and $\mathcal{X}_n^{\pm}$, for $n \geqslant 1$, are Zariski constructible subsets of $\mathbb{R}^{nd}$, hence belong to $\mathcal{AS}$.

Example 3.1. Let $f : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be defined by $f(x) = x^k,\ k \geqslant 1$. Then

$$\mathcal{X}_n = \begin{cases} \{\gamma = a_m t^m + \cdots + a_n t^n;\ a_m \neq 0\} \simeq \mathbb{R}^* \times \mathbb{R}^{n-m} & \text{if n=mk,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore $\beta(\mathcal{X}_n) = (u-1)u^{n-m}$ if $n = mk$ and 0 otherwise, hence

$$Z_f(T) = \sum_{m\geqslant 1} (u-1)u^{mk-m}\big(\frac{T}{u}\big)^{mk} = (u-1)\frac{T^k}{u-T^k}.$$

To compute the zeta functions with sign, we have to consider the case $k = 2p$ and $k = 2p+1$. If $k = 2p$, then $\mathcal{X}_n^- = \emptyset$ and

$$\mathcal{X}_n^+ = \begin{cases} \{\gamma = \pm t^m + \cdots + a_n t^n;\ a_m \neq 0\} \simeq \{\pm 1\} \times \mathbb{R}^{n-m} & \text{if n=mk,} \\ \emptyset & \text{otherwise,} \end{cases}$$

so $Z_f^+(T) = \sum_{m\geqslant 1} 2u^{mk-m}\big(\frac{T}{u}\big)^{mk} = 2\frac{T^k}{u-T^k}$. If $k = 2p+1$, then

$$\mathcal{X}_n^{\pm} = \begin{cases} \{\gamma = \pm t^m + \cdots + a_n t^n;\ a_m \neq 0\} \simeq \{\pm 1\} \times \mathbb{R}^{n-m} & \text{if n=mk,} \\ \emptyset & \text{otherwise,} \end{cases}$$

and thus $Z_f^+(T) = Z_f^-(T) = \sum_{m\geqslant 1} u^{mk-m}\big(\frac{T}{u}\big)^{mk} = \frac{T^k}{u-T^k}$.

It may be convenient to express the zeta functions of a germ f in terms of a modification of f, that is a proper birational map which is an isomorphism over the complement of the zero locus of f, and such that composed with the modification f becomes a normal crossing. Similarly to [DL99], the following proposition, called Denef & Loeser formula, enables to do this in the case the germ is a Nash germ.

Proposition 3.2. *Let $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be a Nash function germ. Let $\sigma : \big(M, \sigma^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$ be a modification of $\mathbb{R}^d$ such that $f\circ\sigma$ and the jacobian determinant $\operatorname{jac}\sigma$ are normal crossings simultaneously, and assume moreover that σ is an isomorphism over the complement of the zero locus of f.*

Let $(f\circ\sigma)^{-1}(0) = \cup_{j\in J} E_j$ be the decomposition into irreducible components of $(f\circ\sigma)^{-1}(0)$, and assume that $\sigma^{-1}(0) = \cup_{k\in K} E_k$ for some $K \subset J$.

Put $N_i = \operatorname{mult}_{E_i} f\circ\sigma$ and $\nu_i = 1 + \operatorname{mult}_{E_i} \operatorname{jac}\sigma$, and for $I \subset J$ denote by E_I^0 the set $(\cap_{i\in I} E_i) \setminus (\cup_{j\in J\setminus I} E_j)$. Then

$$Z_f(T) = \sum_{I\neq\emptyset} (u-1)^{|I|}\beta\big(E_I^0 \cap \sigma^{-1}(0)\big) \prod_{i\in I} \frac{u^{-\nu_i}T^{N_i}}{1-u^{-\nu_i}T^{N_i}}.$$

Remark 3.3. We have to assume that f is Nash, and not only analytic, in order the sets E_I^0 to be arc-symmetric.

Example 3.4. Let $f_k : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R},0)$ be defined by $f_k(x,y) = x^k + y^k$, $k \geqslant 2$. The blowing-up at the origin gives a suitable modification σ for f. Here $(f\circ\sigma)^{-1}(0)$ consists of just the exceptional divisor $\mathbb{P}^1$ in the case k even, and furthermore, in the case k odd, of the strict transform of f which is a smooth curve crossing transversally the exceptional divisor. Then

$$Z_{f_k} = \begin{cases} (u^2-1)u^{-2}\frac{T^k}{1-u^{-2}T^k} & \text{if k is even,} \\ (u-1)\frac{u^{-2}T^k}{1-u^{-2}T^k}(u+(u-1)\frac{u^{-1}T}{1-u^{-1}T}) & \text{if k is odd.} \end{cases}$$

Note in particular that for $k \neq k'$, the zeta functions Z_{f_k} and $Z_{f_{k'}}$ are different.

When we are dealing with signs, one has to define coverings $\widetilde{E_I^{0,\pm}}$ of E_I^0, where $\pm$ designs either $+$ or $-$, in order to state the correponding Denef & Loeser formulae.

Let U be an affine open subset of M such that $f\circ\sigma = u\prod_{i\in I} y_i^{N_i}$ on U, where u is a unit. Let us put

$$R_U^{\pm} = \{(x,t) \in (E_I^0 \cap U)\times\mathbb{R}; t^m = \pm\frac{1}{u(x)}\},$$

where $m = gcd(N_i)$. Then the $R_U^{\pm}$ glue together along the $E_I^0\cap U$ (cf. lemma 3.12) to give $\widetilde{E_I^{0,\pm}}$.

Proposition 3.5. *With the assumptions and notations of proposition 3.2, one can express the zeta functions with sign in terms of a modification as:*

$$Z_f^{\pm}(T) = \sum_{I\neq\emptyset}(u-1)^{|I|-1}\beta\big(\widetilde{E_I^{0,\pm}}\cap\sigma^{-1}(0)\big)\prod_{i\in I}\frac{u^{-\nu_i}T^{N_i}}{1-u^{-\nu_i}T^{N_i}}.$$

These formulae will be proven in section 3.2 below.

Example 3.6.

i) The case of a normal crossings function is particularly simple to handle with. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be defined by $f(x) = u(x)\prod_{i=1}^k x_i^{N_i}$, with $N_i \in \mathbb{N}$. Then

$$Z_f(T) = (u-1)^k\prod_{i=1}^k\frac{u^{-1}T^{N_i}}{1-u^{-1}T^{N_i}}.$$

Now, if there exists at least one N_i odd, then $Z_f^+(T) = Z_f^-(T) = \frac{1}{u-1}Z_f(T)$.
On the other hand, if all the N_i are even, then $Z_f^-(T) = 0$ and $Z_f^+(T) = \frac{2}{u-1}Z_f(T)$ if u is positive, the converse otherwise.

ii) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x^2+y^2$. As f is a positive function, then $Z_f^-(T) = 0$. We obtain a modification in the same way as in example 3.4, and $\widetilde{E_I^{0,+}}$ is here the boundary of a Mobius band, hence homeomorphic to $\mathbb{P}^1$. Therefore

$$Z_f^+(T) = (u+1)\frac{u^{-2}T^2}{1-u^{-2}T^2} = \frac{1}{u-1}Z_f(T).$$

iii) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x^2+y^4$. One can solve the singularities of f by two successive blowings-up, and then one obtains that the exceptional divisor E has two irreducible components E_1 and E_2 with $N_1 = 2,\ \nu_1 = 2,\ N_2 = 4,\ \nu_2 = 3$. Therefore

$$Z_f(T) = (u-1)^2\frac{u^{-2}T^2}{1-u^{-2}T^2}\frac{u^{-3}T^4}{1-u^{-3}T^4} + (u-1)u\frac{u^{-2}T^2}{1-u^{-2}T^2} + (u-1)u\frac{u^{-3}T^4}{1-u^{-3}T^4}.$$

Moreover in this case $\widetilde{E^{0,+}_{\{1\}}}$ and $\widetilde{E^{0,+}_{\{2\}}}$ are homeomorphic to a circle minus two points, so

$$Z_f^+(T) = 2(u-1)\frac{u^{-2}T^2}{1-u^{-2}T^2}\frac{u^{-3}T^4}{1-u^{-3}T^4} + (u-1)\frac{u^{-2}T^2}{1-u^{-2}T^2} + (u-1)\frac{u^{-3}T^4}{1-u^{-3}T^4}.$$

Note that in this particular case one has neither $Z_f(T) = (u-1)Z_f^+(T)$ nor $Z_f(T) = \frac{u-1}{2}Z_f^+(T)$, whereas it was the case in the previous examples.

Remark 3.7. It would be convenient to dispose of a Thom-Sebastiani formula in order to compute the zeta functions of the function $f * g$, which is defined by the formula $f * g(x,y) = f(x) + g(y)$, from the ones of f and g, as it is the case in [KP, DL01, Loo00]. But it seems to be impossible to find in general such formulae with our zeta functions. However in the particular case of two positive (respectively negative) functions, one has the following formulae.

Proposition 3.8. *Let $f : (\mathbb{R}^{d_1}, 0) \longrightarrow (\mathbb{R}, 0)$ and $g : (\mathbb{R}^{d_2}, 0) \longrightarrow (\mathbb{R}, 0)$ be two positive or two negative real analytic function germs. Let us put $Z_f(T) = \sum_{n\geqslant 1} a_n T^n$, $Z_g(T) = \sum_{n\geqslant 1} b_n T^n$ and moreover $A_n = 1 - \sum_{j=1}^n a_j$, $B_n = 1 - \sum_{j=1}^n b_j$.*

Then the naive zeta function of $f * g : \mathbb{R}^{d_1+d_2} \longrightarrow \mathbb{R}$ is $Z_{f*g}(T) = \sum_{n\geqslant 1} c_n T^n$, where

$$c_n = a_n B_n + A_n b_n + a_n b_n.$$

Example 3.9.

i) Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $h(x,y) = x^2 + y^2$. Recall that (cf. example 3.4):

$$Z_f(T) = (u^2-1)\sum_{n\geqslant 1}\frac{T^{2n}}{u^{2n}}.$$

Putting $f(x) = g(x) = x^2$, then $h = f * g$ and by example 3.1 we get that $a_{2n} = b_{2n} = \frac{u-1}{u^n}$ and $a_{2n+1} = b_{2n+1} = 0$, hence $A_{2n} = A_{2n+1} = \frac{1}{u^n}$. Then, by proposition 3.8, we rederive $c_{2n} = \frac{u^2-1}{u^{2n}}$ and $c_{2n+1} = 0$.

ii) Let f and g be defined by $f(x) = x^2$ and $g(y) = y^4$, and consider $f * g(x,y) = x^2 + y^4$. The odd coefficients of the naive zeta function of $f * g$ are zero because f and g are positive, and it is easy to verify that

$$a_{2n} = \frac{u-1}{u^n},\quad A_{2n} = \frac{1}{u^n} \text{ and } b_{4n} = \frac{u-1}{u^n},\quad b_{4n+2} = 0,\quad B_{4n} = \frac{1}{u^n} = B_{4n+2}.$$

Therefore $c_{4n} = \frac{u^2-1}{u^{3n}}$ and $c_{4n+2} = \frac{u-1}{u^{3n+1}}$, which was not so clear on the expression of the naive zeta function of $f * g$ computed with the Denef & Loeser formula, in example 3.6.iii.

Proof of proposition 3.8. Remark first that $u^{nd_1}A_n = \beta(\{\gamma \in \mathcal{L}_n;\ ord(f\circ\gamma) > n\})$.

Actually, the space $\mathcal{L}_n$ can be decomposed into the disjoint union

$$\mathcal{L}_n = \pi_{n,1}^{-1}(\mathcal{X}_1) \sqcup \ldots \sqcup \pi_{n,n}^{-1}(\mathcal{X}_n) \sqcup \{\gamma \in \mathcal{L}_n;\ ord(f\circ\gamma) > n\}.$$

Hence, by additivity of β, one gets $u^{nd_1} = \sum_{j=1}^n a_i u^{nd_1} + \beta(\{\gamma \in \mathcal{L}_n;\ ord(f\circ\gamma) > n\})$, and the remark is proved.

Now take (γ_1,γ_2) in $\mathcal{L}_n(\mathbb{R}^{d_1}) \times \mathcal{L}_n(\mathbb{R}^{d_2}) = \mathcal{L}_n(\mathbb{R}^{d_1+d_2})$. Then $ord(f\circ\gamma_1 + g\circ\gamma_2)$ is greater than n if and only if $ord(f\circ\gamma_1)$ and $ord(g\circ\gamma_2)$ are greater than n, because f and g are of the same sign. Therefore we have to distinguish the three cases:

- $ord(f\circ\gamma_1) = n$ and $ord(g\circ\gamma_2) > n$,
- $ord(f\circ\gamma_1) > n$ and $ord(g\circ\gamma_2) = n$,
- $ord(f\circ\gamma_1) = n$ and $ord(g\circ\gamma_2) = n$.

The computation gives:

$$\beta\big(\mathcal{X}_n(f*g)\big)=\beta\big(\mathcal{X}_n(f)\big)u^{nd_2}B_n+u^{nd_1}A_n\beta\big(\mathcal{X}_n(g)\big)+\beta\big(\mathcal{X}_n(f)\big)\beta\big(\mathcal{X}_n(g)\big).$$

□

3.2 Motivic integration and the proof of Denef & Loeser formula

The proof of Denef & Loeser formula, which is a simplification of the one of [DL98], theorem 2.2.1 to our setting, uses the theory of motivic integration on arc spaces for real algebraic varieties. In particular, we will use the change of variables formula of Kontsevich.

For the convenience of the reader, we recall briefly these notions before proving theorems 3.2 and 3.5.

Take $\sigma:\big(M,\sigma^{-1}(0)\big)\longrightarrow(\mathbb{R}^d,0)$ a real modification, and define the arc space associated to $\big(M,\sigma^{-1}(0)\big)$ by:

$$\mathcal{L}\big(M,\sigma^{-1}(0)\big)=\{\gamma:(\mathbb{R},0)\longrightarrow\big(M,\sigma^{-1}(0)\big);\ \gamma\text{ is formal}\}.$$

The truncated arc space $\mathcal{L}_n$ consists of the arcs of $\mathcal{L}$, but truncated at the order $n+1$, for an integer $n\geqslant 0$.

Denote by $\pi_n:\mathcal{L}\longrightarrow\mathcal{L}_n$ the natural truncation morphism, for $n\in\mathbb{N}$, where $\mathcal{L}$ denotes either $\mathcal{L}\big(M,\sigma^{-1}(0)\big)$ or $\mathcal{L}(\mathbb{R}^d,0)$. A subset $A\subset\mathcal{L}$ is called stable if there exist a constructible set $C\subset\mathcal{L}_n$ and some $n\geqslant 0$ such that $A=\pi_n^{-1}(C)$. Then we can define the measure in $\mathbb{Z}[u,u^{-1}]$ of such a stable set A, with respect to the virtual Poincaré polynomial β, by

$$\beta(A)=u^{-(n+1)d}\beta\big(\pi_n(A)\big),$$

for n large enough (note that $\beta\big(\pi_n(A)\big)$ is well defined since Zariski constructible real algebraic varieties are arc-symmetric sets). Indeed, $\beta(A)$ does not depend on n because the natural projections $\mathcal{L}_{n+1}\longrightarrow\mathcal{L}_n$ are locally trivial fibrations with fiber $\mathbb{R}^d$.

Let us recall now the definition of integrals. Let $\theta:A\longrightarrow\mathbb{Z}[u,u^{-1}]$ be a map with a finite image and whose fibers are stable sets. Then the integral of θ over A with respect to β is defined by:

$$\int_A\theta d\beta=\sum_{c\in\mathbb{Z}[u,u^{-1}]}c\beta\big(\theta^{-1}(c)\big).$$

We can state the Kontsevich change of variables formula. Recall that $\operatorname{jac}\sigma$ denotes the jacobian determinant of $\operatorname{Jac}\sigma$.

Proposition 3.10. *([Kont, DL99]) Let $A\subset\mathcal{L}(\mathbb{R}^d,0)$ be stable, and suppose that the function $\operatorname{ord}_t\operatorname{jac}\sigma$ is bounded on $\sigma^{-1}(A)$. Then*

$$\beta(A)=\int_{\sigma^{-1}(A)}u^{-\operatorname{ord}_t\operatorname{jac}\sigma}d\beta.$$

Before giving the details of the proof of Denef & Loeser formula, and notably a preliminary lemma, we fix some notations. Recall that f is a Nash function germ. The modification σ induces applications σ_* (respectively $\sigma_{*,n}$) between $\mathcal{L}\big(M,\sigma^{-1}(0)\big)$ and $\mathcal{L}(\mathbb{R}^d,0)$ (respectively $\mathcal{L}_n\big(M,\sigma^{-1}(0)\big)$ and $\mathcal{L}_n(\mathbb{R}^d,0)\big)$. Put

$$\mathcal{Z}_n(f)=\pi_n^{-1}(\mathcal{X}_n)\ \text{ and }\ \mathcal{Z}_n(f\circ\sigma)=\sigma_*^{-1}\big(\mathcal{Z}_n(f)\big).$$

Moreover, for $e\geqslant 1$, put $\Delta_e=\{\gamma\in\mathcal{L}\big(M,\sigma^{-1}(0)\big);\ \operatorname{ord}_t\operatorname{jac}\sigma\big(\gamma(t)\big)=e\}$, and finally define $\mathcal{Z}_{n,e}(f\circ\sigma)=\mathcal{Z}_n(f\circ\sigma)\cap\Delta_e$.

Lemma 3.11. *Let $\sigma:\big(M,\sigma^{-1}(0)\big)\longrightarrow(\mathbb{R}^d,0)$ be a modification of $\mathbb{R}^d$ such that $f\circ\sigma$ and the jacobian determinant $\operatorname{jac}\sigma$ are normal crossings simultaneously, and assume moreover that σ is an isomorphism over the complement of the zero locus of f.*

Let $(f\circ\sigma)^{-1}(0)=\cup_{j\in J}E_j$ be the decomposition into irreducible components of $(f\circ\sigma)^{-1}(0)$, and assume that $\sigma^{-1}(0)=\cup_{k\in K}E_k$ for some $K\subset J$.

Put $N_i=\operatorname{mult}_{E_i} f\circ\sigma$ and $\nu_i=1+\operatorname{mult}_{E_i}\operatorname{jac}\sigma$, and for $I\subset J$ denote by E_I^0 the set $(\cap_{i\in I}E_i)\setminus(\cup_{j\in J\setminus I}E_j)$. Then there exists $c\in\mathbb{N}$ such that the naive zeta function $Z_f(T)$ of f equals

$$u^d\sum_{n\geqslant 1}T^n\sum_{e\leqslant cn}u^{-e}\sum_{I\neq\emptyset}\beta(\{\gamma\in\mathcal{L}_n(M,E_I^0)\cap\pi_n(\Delta_e);\ \operatorname{ord} f\circ\sigma\circ\gamma=n\})$$

and the zeta functions with sign $Z_f^{\pm}(T)$ equal

$$u^d\sum_{n\geqslant 1}T^n\sum_{e\leqslant cn}u^{-e}\sum_{I\neq\emptyset}\beta(\{\gamma\in\mathcal{L}_n(M,E_I^0)\cap\pi_n(\Delta_e);\ f\circ\sigma\circ\gamma(t)=\pm t^n+\cdots\}).$$

Proof. Let us prove the lemma for Z_f since the argument is the same for $Z_f^{\pm}$.

For $n\geqslant 1$, $\mathcal{Z}_n(f)$ is stable, so $\beta\big(\mathcal{Z}_n(f)\big)$ is defined and equals $u^{-(n+1)d}\beta(\mathcal{X}_n)$, hence $Z_f(T)=u^d\sum_{n\geqslant 1}\beta\big(\mathcal{Z}_n(f)\big)T^n$.

Moreover $\mathcal{Z}_n(f\circ\sigma)$ equals the disjoint union $\cup_{e\geqslant 1}\mathcal{Z}_{n,e}(f\circ\sigma)$, which is a finite union. Actually, take $\gamma\in\mathcal{Z}_n(f\circ\sigma)$; there exists $I\subset J$ such that $\pi_0(\gamma)\in E_I^0$. Then in a neighbourhood of $\gamma(0)$, one can choose coordinates such that

$$f\circ\sigma=unit\prod_{i\in I}y_i^{N_i}\quad\text{and}\quad\operatorname{jac}\sigma=unit\prod_{i\in I}y_i^{\nu_i-1},$$

where by *unit* we denote a non-vanishing analytic function.

Let us write $\gamma=(\gamma_1,\dots,\gamma_d)$, and $k_i=\operatorname{ord}_t\gamma_i$, for $i=1,\dots,d$. Then the order $\operatorname{ord}_t f\circ\sigma\big(\gamma(t)\big)$ is equal to $\sum_{i=1}^d N_ik_i=n$ and therefore:

$$\operatorname{ord}_t\operatorname{jac}\sigma\big(\gamma(t)\big)=\sum_{i=1}^d(\nu_i-1)k_i\leqslant\max_i(\frac{\nu_i-1}{N_i})\sum_{i=1}^d N_ik_i=\max_i(\frac{\nu_i-1}{N_i})n.$$

Let $c=\max_i(\frac{\nu_i-1}{N_i})$. Then we have shown that $\cup_{e\geqslant 1}\mathcal{Z}_{n,e}(f\circ\sigma)=\cup_{e\leqslant cn}\mathcal{Z}_{n,e}(f\circ\sigma)$, where the union is finite.

Now Kontsevich change of variables formula induces that $\beta\big(\mathcal{Z}_n(f)\big)$ is equal to the finite sum $\sum_{e\leqslant cn}u^{-e}\beta\big(\mathcal{Z}_{n,e}(f\circ\sigma)\big)$, and then $Z_f(T)=u^d\sum_{n\geqslant 1}T^n\sum_{e\leqslant cn}u^{-e}\beta\big(\mathcal{Z}_{n,e}(f\circ\sigma)\big)$.

We are going to compute $\beta\big(\mathcal{Z}_{n,e}(f\circ\sigma)\big)$ using the fact that $\mathcal{Z}_{n,e}(f\circ\sigma)$ equals the disjoint union $\bigsqcup_{I\neq\emptyset}\mathcal{Z}_{n,e}(f\circ\sigma)\cap\pi_0^{-1}\big(E_I^0\cap\sigma^{-1}(0)\big)$. Indeed, by additivity we find:

$$\beta\big(\mathcal{Z}_{n,e}(f\circ\sigma)\big)=\sum_{I\neq\emptyset}\beta\Big(\mathcal{Z}_{n,e}(f\circ\sigma)\cap\pi_0^{-1}\big(E_I^0\cap\sigma^{-1}(0)\big)\Big).$$

Choose $I\neq\emptyset$. Then $\pi_n\big(\mathcal{Z}_{n,e}(f\circ\sigma)\cap\pi_0^{-1}(E_I^0\cap\sigma^{-1}(0)\big)$ is just the set

$$\{\gamma(t)\in\mathcal{L}_n\big(M,\sigma^{-1}(0)\big);\ \gamma(0)\in E_I^0\cap\sigma^{-1}(0),\operatorname{ord}_t f\circ\sigma\big(\gamma\big)=n,\operatorname{ord}_t\operatorname{jac}\sigma\big(\gamma\big)=e\}.$$

The results follows directly from the additivity of β. $\square$

The proof of propositions 3.2 and 3.5 just consists in computing the value of the virtual Poincaré polynomial β on the sets which appear in the formulae of lemma 3.11.

Proof of proposition 3.2. Take $\gamma\in\pi_n\big(\mathcal{Z}_{n,e}(f\circ\sigma)\cap\pi_0^{-1}(E_I^0\cap\sigma^{-1}(0)\big)$. On a neighbourhood of $\gamma(0)$, one can choose coordinates such that

$$f\circ\sigma=unit\prod_{i\in I}y_i^{N_i}\quad\text{and}\quad\operatorname{jac}\sigma=unit\prod_{i\in I}y_i^{\nu_i-1},$$

hence $\pi_n(\mathcal{Z}_{n,e}(f\circ\sigma)\cap\pi_0^{-1}(E_I^0\cap\sigma^{-1}(0))$ is isomorphic to

$$\{\gamma\in\mathcal{L}_n(M,\sigma^{-1}(0));\ \gamma(0)\in E_I^0\cap\sigma^{-1}(0),\sum_{i\in I}k_iN_i=n,\sum_{i\in I}k_i(\nu_i-1)=e\},$$

where $k_i=\operatorname{ord}_t\gamma_i$ for $i\in I$. As a consequence $\pi_n\Big(\mathcal{Z}_{n,e}(f\circ\sigma)\cap\pi_0^{-1}\big(E_I^0\cap\sigma^{-1}(0)\big)\Big)$ is isomorphic to

$$\bigsqcup_{k\in A(n,e)}\big(E_I^0\cap\sigma^{-1}(0)\big)\times(\mathbb{R}^*)^{|I|}\big(\prod_{i\in I}\mathbb{R}^{n-k_i}\big)\times(\mathbb{R}^n)^{d-|I|}$$

where $A(n,e)$ is the subset of $k\in\mathbb{N}^d$ defined by the equations $\sum_{i=1}^d N_ik_i=n$ and $\sum_{i=1}^d(\nu_i-1)k_i=e$.

By taking the image by β, we obtain the equality:

$$\beta\Big(\pi_n\Big(\mathcal{Z}_{n,e}(f\circ\sigma)\cap\pi_0^{-1}\big(E_I^0\cap\sigma^{-1}(0)\big)\Big)\Big)=\sum_{k\in A(n,e)}\beta\big(E_I^0\cap\sigma^{-1}(0)\big)(u-1)^{|I|}u^{nd-\sum_{i=1}^d k_i},$$

hence the naive zeta function of f satisfies:

$$Z_f(T)=\sum_{I\neq\emptyset}(u-1)^{|I|}\beta\big(E_I^0\cap\sigma^{-1}(0)\big)\sum_{n\geqslant1}\sum_{e\leqslant cn}\sum_{k\in A(n,e)}u^{-e-\sum_{i=1}^d k_i}T^n.$$

Remark that $\{k\in A(n,e);\ n\geqslant1,e\leqslant cn\}$ is in bijection with $\mathbb{N}^{|I|}$, therefore

$$\sum_{n\geqslant1}\sum_{e\leqslant cn}\sum_{k\in A(n,e)}u^{-e-\sum_{i=1}^d k_i}T^n=\sum_k\prod_{i\in I}(u^{-\nu_i}T^{N_i})^{k_i}=\prod_{i\in I}\frac{u^{-\nu_i}T^{N_i}}{1-u^{-\nu_i}T^{N_i}},$$

and this ends the proof. □

The proof of proposition 3.5 is a little bit more complicated due to the fact that we have to introduce a covering $\widetilde{E_I^{0,\pm}}$ of E_I^0 in order to compute $Z_f^{\pm}(T)$. Recall that if U is an affine open subset of M such that $f\circ\sigma=u\prod_{i\in I}y_i^{N_i}$ on U, where u denotes a unit, then by $R_U^{\pm}$ we mean the set $R_U^{\pm}=\{(x,t)\in(E_I^0\cap U)\times\mathbb{R};\ t^{m_I}=\pm\frac{1}{u(x)}\}$, where $m_I=gcd_{i\in I}(N_i)$. Then $\widetilde{E_I^{0,\pm}}$ is the gluing of the $R_U^{\pm}$ along the $E_I^0\cap U$.

LEMMA 3.12. *The $R_U^{\pm}$ glue together along $E_I^0\cap U$.*

Proof. It suffices to prove that the definition of $\widetilde{E_I^{0,\pm}}$ does not depend on the choice of the local coordinates. Let z_i be another local system of coordinates on U such that $f\circ\sigma=v\prod_{i\in I}z_i^{N_i}$. Then z_i is proportional to y_i for the indices i in I, therefore $z_i=\alpha_iy_i$ for a non vanishing analytic function α_i. So $v(y)\prod_{i\in I}\alpha_i^{N_i}=u(y)$ and therefore the sets $\{(x,t)\in(E_I^0\cap U)\times\mathbb{R};\ t^{m_I}=\pm\frac{1}{u(x)}\}$ and $\{(x,t)\in(E_I^0\cap U)\times\mathbb{R};\ t^{m_I}=\pm\frac{1}{v(x)}\}$ are isomorphic by the application $(x,t)\longmapsto(x,t\prod_{i\in I}\alpha_i^{\frac{N_i}{m}})$. □

Proof of proposition 3.5. Let U be an affine open subset of M such that $f\circ\sigma=u\prod_{i\in I}y_i^{N_i}$ on U, where u is a unit. What we have to compute is the value of β on

$$W^{\pm}=\{(x,y)\in(E_I^0\cap U)\times(\mathbb{R}^*)^{|I|};\ u(x)\prod_{i\in I}y_i^{N_i}=\pm1\}.$$

Denote by m the greatest common divisor of the N_i, $i\in I$, and choose n_i, $i\in I$ such that $\sum_{i\in I}n_iN_i=m$. Assume that $I=\{1,\dots,s\}$. Remark that $W^{\pm}$ is isomorphic to

$$W'^{,\pm}=\{(x,y,t)\in(E_I^0\cap U)\times(\mathbb{R}^*)^{|I|}\times\mathbb{R}^*;\ t^m=\frac{\pm1}{u(x)},\ \prod_{i\in I}y_i^{N_i/m}=1\},$$

by the application $(x,y,t) \longmapsto (x, t^{n_1}y_1, \dots, t^{n_s}y_s)$ from $W'^{,\pm}$ to $W^{\pm}$. The inverse is the morphism given by

$$(x,y) \longmapsto (x, (\prod_{i\in I} y_i^{N_i/m})^{-n_1} y_1, \dots, (\prod_{i\in I} y_i^{N_i/m})^{-n_s} y_s, \prod_{i\in I} y_i^{N_i/m}).$$

Now it is easier to compute $\beta(W'^{,\pm})$ because $W'^{,\pm} \simeq R_U^{\pm} \times (\mathbb{R}^*)^{|I|-1}$. Actually, this last isomorphism comes from the fact that at least one $\frac{N_i}{m}$ is odd. Therefore $\beta(W^{\pm}) = (u-1)^{|I|-1}\beta(R_U^{\pm})$, and the same computation as in the naive case gives the formula. □

4. An invariant of the blow-Nash equivalence

In this part, we define the blow-Nash equivalence of Nash function germs, by analogy with the blow-analytic equivalence due to T.-C. Kuo [Kuo85]. He introduced this equivalence relation after noticing that the Whitney family, which is topologically trivial but has infinitely many C^1-equivalence classes, is analytically trivial after one blowing-up.

For a survey on blow-analytic equivalence, we refer to [FKK98]. Note in particular that this equivalence relation between real analytic function germs does not admit moduli for a family with isolated singularities. Moreover, there exist efficient tools to prove that two given germs are blow-analytically equivalent, for example via toric modifications. Note that there exist some recent invariants of this equivalence relation which enable to distinguish blow-analytic types ([Fu97, KP]).

We prove that similar triviality results hold for this new equivalence relation. Moreover, we state and prove the main result of this part which is the invariance of the zeta functions, constructed in part 3, with respect to the blow-Nash equivalence. As an application, we finally state in section 4.2 the blow-Nash equivalence classes of the two variables Brieskorn polynomials.

4.1 Blow-Nash equivalence

4.1.1 *Definition* To begin with, let us recall that by an algebraic modification of a real analytic function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$, we mean a proper birational algebraic morphism $\sigma_f : \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$, between analytic neighbourhoods of 0 in $\mathbb{R}^d$ and the exceptional divisor $\sigma_f^{-1}(0)$ in M_f, which is an isomorphism over the complement of the zero locus of f and for which $f \circ \sigma$ is in normal crossing.

Definition 4.1.

i) A map $\phi : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}^d, 0)$ is blow-Nash if there exists a proper birational algebraic morphism $\sigma : \big(M, \sigma^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$ such that $\phi \circ \sigma$ is Nash (i.e. semi-algebraic and analytic).

ii) Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be two germs of Nash functions. They are said to be blow-Nash equivalent if there exist two algebraic modifications

$$\sigma_f \;:\; \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0) \text{ and } \sigma_g : \big(M_g, \sigma_g^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0),$$

such that $f \circ \sigma_f$ and $\operatorname{jac} \sigma_f$ (respectively $g \circ \sigma_g$ and $\operatorname{jac} \sigma_g$) are in normal crossings and a Nash isomorphism (i.e. a semi-algebraic map which is an analytic isomorphism) Φ between analytic neighbourhoods $\big(M_f, \sigma_f^{-1}(0)\big)$ and $\big(M_g, \sigma_g^{-1}(0)\big)$ which preserves the multiplicities of the jacobian determinants of σ_f and σ_g along the components of the exceptional divisors, and which induces a homeomorphism $\phi : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}^d, 0)$ such that $f = g \circ \phi$, as illustrated by the commutative diagram:

$$\begin{tikzcd}
\left(M_f, \sigma_f^{-1}(0)\right) \arrow[rr, "\Phi"] \arrow[d, "\sigma_f"'] & & \left(M_g, \sigma_g^{-1}(0)\right) \arrow[d, "\sigma_g"] \\
(\mathbb{R}^d, 0) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (\mathbb{R}^d, 0) \arrow[dl, "g"] \\
& (\mathbb{R}, 0) &
\end{tikzcd}$$

Remark 4.2.

i) We do not know whether the blow-Nash equivalence is an equivalence relation or not. When trying to adapt the corresponding proof for the blow-analytic equivalence [Kuo85], the problem comes from the transitivity property. Namely, the fiber product of an algebraic map and a Nash map needs not to be algebraic.
In the following, we consider the equivalence relation generated by this relation, which still we call blow-Nash equivalence.

ii) Apart from the fact that the blow-Nash equivalence only concerns Nash germs, the differences between the blow-analytic equivalence (cf. [Kuo85]) and the blow-Nash equivalence are the following. In the definition of blow-Nash equivalence, we ask:
   - the modifications to be algebraic, and not only analytic, and their jacobian determinant to be in normal crossings,
   - the isomorphism Φ upstairs to be Nash, and not only analytic,
   - Φ to preserve the jacobian determinant orders of the modifications.

   These additional assumptions are necessary in order to adapt techniques coming from motivic integration to this more analytic framework, and in particular to obtain the central result of this paper, theorem 4.8.
   Note however that different definitions of a blow-analytic homeomorphism have occured since the original article of T.-C. Kuo [Kuo85] appeared, and notably T. Fukui, T.-C. Kuo and L. Paunescu propose in [FKP01] a definition closer to our one. Actually, they define a blow-analytic isomorphism to be a homeomorphism such that there exists an analytic isomorphism upstairs which is moreover an isomorphism between the critical loci of the modifications.

iii) Another relation close to blow-analytic equivalence, called Blow-analytic equivalence, is also studied (cf. [FKK98]). It is defined in a similar way as blow-analytic equivalence, with the difference that the real analytic modifications are required to be compositions of blowings-up along smooth centers. Up to now, it is not known whether Blow-analytic equivalence is an equivalence relation or not as soon as $d > 2$.

4.1.2 *Properties* In this part, we are concerned with triviality results and classification tools.

First, it is obvious that two real analytic function germs that are analytically equivalent are blow-analytically equivalent! But note also that two Nash germs that are only analytically equivalent are also Nash equivalent, as proven by M. Shiota [Sh98], and therefore blow-Nash equivalent. Therefore the blow-Nash equivalence is a particular case of analytic equivalence between Nash germs.

Now, the question of moduli is a natural question when one studies an equivalence relation of germs. The following theorem states that there is no moduli for a Nash family with isolated singularities under some algebraicity assumptions on modifications.

Theorem 4.3. *Let $F : (\mathbb{R}^d, 0) \times P \longrightarrow (\mathbb{R}, 0)$ be Nash, where P is a Nash set diffeomorphic to an* *open simplex in an Euclidean space. Assume that $F(.,p) : (\mathbb{R}^d, 0) \longrightarrow \mathbb{R}$ has an isolated singularity at 0 for each $p \in P$, and assume moreover that F admits an algebraic resolution of singularities.*

Then the family $F(.,p)$, for $p \in P$, consists of a finite number of blow-Nash equivalence classes.

REMARK 4.4.

i) By an algebraic resolution of singularities for F, we mean a finite composition $\beta : M \longrightarrow \mathbb{R}^d \times P$ of blowings-up with smooth algebraic centers such that β is an isomorphism outside $0 \times P$, and $F \circ \beta$ is in normal crossing.
Note that in the particular case where F is polynomial, such a resolution exists by Hironaka's Desingularization theorem [Hir64].

ii) The proof of theorem 4.3, which is postponed to section 4.3, is inspired by [Kuo85] where T.-C. Kuo proved the finiteness of the number of blow-analytic equivalence classes for an analytically parametrized family of isolated singularities. However, the key argument of integration along a vector field does no longer apply in the Nash category, so we need new arguments.

The following particular case, which is a consequence of the proof of theorem 4.3, gives a tool to prove blow-Nash triviality. It will help us in classifying the blow-Nash type of Brieskorn polynomials (see section 4.2).

COROLLARY 4.5. *Let $f_t : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$, $t \in I$, with I an interval of $\mathbb{R}$, be a Nash parametrized family of weighted homogeneous polynomials of the same weight with an isolated singularity at the origin. Then the family $\{f_t\}_{t \in I}$ is blow-Nash trivial.*

Proof. It is well-known (see [FKK98] for example) that, in that case, one can find a toric modification, given by the weight of the weighted homogeneous polynomials, $\beta : (M, E) \longrightarrow (\mathbb{R}^d, 0)$ of $(\mathbb{R}^d, 0)$ such that $\beta \times id$ fulfils the assumptions of proposition 4.17. $\square$

EXAMPLE 4.6. Consider the Nash function germs $x^p + y^{kp}$ and $x^p - y^{kp}$ from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}, 0)$. They are blow-Nash equivalent. Indeed,

$$f_t(x, y) = x^p + \frac{1 - t^2}{1 + t^2} pxy^{k(p-1)} + \frac{2t}{1 + t^2} y^{kp}, \quad t \in [-1, 1],$$

is a weighted homogeneous polynomial of weight $(k, 1)$ with an isolated singularity at the origin for each $t \in [-1, 1]$, and therefore corollary 4.5 implies that f_{-1} and f_1 are blow-Nash equivalent.

The counterpart is now to be able to distinguish blow-Nash types. Actually, it is a difficult issue with this kind of relations to find invariants. More precisely, for the blow-analytic equivalence, only the Fukui invariants [IKK02] and the zeta functions of S. Koike and A. Parusiński [KP], defined with the Euler characteristic with compact supports, are known. Note that by evaluating the virtual Poincaré polynomial at $u = -1$ in the zeta functions defined in section 3, one recovers these zeta functions, and moreover our zeta functions generalize the Fukui invariants. Indeed, it follows directly from remark 2.11 that:

PROPOSITION 4.7. *The Fukui invariants are the exponents of the naive zeta function $Z_f(T)$ with non-zero coefficients. Similarly, the Fukui invariants with sign are the exponents of the zeta functions with sign $Z_f^{\pm}(T)$ with non-zero coefficients.*

The following theorem is the main one of this paper.

THEOREM 4.8. *The naive zeta function $Z_f(T)$ and the zeta functions with sign $Z_f^{\pm}(T)$ of germs of Nash functions are invariants of the blow-Nash equivalence.*

Proof. Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be two blow-Nash equivalent Nash function germs. By definition, there exist modifications

$$\sigma_f : \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0) \quad \text{and} \quad \sigma_g : \big(M_g, \sigma_g^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0),$$

and a Nash isomorphism $\Phi : \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow \big(M_g, \sigma_g^{-1}(0)\big)$ as in definition 4.1. Then the assumptions of proposition 3.2 and 3.5 are satisfied.

Now, it suffices to prove that the expression of the zeta functions given by the Denef & Loeser formulae coincide. But β is invariant under Nash isomorphisms by theorem 2.16, and moreover Φ preserves

-the multiplicities of $f \circ \sigma_f$ and $g \circ \sigma_g$, because it is an isomorphism,

-the multiplicities of the jacobians of σ_f and σ_g along the components of the exceptional divisors, by definition of blow-Nash equivalence.

Therefore the zeta functions of f and g coincide. □

Remark 4.9.

i) If the zeta functions generalize the Fukui invariant and the zeta functions defined with the Euler characteristic with compact supports, they are invariant for a more restrictive relation between germs.

ii) However, our zeta functions are also invariants for the analytic equivalence of real analytic germs.
Actually, the constructible sets $\mathcal{X}_n$ and $\mathcal{X}_n^{\pm}$ associated with two analytically equivalent function germs f, g are isomorphic. Indeed, let h be a local analytic isomorphism such that $f = g \circ h$. Then, the application $\gamma \longmapsto \pi_n(h(\gamma))$ from $\mathcal{X}_n(f)$ to $\mathcal{X}_n(g)$ is an algebraic isomorphism because, after truncation at the level of the space of arcs, the local analytic isomorphism becomes algebraic. Therefore the naive zeta functions of f and g coincide. The proof in the case with sign is similar.

4.2 Application to Brieskorn polynomials

We apply our zeta functions to sketch the classification of two variables Brieskorn polynomials under blow-Nash equivalence, and to give examples in three variables. Brieskorn polynomials in two or three variables are polynomials of the type

$$\varepsilon_p x^p + \varepsilon_q y^q (+\varepsilon_r z^r), \ p \leqslant q \leqslant r \in \mathbb{N}, \ \varepsilon_p, \varepsilon_q, \varepsilon_r \in \{\pm 1\}.$$

Remark that if $p = 1$, then $\varepsilon_p x + \varepsilon_q y^q (+\varepsilon_r z^r)$ is Nash isomorphic to x. Therefore we will restrict our attention to the case $p \geqslant 2$.

Actually, the classification under blow-analytic equivalence has been done completely in the two variables case, and almost completely in the three variables case in [KP] using the zeta functions defined with Euler characteristic with compact supports and the Fukui invariants (see [KP], theorem 7.3; for the Fukui invariants, see [IKK02]). There exists only one case that can not be decided, and the following example shows that we can distinguish it under blow-Nash equivalence. However this it is not sufficient to conclude for it in the blow-analytic context.

Example 4.10. Let $f_{p,k}$ be the Brieskorn polynomial defined by $f_{p,k} = \pm(x^p + y^{kp} + z^{kp})$, with p even, $k \in \mathbb{N}$. We prove that for fixed p and different k, two such polynomials are not blow-Nash equivalent.

In order to do this, we calculate directly the naive zeta function of $f_{p,k}$. For $n \in \mathbb{N}$, we have to compute $\beta(\mathcal{X}_n)$. First, it is clear that $\mathcal{X}_n = \emptyset$ when n is not a multiple of p. If n is a multiple of

p, write $n = p(mk + r)$ where $mk + r$ represents the euclidean division of $\frac{n}{p}$ by k. If $\gamma \in \mathcal{L}_n$, put $\gamma = (a_1 t + \cdots + a_n t^n, b_1 t + \cdots + b_n t^n, c_1 t + \cdots + c_n t^n)$.

Then if $r \neq 0$, the first non zero term of $f \circ \gamma$ is given by the first component of γ, hence $\mathcal{X}_n$ equals $\{\gamma;\ a_{mk+r} \neq 0, a_1 = \cdots = a_{mk+r-1} = b_1 = \cdots = b_m = c_1 = \cdots = c_m = 0\}$.

In the case where $r = 0$, the three components of γ play a part, and $\mathcal{X}_n$ equals

$$\{\gamma;\ (a_{mk}, b_m, c_m) \neq 0, a_1 = \cdots = a_{mk-1} = b_1 = \cdots = b_{m-1} = c_1 = \cdots = c_{m-1} = 0\}.$$

Therefore

$$\mathcal{X}_n \simeq \begin{cases} \mathbb{R}^* \times \mathbb{R}^{(p-1)(mk+r)} \times (\mathbb{R}^{p(mk+r)-m})^2 & \text{if } r \neq 0 \\ (\mathbb{R}^3)^* \times \mathbb{R}^{(p-1)mk} \times (\mathbb{R}^{pmk-m})^2 & \text{if } r = 0 \end{cases}$$

hence the coefficient of T^n is

$$\beta(\mathcal{X}_n) u^{-3n} = \begin{cases} (u-1)u^{-(mk+r)-2m} & \text{if } n = p(mk+r), 0 < r < k \\ (u^3-1)u^{-mk-2m} & \text{if } n = pmk \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the zeta function of $f_{p,k}$ looks like

$$\begin{aligned} Z_{f_{p,k}} &= (u-1)\big(u^{-1}T^p + u^{-2}T^{2p} + \cdots + u^{-(k-1)}T^{(k-1)p}\big) + (u^3-1)u^{-k-2}T^{kp} \\ &+ (u-1)\big(u^{-(k+3)}T^{(k+1)p} + u^{-(k+4)}T^{(k+2)p} + \cdots + u^{-(2k+1)}T^{(2k-1)p}\big) \\ &+ (u^3-1)u^{-2(k-2)}T^{2kp} + \cdots. \end{aligned}$$

Now it suffices to note that, for p fixed and $k < k'$ the pk-coefficient of $Z_{f_{p,k}}$ is $(u^3-1)u^{-k-2}$ whereas the one of $Z_{f_{p,k'}}$ is $(u-1)u^{-k}$.

REMARK 4.11. The case of two variables Brieskorn polynomials have been dealt with in [KP], using their zeta functions and the Fukui invariants. Actually the only case where the equivalence class of Brieskorn polynomials of two variables can not be distinguished using only their zeta functions, and which requires the use of the Fukui invariants, is the following: $f_k(x, y) = \pm(x^k + y^k)$, $k \geqslant 2$ even. Remark that we have seen in example 3.4 that for $k \neq k'$ the naive zeta functions Z_{f_k} and $Z_{f'_k}$ are different, therefore our zeta function distinguishes this case, for blow-Nash equivalence.

Actually, one can say more. Indeed, for two variables Brieskorn polynomials the naive zeta function determines the exponents p and q.

PROPOSITION 4.12. *Let $g = \pm x^p \pm y^q$ be a two variables Brieskorn polynomial. Then the exponents p and q are uniquely determined by the naive zeta function $Z_g(T) = \sum_{n\geqslant 1} g_n T^n$. More precisely*

$$p = min\{n;\ g_n \neq 0\}$$

and if $l = min\{n;\ g_n \neq a_n u^n\}$, where $\sum_{n \geqslant 1} a_n T^n$ denotes the naive zeta function of $\pm x^p$, then

$$q = \begin{cases} l-1 & \textit{if } p \textit{ is odd, } p \textit{ divides } l-1 \textit{ and } g_l \neq (u-1)u^k \textit{ for any } k \in \mathbb{N} \\ l & \textit{otherwise.} \end{cases}$$

Proof. The characterization of p is clear. Now, if $p \nmid q$, then $q = l$ and $g_q = (u-1)u^k$ for some $k \in \mathbb{N}$.

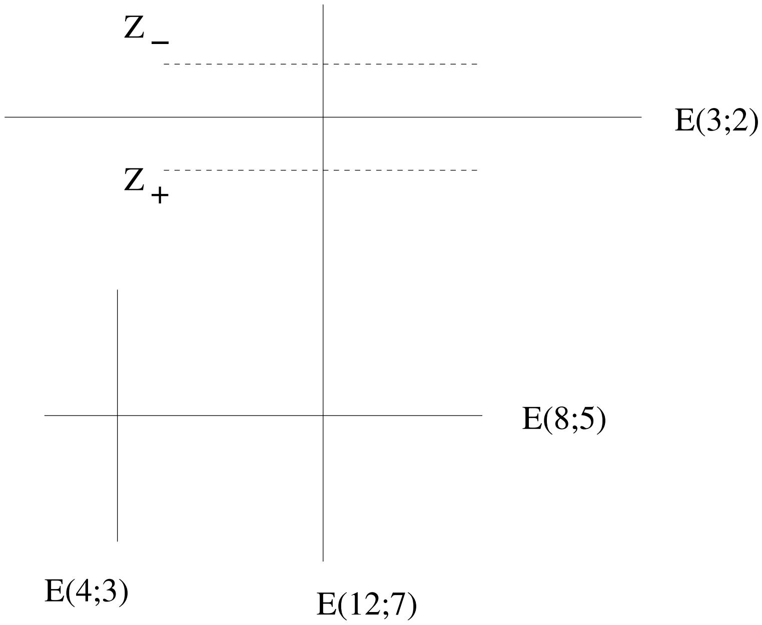
If $q = kp$ for some $k \in \mathbb{N}$, then $g_{kp} = \beta(\{\pm a^p \pm b^{kp})\})u^{2kp-k-1}$ and thus

$$g_{kp} = a_{kp}u^{kp} \iff \beta\big(\{\pm a^p \pm b^{kp} \neq 0\}\big) = u(u-1) \iff p \text{ odd}.$$

In that case $g_{kp+1} = (u-1)^2 u^{2kp-k-2}$, so $l = kp + 1$.

Therefore if p is even then $q = l$, and if p is odd, then either $q = l$ or $q = l - 1$. More precisely, if $g_l = (u-1)u^k$ then $p \nmid q$ and $q = l$, whereas if $g_l = (u-1)^2 u^k$, then $p | l-1$ and $q = l-1$. $\square$

Figure 1. Resolution tree of $x^3 \pm y^4$.



To find the signs in front of x^p and y^q, the naive zeta function is not sufficient as shown by the following example.

Example 4.13. Let $f_\pm : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ be defined by $f_\pm(x,y) = x^3 \pm y^4$. One can solve the singularities of $f_\pm$ by a succession of four blowings-up. The resolution tree of this modification σ is drawn in figure 1, where $Z_\pm$ denotes the strict transform of $f_\pm$, and $E(N,\nu)$ denote the irreducible component of the exceptional divisor such that $\mathrm{mult}_E\, f_\pm \circ \sigma = N$ and $1 + \mathrm{mult}_E\, \mathrm{jac}\, \sigma = \nu$. The Denef & Loeser formulae imply that Z_{f_+} is equal to Z_{f_-}, but f_+ and f_- are not blow-Nash equivalent (they are not even blow-analytically equivalent, see [KP], theorem 6.1 for example).

Actually the zeta functions with sign enable to discuss the signs for two variables Brieskorn polynomials except in one case. The following proposition details the possibilities. Note that $(x,y) \longrightarrow (\pm x, \pm y)$ gives an action on the blow-Nash classes, hence when the power p (or q) is odd, the corresponding sign can not be determined. By convention, in the case where $p = q$ and the signs are opposite, we consider $x^p - y^p$ rather than $-x^p + y^p$.

Proposition 4.14. *Let $Z_g^\pm(T) = \sum_{n \geqslant 1} g_n^\pm T^n$ be the zeta functions with sign of $\varepsilon_p x^p + \varepsilon_q y^q$, with $\varepsilon_p, \varepsilon_q \in \{\pm 1\}$. If p is even, then*

$$\varepsilon_p = \begin{cases} +1 & \text{if } g_p^+ \neq 0 \\ -1 & \text{otherwise,} \end{cases}$$

and if q is even, but not multiple of an odd p, then

$$\varepsilon_q = \begin{cases} +1 & \text{if } g_p^- = 0 \\ -1 & \text{otherwise.} \end{cases}$$

The remaining case is when p is odd and $q = kp$ with k even. But in example 4.6 we proved that in that case $x^p + y^{kp}$ and $x^p - y^{kp}$ are blow-Nash equivalent, therefore:

Proposition 4.15. *For two Brieskorn polynomials in two variables, the three following statements are equivalent:*

– *they are blow-Nash equivalent,*
– *they are blow-analytically equivalent,*
– *their naive zeta function and zeta functions with sign coincide.*

4.3 Proof of theorem 4.3

T.-C. Kuo [Kuo85] proved the finiteness of the number of blow-analytic equivalence classes for an analytically parametrized family of real analytic germs with an isolated singularity. In that setting, he used integration along vector fields to construct a trivialization of a modification of the zero set of the family. Unfortunately, in the Nash situation, this efficient method is forbidden for we go out from the Nash world...

Fortunately T. Fukui, S. Koike and M. Shiota have given an effective tool to show Nash triviality: the Nash Isotopy Lemma [FKS98]. It gives a trivialization of Nash submanifolds, possibly with boundary, with normal crossings, and also to their arbitrary intersections:

Theorem 4.16 Nash Isotopy Lemma. *Let M be a Nash manifold possibly with boundary and $N_1, \dots, N_k$ be Nash submanifolds of M possibly with boundary which together with $N_0 = \partial M$ are normal crossing. Assume that $\partial N_i \subset N_0$ for $i \in \{1, \dots, k\}$. Let P be a Nash manifold diffeomorphic to an open simplex in an Euclidean space, and $\omega : M \longrightarrow P$ be a proper onto Nash submersion such that the restrictions of ω*

$$\omega : N_{i_1} \cap \dots \cap N_{i_s} \longrightarrow P,$$

for $0 \leqslant i_1 < \dots < i_s \leqslant k$, are also proper onto submersions.

Then there exists a Nash isomorphism

$$\phi : (M; N_1, \dots, N_k) \longrightarrow \big(M \cap \omega^{-1}(0); N_1 \cap \omega^{-1}(0), \dots, N_k \cap \omega^{-1}(0)\big) \times P$$

such that $\omega \circ \phi^{-1} : \big(M \cap \omega^{-1}(0)\big) \times P \longrightarrow P$ is the canonical projection.

This result does not replace totally the integration along vector fields because it works just at the level of manifolds, and not of functions. Therefore it enables us to obtain a Nash trivialization of the zero sets of our Nash parametrized family of Nash germs, but not of the nonzero levels of the functions of the family. Now, in order to show the Nash triviality, we have recourse here to orthogonal projections between levels of functions (cf. lemma 4.18) in order to force the trivialization. And the point is that, if this technique does not allow us to keep a Nash isomorphism, it gives us a blow-Nash isomorphism which is sufficient for our matter!

Proof of theorem 4.3. Following T.-C. Kuo proof [Kuo85], we can subdivide P into a finite number of Nash sets P', diffeomorphic to open simplices in an Euclidean space, such that there exists an algebraic modification $\beta : M \longrightarrow \mathbb{R}^d \times P'$ which satisfies:

– β is an isomorphism outside $0 \times P'$,
– $F \circ \beta$ is in normal crossing,
– if $r : \mathbb{R}^d \times P' \longrightarrow P'$ denotes the canonical projection, then $r \circ \beta$ maps the canonical strata of $(F \circ \beta)^{-1}(0)$ submersively onto P'.

Now the Nash Isotopy Lemma enables to apply proposition 4.17 below, and thus to prove that two germs $F(.,p)$ and $F(.,q)$ with $p, q \in P'$ are blow-Nash equivalent. □

So, thanks to the Nash Isotopy Lemma, we are lead to study the particular case where the family $F : (\mathbb{R}^d, 0) \times P \longrightarrow \mathbb{R}$ admits, as a resolution of singularities, a product $\beta \times id_P$, where $\beta : (M, E) \longrightarrow (\mathbb{R}^d, 0)$ is a proper birational morphism outside 0 and $F \circ (\beta \times id_P)$ is in normal crossing. More precisely, we mean that, in that case, there exist local systems of parameters $(x_1, \dots, x_d)$ centered on a point in $E \times P$ such that

$$F \circ (\beta \times id_P)(x_1, \dots, x_d, p) = u_p(x) \prod_{i=1}^{d} x_i^{r_i},$$

where u_p is a nonvanishing Nash function.

In that setting, we have the following triviality result:

PROPOSITION 4.17. *Let $F : (\mathbb{R}^d, 0) \times P \longrightarrow \mathbb{R}$ be a Nash mapping, where P is a connected Nash manifold and assume that $f_p = F(.,p) : (\mathbb{R}^d, 0) \longrightarrow \mathbb{R}$ has an isolated singularity at 0 for each $p \in P$. Assume moreover that there exists a proper birational morphism $\beta : (M, E) \longrightarrow (\mathbb{R}^d, 0)$, with E the exceptional divisor, such that the product $\beta \times id_P$ is a resolution for F. Then the family $F(.,p)$ consists of a unique blow-Nash equivalence class.*

We prove proposition 4.17 in several steps. By assumption we dispose of a trivialization (the identity map) of the zero sets of $F(.,p)$ for $p \in P$. But there is no chance that this trivialization (in fact, the identity of M) respects the levels of $F(.,p)$. Actually we are going to project the trivialisation of the zero level in order to force it to trivialize also the others levels.

Note that M, as a modification of an affine space, is affine, and so one can assume $M \subset \mathbb{R}^N$. Denote $d = \dim M$.

We show first that we can define locally the projection directly on M, without blowing-up. For simplicity, denote $F(.,p)$ by F_p.

LEMMA 4.18. *Take $x \in M$ and $p_0 \in P$. Then there exists $\eta_x > 0$ such that the orthogonal projection $\phi(x,p)$ of x onto the level $\{F_p = F_{p_0}(x)\}$ is well defined for $|p - p_0| < \eta_x$.*

Proof. If $x \in E$, then put $\phi(x,p) = x$. Now, if $x \notin E$, then $F_p(x) \neq 0$ for all p in P. But for $c \neq 0$, there exists a Nash tubular neighbourhood (cf. [BCR], corollary 8.9.5) of the level $\{F_{p_0} = c\}$. Therefore for p sufficiently closed to p_0, x belongs to the Nash tubular neighbourhood of $\{F_{p_0} = F_p(x)\}$, and then $\phi(x,p)$ is defined as the unique orthogonal projection of x onto $\{F_{p_0} = F_p(x)\}$. $\square$

REMARK 4.19. Such a construction seems not to be so easy to perform in a global situation because the width of the Nash tubular neighbourhood $\{F_{p_0} = c\}$, for a fixed function F_{p_0}, tends to 0 as c tends to 0. Therefore it does not seem to be reasonable to hope for a global stricly positive η.

Now let us perform the computation of the projection in the local case. For simplicity of notation, we assume in the following that M is an hypersurface of $\mathbb{R}^N$, that is $N = d + 1$. Take $x_0 \in E \subset M \subset \mathbb{R}^N$, and $p_0 \in P$. For simplicity assume also that $p_0 = 0$. There exists a system of parameters centered at x_0 such that:

- M is locally the graph $z = G(x_1, \dots, x_d)$ of a Nash function G,
- if we extend in the trivial way $F \circ (\beta \times id_P)$ to a function on $\mathbb{R}^N$, then $F \circ (\beta \times id_P)$ is of the form

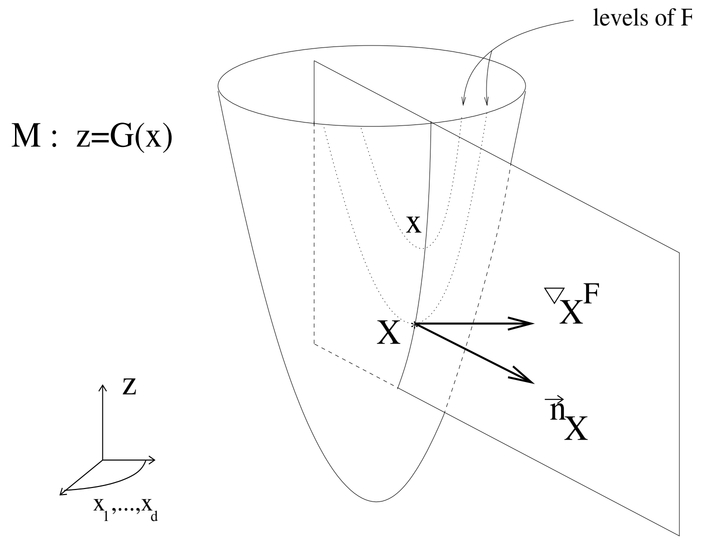
$$F \circ (\beta \times id_P)(x, z, p) = u_p(x) \prod_{i=1}^{d} f_i(x)^{r_i},$$

  where u_p is a Nash function in (x,p) which does not vanish and $f_1, \dots, f_d$ are normal crossings functions.

For $(x,p) \in M \times P$, the orthogonal projection $\phi(x,p) = X$ satisfies:

$$\begin{cases} x = X + c\nabla_X F - k\nabla_X G \\ z = Z + k \\ F(x, z, 0) = F(X, Z, p), \end{cases} \qquad (1)$$

Figure 2. Local situation.



where $\nabla_X G$ (respectively $\nabla_X F$) denotes the gradient vector of G (respectively F) at X.

Lemma 4.20. *ϕ is a blow-Nash isomorphism from a neighbourhood of $(0,0)$ in $\mathbb{R}^d \times P$ to a neighbourhood of $(0,0)$ in $\mathbb{R}^d \times P$.*

Proof. Let us prove the result in two steps. As a first step we prove that ϕ is blow-Nash, and after we show that ϕ is a blow-Nash isomorphism.

First step. In order to prove the lemma, it suffices to show that the application

$$(X,p) \qquad \xrightarrow{\psi} \qquad (X + c(X,p)\nabla_X F - k(X,p)\nabla_X G, p),$$

which is by (1) the inverse of ϕ, is a blow-Nash isomorphism between two neighbourhoods of $(0,0)$ in $\mathbb{R}^d \times P$. As c and k are given by implicit formulae, we want to apply the implicit function theorem, which makes sense in the Nash setting (cf. [BCR]). However the computation downstairs shows us that we can not apply directly this theorem, and actually we need to separate the divisors given by $f_i = 0$ by several blowings-up, for each i such that $r_i \neq 0$.

To begin with, let us explicit the members of the last equation of system (1) in coordinates. The left one is

$$F(x,z,0) = u_0(X + c\nabla_X F - k\nabla_X G)\prod_{i=1}^{d} f_i\big(X + c\nabla_X F - k\nabla_X G\big)^{r_i} \tag{2}$$

and the right one is $F(X,Z,p) = u_p(X)\prod_{i=1}^{d} f_i(X)^{r_i}$.

Remark that the Taylor Formula applied to f_i implies that

$$f_i(X+H) = f_i(X) + < \nabla_X f_i, H > + h_{i,X}(H),$$

where h_i is a Nash function such that $\|h_{i,X}(H)\| \leqslant K\|H\|^2$ on some neighbourhood of 0, for some positive constant K.

Therefore $f_i(X + c\nabla_X F - k\nabla_X G)$ is equal to

$$f_i(X) + c < \nabla_X f_i, \nabla_X F > -k < \nabla_X f_i, \nabla_X G > + h_{i,X}(c\nabla_X F - k\nabla_X G).$$

Assume now, for simplicity, that the numbers of divisors $f_i = 0$ is equal to d (that is $r_i > 0$ for all i). We separate these divisors $f_i = 0$ by a succession of d blowings-up with respect to ideal sheaves corresponding to blow-up the origin, then the total transform of 1-dimensional intersections, etc... By symmetry, it suffices to perform the computation in one chart, and therefore one can assume this modification π to be given in the chart $\mathcal{U}$ by

$$(f_1, f_2, \dots, f_d) = (Y_1, Y_1Y_2, \dots, Y_1Y_2\cdots Y_d).$$

Let us denote respectively by c', k' the modified form of c, k. Put

$$e_i = \sum_{j=i}^{d} r_j, \qquad \widetilde{c}(Y,p) = \big(\prod_{i=1}^{d} Y_i^{e_i-2}\big)c', \qquad \widetilde{k}(Y,p) = \frac{k'}{\big(\prod_{i=1}^{d} Y_i\big)\widetilde{c}(Y,p)}$$

and

$$V = \sum_{i=1}^{d} r_i\big(\prod_{j=i+1}^{d} Y_j\big)\nabla_{\pi(Y)} f_i.$$

Remark that V can not vanish since the f_i form a system of coordinates.

Then, after the modification, the term $c\nabla_X F$ is transformed into

$$u_p\widetilde{c}\big(\prod_{i=1}^{d} Y_i\big)V + \widetilde{c}\big(\prod_{i=1}^{d} Y_i\big)^2\nabla_{\pi(Y)}u_p.$$

As a consequence $f_i(X + c\nabla_X F - k\nabla_X G)$ becomes: $Y_1\cdots Y_i + \widetilde{c}\big(\prod_{i=1}^{d} Y_i\big)\Lambda_i$, where

$$\Lambda_i = u_p < \nabla_{\pi(Y)} f_i, V > + \widetilde{c}\big(\prod_{i=1}^{d} Y_i\big) < \nabla_{\pi(Y)} f_i, \nabla_{\pi(Y)} u_p > - \widetilde{k} < \nabla_{\pi(Y)} f_i, \nabla_{\pi(Y)} G >$$

$$+ \frac{h_{i,X}\Big(\widetilde{c}(\prod_{i=1}^{d} Y_i)\big(V + (\prod_{i=1}^{d} Y_i)\nabla_{\pi(Y)}u_p - \widetilde{k}\nabla_{\pi(Y)}G\big)\Big)}{\widetilde{c}\big(\prod_{i=1}^{d} Y_i\big)}.$$

Therefore equation (2) is transformed into

$$\sum_{l=1}^{r} \widetilde{c}^l \sum_{\substack{j_1+\cdots+j_d=l\\ j_i\in\{1,\dots,r_i\}}} \prod_{i=1}^{d} \binom{j_i}{r_i} \Lambda_i^{j_i} Y_i^{j_1+\cdots+j_{i-1}} = v(\widetilde{c}, \widetilde{k}, Y), \tag{3}$$

with

$$v(\widetilde{c}, \widetilde{k}, Y) = \frac{u_p\big(\pi(Y)\big)}{u_0\Big(\pi(Y) + \widetilde{c}(\prod_{i=1}^{d} Y_i)\big(V - \widetilde{k}\nabla_{\pi(Y)}G\big)\Big)} - 1.$$

So $\widetilde{c}$ and $\widetilde{k}$ are given implicitly by the following equations:

$$E_1(\widetilde{c}, \widetilde{k}, Y_1, p) = \sum_{l=1}^{r} \widetilde{c}^l \sum_{\substack{j_1+\cdots+j_d=l\\ j_i\in\{1,\dots,r_i\}}} \prod_{i=1}^{d} \binom{j_i}{r_i} \Lambda_i^{j_i} Y_i^{j_1+\cdots+j_{i-1}} - v(\widetilde{c}, \widetilde{k}, Y) = 0 \tag{4}$$

$$E_2(\widetilde{c}, \widetilde{k}, Y_1, p) = \widetilde{k} - \Big(G\big(\pi(Y) + (\prod_{i=1}^{d} Y_i)\widetilde{c}V - \widetilde{k}\nabla_{\pi(Y)}G\big) - G\circ\pi(Y)\Big) = 0, \tag{5}$$

where we consider $Y_2, \dots, Y_d$ as parameters. Now we can apply the implicit function theorem. First, remark that $E_1(0,0,0,0) = 0$ and $E_2(0,0,0,0) = 0$. Now, let us explicit the coefficient of $\widetilde{c}$ in equation (4). This coefficient is just $\sum_{i=1}^{d} r_i(\prod_{j=i+1}^{d} Y_j)\Lambda_i - a$, where a is the contribution coming from $-v(\widetilde{c},\widetilde{k},Y)$, thus the coefficient of $\widetilde{c}$ equals

$$u_p\|V\|^2+(\prod_{i=1}^d Y_i) < V,\nabla_{\pi(Y)}u_p > -\widetilde{k} < V,\nabla_{\pi(Y)}G >$$

$$+\frac{\sum_{i=1}^d r_i(\prod_{j=i+1}^d Y_j)h_{i,X}\Big(\widetilde{c}(\prod_{i=1}^d Y_i)\big(V+(\prod_{i=1}^d Y_i)\nabla_{\pi(Y)}u_p-\widetilde{k}\nabla_{\pi(Y)}G\big)\Big)}{\widetilde{c}\big(\prod_{i=1}^d Y_i\big)}-a.$$

Note that a tends to zero as (Y_1,p) tends to $(0,0)$.

Now, it is easy to compute the jacobian matrix of (E_1,E_2) with respect to the variables $(\widetilde{c},\widetilde{k})$ at the point $(0,0,0,0)$. The result is:

$$\begin{pmatrix} \frac{\partial E_1}{\partial \widetilde{c}}(0,0,0,0) & \frac{\partial E_1}{\partial \widetilde{k}}(0,0,0,0) \\ \frac{\partial E_2}{\partial \widetilde{c}}(0,0,0,0) & \frac{\partial E_2}{\partial \widetilde{k}}(0,0,0,0)\end{pmatrix}=\begin{pmatrix} u_p(0)\|V\|^2 & 0\\ 0 & 1\end{pmatrix},$$

which is an invertible matrix because u_p and V do not vanish. Therefore $\widetilde{c}$ and $\widetilde{k}$ are defined and analytic in a neighbourhood of $(0,0)$ in $M\times P$.

Now, let us come back to ψ. For a fixed p, write $\psi_p(.)$ instead of $\psi(.,p)$. Then ψ_p is defined in a neighbourhood $\mathcal{V}(\mathbb{R}^d)$ of 0 in $\mathbb{R}^d$, and in restriction to a neighbourhood of $Y_1=0$ in the chart $\mathcal{U}$, one has

$$\psi_p\circ\pi(Y)=\Big(Y_1+\widetilde{c}(\prod_{i=1}^d Y_i)W_1, Y_1Y_2+\widetilde{c}(\prod_{i=1}^d Y_i)W_2,\ldots,Y_1Y_2\cdots Y_d+\widetilde{c}(\prod_{i=1}^d Y_i)W_d\Big),$$

where W is the vector $W=V-\widetilde{k}\nabla_{\pi(Y)}G$.

Then ψ_p lifts in a function $\widetilde{\psi_p}$:

$$\begin{array}{ccc}\mathcal{V}(\mathcal{U}) & \overset{\widetilde{\psi_p}}{\dashrightarrow} & \mathcal{V}(\mathcal{U})\\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi}\\ \mathcal{V}(\mathbb{R}^d) & \xrightarrow[\psi_p]{} & \mathcal{V}(\mathbb{R}^d)\end{array}$$

between neighbourhoods of $Y_1=0$ in $\mathcal{U}$, with $\widetilde{\psi_p}$ being given in coordinates by

$$\widetilde{\psi_p}(Y)=\Big(Y_1\big(1+\widetilde{c}(\prod_{i=2}^d Y_i)W_1\big),Y_2\frac{1+\widetilde{c}(\prod_{i=3}^d Y_i)W_2}{1+\widetilde{c}(\prod_{i=2}^d Y_i)W_1},\ldots,Y_d\frac{1+\widetilde{c}W_d}{1+\widetilde{c}Y_dW_{d-1}}\Big).$$

Note that the denominators can not vanish in a neighbourhood of $(Y_1,p)=(0,0)$ because $\widetilde{c}$ is small for (Y_1,p) sufficiently small.

Second step. It suffices to prove that the jacobian determinant of $\widetilde{\psi}$ is non zero for Y_1 and p small because $\widetilde{\psi}$ is a bijection. Indeed, $\widetilde{\psi}$ is a bijection outside E because so is ψ, and restricted to E, it is just the identity.

Note that $\widetilde{c}$ and $\frac{\partial\widetilde{c}}{\partial Y_i}$, for $i\in\{2,\ldots,d\}$, vanish when $(Y_1,p)=(0,0)$. Therefore, evaluated at $(Y_1,p)=(0,0)$,

$$\frac{\partial\widetilde{\psi}^i}{\partial Y_i}=1,\ \ \frac{\partial\widetilde{\psi}^i}{\partial Y_j}=0$$

if $i\neq j$ and $j\neq 1$. So the jacobian determinant of $\widetilde{\psi}$ equals 1 at $(Y_1,p)=(0,0)$ and thus is nonzero for Y_1 and p small.

□

REMARK 4.21. The projection constructed is locally the minimal one because it is trivial if $x \in E$ or $p = 0$.

Now, we have to prove that these projections glue together.

LEMMA 4.22. *The projections of lemma 4.20 glue together.*

Proof. Cover E by a finite number of neighbourhoods of the kind of lemma 4.20 using the compactness of E. We can assume that the intersections two by two of these neighbourhoods are connected. Then, we dispose of a finite number of projections ϕ^i defined on neighbourhoods of the form $V^i \times B(p_0, \eta_i)$, with $\eta_i > 0$. Denote by η the minimal of the rays η_i and by U the union of the neighbourhoods. We are going to prove that the projections ϕ^i glue together on $U \times B(p_0, \eta)$.

Assume that there exists a point x in the intersection $V_1 \cap V_2$. Then, on a neighbourhood of (x, p_0) in $M \times P$, the three projections ϕ^1, ϕ^2 and the one of lemma 4.18 coincide by minimality. Therefore the analytic functions ϕ^1 and ϕ^2, which coincide on a nonisolated set of points, are equal on the connected set $V_1 \cap V_2$. □

The proof of proposition 4.17 follows now easily from these lemmas. Actually, it suffices to prove that two germs F_p and F_q are blow-Nash equivalent for p and q sufficiently closed, and the lemmas give a relevant blow-Nash isomorphism in that case.

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