VIRTUAL BETTI NUMBERS OF ARC-SYMMETRIC SETS AND BLOW-NASH EQUIVALENCE

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The aim of this talk is to present an application of motivic integration to the study of real analytic function germs. This application can be divided into three main parts:

(1) the virtual Poincaré polynomial of arc-symmetric sets,
(2) the zeta functions of a real analytic function germ,
(3) an application to blow-Nash equivalence of real analytic function germs.

The link between these three topics is the motivic integration, a theory due to M. Kontsevich [6], and developped essentially by J. Denef and F. Loeser (for a survey, one can see [2]). It is a powerful tool to construct invariants for singularities, based on an integration on the space of arcs of an algebraic variety.

In order to apply motivic integration in real geometry, one needs a computable mesure given by a generalized Euler characteristic. If such measures abound in complex algebraic geometry, this is not the case in real algebraic geometry where the only one known until now was the Euler characteristic with compact supports. However, recently, C. McCrory and A. Parusiński [9], and independently B. Totaro [11], have constructed virtual Betti numbers of a real algebraic variety which give such an example of a generalized Euler characteristic in the real algebraic case, called the virtual Poincaré polynomial. In this talk, we present an extension of these virtual Betti numbers, and of the virtual Poincaré polynomial, to the larger category of arc-symmetric sets.

Then, following ideas of J. Denef and F. Loeser, we construct zeta functions of a real analytic function germs, by considering the measure under our virtual Poincaré polynomial of certain subspaces of the spaces of truncated arcs. These zeta functions are invariants for the blow-Nash equivalence of real analytic function germs, which is a particular case of the blow-analytic equivalence, a notion introduced by T.-C. Kuo [7] (one can consult also [3] for a survey).

1. **Virtual Poincaré polynomial of arc-symmetric sets.** Arc-symmetric sets have been introduced by K. Kurduka [8] in 1988 in order to study "rigid components" of real algebraic varieties. The category of arc-symmetric sets contains the real algebraic varieties and, in some sense, this category has a better behaviour that the one of real algebraic varieties, maybe closer to complex algebraic varieties. We can think about arc-symmetric sets as the biggest category, noted $\mathcal{AS}$, stable under boolean operations and containing the compact real algebraic varieties and their connected components. In particular:

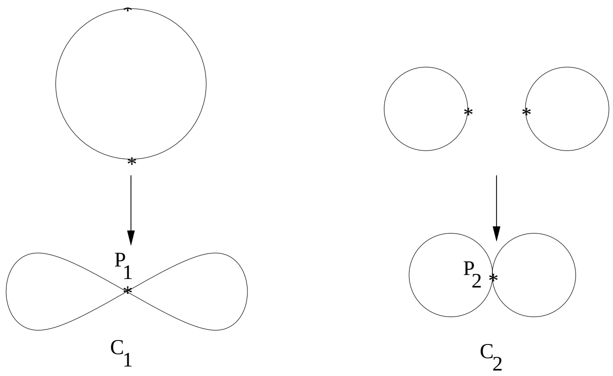
Lemma 0.1. *Compact nonsingular arc-symmetric sets are unions of connected components of compact nonsingular real algebraic varieties.*

A generalized Euler characteristic is defined as follows. Note that here, by an isomorphism between arc-symmetric sets, we mean a birational map containing the arc-symmetric sets in the support.

Definition 0.2. An additive map on $\mathcal{AS}$ with values in an abelian group is a map χ defined on $\mathcal{AS}$ such that

(1) for arc-symmetric sets A and B which are isomorphic, $\chi(A) = \chi(B)$,
(2) for a closed arc-symmetric subset B of A, $\chi(A) = \chi(B) + \chi(A \setminus B)$.

FIGURE 1. resolution of C_1 and C_2



If moreover χ takes values in a commutative ring and satisfies $\chi(A \times B) = \chi(A) \cdot \chi(B)$ for arc-symmetric sets A, B, then we say that χ is a generalized Euler characteristic on $\mathcal{AS}$.

Following an idea of F. Bittner [1], associated with the particular behaviour of homology with coefficients in $\frac{\mathbb{Z}}{2\mathbb{Z}}$, one can state the existence of a new generalized Euler characteristic for $\mathcal{AS}$:

Theorem 0.3. *There exist additive maps, noted β_i and called virtual Betti numbers, on $\mathcal{AS}$ with values in $\mathbb{Z}$ such that β_i coincides with the classical Betti numbers $\dim H_i(\cdot, \frac{\mathbb{Z}}{2\mathbb{Z}})$ on the connected component of compact nonsingular real algebraic varieties.*

Moreover $\beta(\cdot) = \sum_{i \geq 0} \beta_i(\cdot) u^i$ is a generalized Euler characteristic on $\mathcal{AS}$, with values in $\mathbb{Z}[u]$.

Example 0.4.

(1) If $\mathbb{P}^k$ denotes the real projective space of dimension k, which is nonsingular and compact, then $\beta(\mathbb{P}^k) = 1 + u + \cdots + u^k$. Now, compactify the affine line $\mathbb{A}^1_{\mathbb{R}}$ in $\mathbb{P}^1$ by adding one point at the infinity. By additivity $\beta(\mathbb{A}^1_{\mathbb{R}}) = \beta(\mathbb{P}^1) - \beta(point) = u$, and so $\beta(\mathbb{A}^k_{\mathbb{R}}) = u^k$.

(2) The real algebraic varieties of figure 1, defined by $y^2 = x^2(1 - x^2)$ for C_1 and by $\big((x+1)^2 + y^2 - 1\big)\big((x-1)^2 + y^2 - 1\big) = 0$ for C_2, are not isomorphic whereas they are clearly homeomorphic. Indeed, one can compute $\beta(C_1)$ and $\beta(C_2)$ by considering the resolutions given by blowing-up the singular point of C_1 and that of C_2. In this manner $\beta(C_1) = \beta(C_1 \setminus P_1) + \beta(P_1)$ by additivity, and $\beta(C_1 \setminus P_1) = \beta(\mathbb{P}^1 \setminus \{\text{two points}\})$ by isomorphism outside the singular locus of C_1. Therefore $\beta(C_1) = \beta(\mathbb{P}^1) - \beta(\text{point}) = u$, by additivity once more.

In the same way: $\beta(C_2) = \beta(C_2 \setminus P_2) + \beta(P_2) = 2\beta(\mathbb{P}^1 \setminus \{\text{point}\}) + \beta(P_2) = 2u + 1$.

Therefore β is not a topological invariant! Remark that the Euler characteristic with compact supports (one can recover it by evaluating u at -1 in this example) does not distinguish these two curves, as it is a topological invariant.

Remark 0.5. The virtual Poincaré polynomial β has the property that, for $A \in \mathcal{AS}$, $\dim(A) = \deg\big(\beta(A)\big)$. This property is interesting for motivic integration because it induces a measure which respects the dimension of mesurable sets.

In order to relate the analytic aspect of arc-symmetric sets and the behaviour of β, we propose the following definition of Nash-isomorphism between arc-symmetric sets. We state after the corresponding invariance theorem concerning β.

Definition 0.6. Let $A, B \in \mathcal{AS}$. Assume that there exist compact analytic varieties V_1, V_2 containing A, B respectively, and also an analytic isomorphism ϕ from V_1 to V_2

such that $\phi(A) = B$. If moreover one can choose V_1, V_2 to be semi-algebraic sets and ϕ a semi-algebraic map, then one say that A and B are Nash-isomorphic.

Theorem 0.7. *Let $A_1, A_2 \in \mathcal{AS}$ be Nash-isomorphic arc-symmetric sets. Then $\beta(A_1) = \beta(A_2)$.*

Remark 0.8. This result is one of the two key ingredients, with the change of variable formula in motivic integration, of the proof of the fact that the zeta functions of a real analytic germ of functions, that we construct below, are invariants of the blow-Nash equivalence.

2. **Zeta functions.** Denote by $\mathcal{L}$ the space of arcs at the origin $0 \in \mathbb{R}^d$:

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d, 0) = \{\gamma : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^d, 0) : \gamma \text{ formal}\},$$

and by $\mathcal{L}_n$ the space of truncated arcs:

$$\mathcal{L}_n = \mathcal{L}_n(\mathbb{R}^d, 0) = \{\gamma \in \mathcal{L} : \gamma(t) = a_1 t + a_2 t^2 + \cdots a_n t^n,\ a_i \in \mathbb{R}^d\}.$$

Consider $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ a real analytic function germ. We define the naive zeta function $Z_f(T)$ of f as the following element of $\mathbb{Z}[u, u^{-1}]((T))$:

$$Z_f(T) = \sum_{n \geq 1} \beta(\chi_n) u^{-nd} T^n,$$

where

$$\chi_n = \{\gamma \in \mathcal{L}_n : ord(f \circ \gamma) = n\} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = bt^n + \cdots, b \neq 0\}.$$

Remark that χ_n, for $n \geq 1$, is a constructible subset of $\mathbb{R}^{nd}$, hence belong to $\mathcal{AS}$. We can define in a similar way zeta functions with sign.

These zeta functions of f contain information on the singularity of f at 0. In particular, we may express the zeta functions in terms of a modification of f, that is a proper birational map which is an isomorphism over the complement of the zero locus of f. As we are dealing with analytic germs, note that we are only interested in analytic neighbourhoods of 0 in $\mathbb{R}^d$ and of the exceptionnal divisor in the modified space.

Proposition 0.9. *(Denef & Loeser formula) Let $\sigma : \left(M, \sigma^{-1}(0)\right) \longrightarrow (\mathbb{R}^d, 0)$ be a modification of $\mathbb{R}^d$ such that $f \circ \sigma$ and the jacobian determinant $\operatorname{jac} \sigma$ are normal crossings simultaneously, and assume moreover that σ is an isomorphism over the complement of the zero locus of f.*

Let $(f \circ \sigma)^{-1}(0) = \cup_{j \in J} E_j$ be the decomposition into irreducible components of $(f \circ \sigma)^{-1}(0)$, and assume that $\sigma^{-1}(0) = \cup_{k \in K} E_k$ for some $K \subset J$.

Put $N_i = \operatorname{mult}_{E_i} f \circ \sigma$ and $\nu_i = 1 + \operatorname{mult}_{E_i} \operatorname{jac} \sigma$, and for $I \subset J$ denote by E_I^0 the set $(\cap_{i \in I} E_i) \setminus (\cup_{j \in J \setminus I} E_j)$. Then

$$Z_f(T) = \sum_{I \neq \emptyset} (u-1)^{|I|} \beta\big(E_I^0 \cap \sigma^{-1}(0)\big) \prod_{i \in I} \frac{u^{-\nu_i} T^{N_i}}{1 - u^{-\nu_i} T^{N_i}}.$$

Example 0.10. Let $f_k : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ be defined by $f_k(x,y) = x^k + y^k$, $k \geq 2$. The blowing-up at the origin gives a suitable modification σ for f. Here $(f \circ \sigma)^{-1}(0)$ consists of just the exceptional divisor $\mathbb{P}^1$ in the case k even, and furthermore, in the case k odd, of the strict transform of f which is a smooth curve crossing transversally the exceptional divisor. Then

$$Z_{f_k} = \begin{cases} (u^2-1)u^{-2} \frac{T^k}{1-u^{-2}T^k} & \text{if k is even} \\ (u-1)\frac{u^{-2}T^k}{1-u^{-2}T^k}\left(u + (u-1)\frac{u^{-1}T}{1-u^{-1}T}\right) & \text{if k is odd} \end{cases}$$

Note in particular that for $k \neq k'$, the zeta functions Z_{f_k} and $Z_{f_{k'}}$ are different.

3. **Blow-Nash equivalence.** By an algebraic modification of a real analytic function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$, we mean a proper birational algebraic morphism $\sigma_f : \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$, between analytic neighbourhoods of 0 in $\mathbb{R}^d$ and the exceptionnal divisor $\sigma_f^{-1}(0)$ in M_f, which is an isomorphism over the complement of the zero locus of f and for which $f \circ \sigma$ is in normal crossing.

Definition 0.11. Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be two germs of real analytic functions. They are said to be blow-Nash equivalent if there exist two algebraic modifications

$$\sigma_f \ : \ \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0) \text{ and } \sigma_g : \big(M_g, \sigma_g^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0),$$

and a Nash-isomorphism (i.e. a semi-algebraic map which is an analytic isomorphism) Φ between analytic neighbourhoods $\big(M_f, \sigma_f^{-1}(0)\big)$ and $\big(M_g, \sigma_g^{-1}(0)\big)$ which preserves the multiplicities of the jacobian determinants of σ_f and σ_g along the components of the exceptional divisors, and which induces a homeomorphism $\phi : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}^d, 0)$ such that $f = g \circ \phi$, as illustrated by the diagram:

$$\begin{tikzcd}
\big(M_f, \sigma_f^{-1}(0)\big) \arrow[rr, "\Phi"] \arrow[d, "\sigma_f"'] & & \big(M_g, \sigma_g^{-1}(0)\big) \arrow[d, "\sigma_g"] \\
(\mathbb{R}^d, 0) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (\mathbb{R}^d, 0) \arrow[dl, "g"] \\
& (\mathbb{R}, 0) &
\end{tikzcd}$$

which is commutative.

We do not know whether the blow-Nash equivalence is an equivalence relation or not. But this is not so important since the equivalence relation we truly want to study is the blow-analytic equivalence. Actually, we can see the blow-Nash equivalence as an algebraic approximation of the blow-analytic equivalence.

Note that the examples currently studied under blow-analytic equivalence (the Whitney family, the Briançon-Speder family, Brieskorn polynomials,...) are usually algebraic, and so it is relevant to study them under blow-Nash equivalence.

Note that two real analytic function germs that are analytically equivalent are, of course, blow-analytically equivalent! Note also that two Nash germs that are analytically equivalent are also Nash equivalent, as proven by M. Shiota [10], and therefore blow-Nash equivalent.

Now, the question of moduli is a natural question when one studies an equivalence relation of germs. The following theorem gives a result in this direction. It states that, for an algebraic family of algebraic functions with an isolated singularity, blow-analytic equivalence implies blow-Nash equivalence. Actually we prove more, namely there is a finite number of blow-Nash equivalence classes, and they coincide with the blow-analytic ones.

Theorem 0.12. *Let $F : (\mathbb{R}^d, 0) \times \mathbb{R} \longrightarrow (\mathbb{R}, 0)$ be algebraic and assume that $F(., p) : (\mathbb{R}^d, 0) \longrightarrow \mathbb{R}$ has an isolated singularity at 0 for each $p \in \mathbb{R}$.*

Then the family $F(., p)$, for $p \in \mathbb{R}$, consists of a finite number of blow-Nash equivalence classes. Moreover each of these classes is blow-analytically trivial.

The proof of this result requires different arguments that in the analytic setting, because the integration along vector fields, used in that case (cf. [7] for example), is not a Nash argument. In particular we use the Nash Isotopy Lemma of T. Fukui, S. Koike, M. Shiota [4].

A difficult issue with this kind of relations is to find invariants, and actually for the blow-analytic equivalence, only the Fukui invariants and the zeta functions of S. Koike and A. Parusiński [5], defined in the same way with the Euler characteristic with compact supports, are known.

Note that our zeta functions generalize the Fukui invariants $A(f)$, which asign to $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ the possible orders of series $f \circ \gamma$ where γ is a real analytic arc at the origin. This is due to the property $\dim(A) = \deg\big(\beta(A)\big)$ that we have already noticed.

Finally, we can state the invariance of the zeta functions.

Theorem 0.13. *The zeta functions $Z_f(T), Z_f^{\pm}(T)$ of a germ of analytic functions f are invariants of the blow-Nash equivalence.*

Example 0.14. Two variables Brieskorn polynomials are polynomials of the type $\varepsilon_p x^p + \varepsilon_q y^q$, $p \leq q \in \mathbb{N}, \varepsilon_p, \varepsilon_q \in \{\pm 1\}$.

Their classification under blow-analytic equivalence have been dealt with by S. Koike and A. Parusiński [5], using their zeta functions and the Fukui invariants. Actually the only case where the equivalence class of Brieskorn polynomials of two variables can not be distinguished using only their zeta functions, and which requires the use of the Fukui invariants, is the following: $f_k(x,y) = \pm(x^k + y^k)$, $k \geq 2$ even. Remark that we have seen in example 0.10 that for $k \neq k'$ the naive zeta functions Z_{f_k} and $Z_{f'_k}$ are different, therefore our zeta function distinguishes this case, for blow-Nash equivalence.

Moreover we can prove, using only our zeta functions, that the blow-Nash types of the two variables Brieskorn polynomials coincide with their blow-analytic types.

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