VIRTUAL POINCARÉ POLYNOMIAL

$\longrightarrow$ for X real algebraic variety, $\beta(X) \in \mathbb{Z}[u]$.

$\longrightarrow$ generalization of χ^{BM}:

- additive: $Y \subseteq X \Rightarrow \beta(X) = \beta(Y) + \beta(X \setminus Y)$
- multiplicative: $\beta(X \times Y) = \beta(X)\beta(Y)$
- if $u = -1$, $\beta(X)(-1) = \chi^{BM}(X)$.

$\longrightarrow$ invariant under isomorphisms

I WEAK FACTORIZATION THEOREM

- An example of birational morphism: the blowing-up

$$\sigma : Bl_C X \longrightarrow X \quad \text{morphism } (= \text{regular})$$

σ^{-1} rational, called "blowing-down"

- One can compose blowings-up, like in resolution of singularities

$$X_m \xrightarrow{\sigma_m} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X$$

Then $\sigma_1 \circ \cdots \circ \sigma_m$ is still a birational morphism.

- One can compose blowings-up and their inverse:

$$\begin{tikzcd}
 & & & X_4 \arrow[dl, "\sigma_4"'] \arrow[dr, "\sigma_4"] & & X_2 \arrow[dl, "\sigma_3"'] \arrow[dr, "\sigma_2"] & & \\
X_7 \arrow[dr, "\sigma_7"] & & X_5 \arrow[dl, "\sigma_6"'] & & X_3 & & X_1 \arrow[dr, "\sigma_1"] & \\
 & X_6 & & & & & & X
\end{tikzcd}$$

Then $\sigma_7 \circ \sigma_6^{-1} \circ \sigma_5^{-1} \circ \sigma_4 \circ \sigma_3^{-1} \circ \sigma_2 \circ \sigma_1 : X_7 \dashrightarrow X$

is no longer a morphism, but still a birational map

- The WFT says they are all like that!

<u>THEOREM</u> (Wlodarczyk, 2003)

Let X and Y be algebraic varieties over k with char $k = 0$.

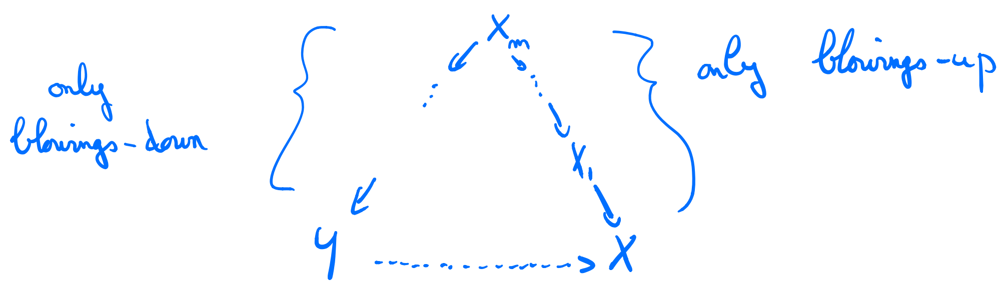
Let $\varphi : Y \dashrightarrow X$ be a birational map, and $U \subseteq X$, $V \subseteq Y$ Zariski dense subsets such that $\varphi_{|V} : V \to U$ is an isomorphism.

Then φ can be factored as

$$Y = X_n \overset{\sigma_n}{\dashrightarrow} X_{n-1} \dashrightarrow \cdots \overset{\sigma_1}{\dashrightarrow} X_0 = X$$

where each σ_i is a blowing-up, or blowing-down, along a non singular center.

<u>Remark</u>: the strong factorization conjecture is still open:



<u>History</u>
- Zariski: for smooth surfaces, strong factorization with blowings-up along points.
- Hironaka stated the strong factorization conjecture, 1960
- Miyake and Oda stated the WFC, 1978

II GROTHENDIECK RING OF VARIETIES

<u>Definition</u> • The grothendieck group of varieties is the free abelian group generated by isomorphism classes of real algebraic varieties, subject to the relation

$$[X] = [Y] + [X \setminus Y] \quad \text{if } Y \subseteq X \text{ is a closed subvariety}$$

• The grothendieck ring is defined adding the relation

$$[X \times Y] = [X] \cdot [Y]$$

<u>Notation</u> $K^0(\mathbb{R}Var)$

<u>Remarks</u> ① Similar definition for algebraic varieties over any field k $\rightsquigarrow$ $K^0(Var_k)$

② Similar definition of semi-algebraic sets $\rightsquigarrow K^0(SA)$

③ K^0 plays the role of an universal additive and multiplicative invariant: if e is an invariant in a ring A, additive and multiplicative, the e induces a ring morphism

$$e : K^0 \longrightarrow A$$

<u>Examples</u> ① $\chi^{BM} : K^0(\mathbb{R}Var) \longrightarrow \mathbb{Z}$

or $\chi^{BM} : K^0(SA) \longrightarrow \mathbb{Z}$

② If k is finite

$$\begin{aligned} K^0(Var_k) &\longrightarrow \mathbb{Z} \\ [X] &\longmapsto \# X(k) \end{aligned}$$

THEOREM (Quarez, 2001)

$\chi^{BM}: K^0(SA) \xrightarrow{\sim} \mathbb{Z}$ is an isomorphism

<u>Proof</u> Let $A \in SA$. Then $A = \coprod_k C_k$ cellular decomposition.

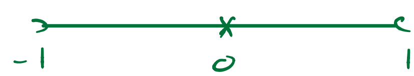
- So $[A] = \sum_k [C_k]$ by additivity

$= \sum_k [(-1,1)^{d_k}]$ by invariance

$= \sum_k [(-1,1)]^{d_k}$ by multiplicativity.

- So $[A]$ is completely determined by $[(-1,1)]$.

- Moreover



$$[(-1,1)] = [(-1,0)] + 1 + [(0,1)]$$
$$= 2[(-1,1)] + 1$$

so

$$[(-1,1)] = -1$$

- As a consequence $[\cdot] = \chi^{BM}$

□

<u>Remark</u> In general, $K^0(\mathrm{Var}_k)$ is difficult to understand.

Some facts:

- $K^0(\mathrm{Var}_k)$ is not a domain (Poonen, 2002)
- the class of the affine line is a zero divisor. (Borisov, 2015)

$$[\mathbb{A}^1]^n([X]-[Y]) = 0$$

III VIRTUAL POINCARÉ POLYNOMIAL

The definition shares some similarities with χ.

Definition Let $X \in \mathbb{R}\mathrm{Var}$ be nonsingular and compact. The Poincaré polynomial of X is

$$b(X) = \sum_{i \geq 0} \underbrace{\dim H_i(X, \mathbb{F}_2)}_{i\text{-th Betti number}} u^i \in \mathbb{N}[u]$$

For instance:

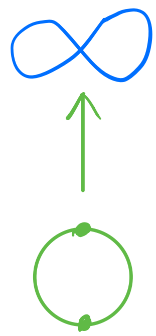
- $b(S^1) = 1 + u = b(\mathbb{P}^1)$
- $b(S^n) = 1 + u^n \neq 1 + u + u^2 + \dots + u^n = b(\mathbb{P}^n)$

To define β, we extend b to any variety using additivity.

For instance: (1) The line $\mathbb{R}$:

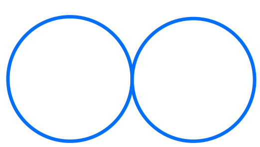
$$\mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{P}^1) - \mathcal{B}(\infty) = b(\mathbb{P}^1) - b(\infty)$$
$$= 1 + u - 1$$
$$= u.$$

Independant of the compactification ?

(2) $X =$  $y^2 = x^2(x^2-1)$

$$\mathcal{B}(X) = \mathcal{B}(X \setminus 0) + \mathcal{B}(0) = \mathcal{B}(S^1 \setminus \{2 \text{ points}\}) + \mathcal{B}(0)$$
$$= \mathcal{B}(S^1) - \mathcal{B}(\text{point})$$
$$= b(S^1) - b(\text{point})$$
$$= u$$

Independent of the resolution ?

(3) $Y =$  union of two circles.

$$\mathcal{B}(Y) = 2\,\mathcal{B}(S^1) - \mathcal{B}(\text{point}) = 2u - 1$$

Note that $\mathcal{B}(Y) \neq \mathcal{B}(X)$

whereas $\chi(Y) = \chi(X)$

$\rightsquigarrow \mathcal{B}$ is not a topological invariant.

THEOREM (McCrory - Parusiński, 2003)

There exists a unique extension

$$\beta : K^0(\mathbb{R}\,\mathrm{Var}) \longrightarrow \mathbb{Z}[u]$$

of the Poincaré polynomial.

Remark: If we evaluate $u=-1$, then we recover χ^{BM}.

Indeed

- $\beta(u=-1)$ is additive, multiplicative
- if X is compact nonsingular $\beta(X)(-1) = \chi^{BM}(X)$

by definition

Corollary Let $X \in \mathbb{R}\,\mathrm{Var}$. Then

$$\boxed{\deg \beta(X) = \dim X}$$

Remark: much better than χ^{BM} !

Proof • if X is compact nonsingular, then

$H_d(X, \mathbb{F}_2) \neq 0$ where $d = \dim X$

so the result is true.

- For X general : compactify $X \hookrightarrow \bar{X}$ via Alexandroff compactification (for instance). So

$$\beta(X) = \beta(\bar{X}) - 1$$

$\rightsquigarrow$ it suffices to deal with X compact.

- For X compact, use resolution of singularities:

$$\begin{array}{ccc} \pi : \tilde{X} & \longrightarrow & X \\ \cup\!\mathsf{I} & & \cup\!\mathsf{I} \\ E & & D \end{array} \quad \text{with} \quad \tilde{X} \setminus E \xrightarrow{\sim} X \setminus D$$

Then

$$\begin{aligned} \beta(X) &= \beta(X \setminus D) + \beta(D) \\ &= \beta(\tilde{X} \setminus E) + \beta(D) \\ &= \underbrace{\beta(\tilde{X})}_{\text{compact non singular}} - \underbrace{\beta(E) + \beta(D)}_{\substack{\dim E < d \\ \dim D < d}} \end{aligned}$$

$\rightsquigarrow$ the result holds by induction on dimension

$\square$

<u>Remark</u>: the proof implies that the leading coefficient of $\beta(X)$ is equal to the number of connected components of a desingularization of a compactification of X:

$$\begin{array}{ccc} X & \longrightarrow & \bar{X} \\ & & \uparrow \\ & & \tilde{X} \end{array}$$

- Towards the proof of the theorem: the proof uses
  - * the weak factorization theorem
  - * a topological result via Poincaré duality.

<u>Proposition</u> Let $X \in \mathbb{R}$ Var compact nonsingular and $C \subseteq X$ compact nonsingular. Consider the blowing-up

$$\begin{array}{ccc} Bl_C X = \tilde{X} & \xrightarrow{\pi} & X \\ \uparrow & & \uparrow \\ E & \longrightarrow & C \end{array}$$

Then $$b(\tilde{X}) - b(E) = b(X) - b(C)$$

<u>Proof</u> Consider the long exact sequence in relative homology:

$$\begin{array}{ccccccccc} \cdots \to & H_{i-1}(\tilde{X}, E) & \to & H_i(E) & \to & H_i(\tilde{X}) & \to & H_i(\tilde{X}, E) & \to \cdots \\ & \bullet \downarrow \pi_* & & \downarrow \pi_* & & \bullet \downarrow \pi_* & & \bullet \downarrow \pi_* & \\ \cdots \to & H_{i-1}(X, C) & \to & H_i(C) & \to & H_i(X) & \to & H_i(X, C) & \to \cdots \end{array}$$

- isomorphisms because $\tilde{X} \setminus E \simeq X \setminus C$
- surjective by Poincaré duality over $\mathbb{F}_2$

Idea:

$$\begin{tikzcd}
H^{d-i}(X) \arrow[r, "\sim", "\text{duality}"'] \arrow[d, "\pi^*"'] \arrow[dr, phantom, ""] & H_i(X) \\
H^{d-i}(\tilde{X}) \arrow[r, "\sim"', "\text{duality}"] & H_i(\tilde{X}) \arrow[u, "\pi_*"']
\end{tikzcd}$$

Hence

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i-1}(\tilde{X}, E) \arrow[r] \arrow[d, "\wr"', "\pi_*"] & H_i(E) \arrow[r] \arrow[d, "\pi_*"] & H_i(\tilde{X}) \arrow[r] \arrow[d, "\pi_*"] & H_i(\tilde{X}, E) \arrow[r] \arrow[d, "\wr"', "\pi_*"] & \cdots \\
\cdots \arrow[r] & H_{i-1}(X, C) \arrow[r] & H_i(C) \arrow[r] & H_i(X) \arrow[r] & H_i(X, C) \arrow[r] & \cdots
\end{tikzcd}$$

induces an exact sequence

$$0 \longrightarrow H_i(E) \longrightarrow H_i(C) \oplus H_i(\tilde{X}) \longrightarrow H_i(X) \longrightarrow 0$$

by diagram chasing. As a consequence:

$$\dim H_i(E) + \dim H_i(X) = \dim H_i(C) + \dim H_i(\tilde{X})$$

so that

$$b(E) + b(X) = b(C) + b(\tilde{X})$$

$\square$

Now we can sketch the proof of the theorem.

The general idea is:

* define $\mathcal{B}$ by induction on dimension
* if X non singular, compactify $X \hookrightarrow \tilde{X}$

in a non singular $\tilde{X}$ (via resolution of singularities) and set

$$\mathcal{B}(X) = \mathcal{B}(\tilde{X}) - \mathcal{B}(\tilde{X} \setminus X)$$

* prove independence on the choice of the resolution using weak factorization theorem.

* general X: $X = \mathrm{Reg}\, X \sqcup \mathrm{Sing}\, X$, use the preceding by additivity.

(Sketch of) <u>Proof</u>

- Let $X \in \mathbb{R}\mathrm{Var}$ be nonsingular.
- Compactify $X \hookrightarrow \overline{X}$
- A priori $\overline{X}$ is singular inside $\overline{X} \setminus X$. Resolve the singularities:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \overline{X} \\ & \tilde{X} \arrow[u] \end{tikzcd}$$

so that we can assume $X \hookrightarrow \tilde{X}$ compact nonsingular

- Take another nonsingular compactification $X \hookrightarrow \tilde{\tilde{X}}$.

We want to prove $\mathcal{B}(\tilde{X}) - \mathcal{B}(\tilde{X} \setminus X) = \mathcal{B}(\tilde{\tilde{X}}) - \mathcal{B}(\tilde{\tilde{X}} \setminus X)$.

Then

$$\begin{tikzcd} & \tilde{X} \arrow[dd, dotted, "\wr\ \varphi \text{ birational}"] \\ X \arrow[ur, hook] \arrow[dr, hook] & \\ & \tilde{\tilde{X}} \end{tikzcd}$$

• φ is a composition of blowings-up and blowings-down by WFT.

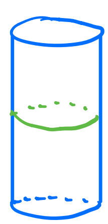
It suffices to prove the result in the case of one blowing-up:

$$\begin{tikzcd} & \tilde{X} \arrow[d] & E \arrow[l, hook'] \arrow[d] \\ X \arrow[ur, hook] \arrow[dr, hook] & \downarrow & \\ & \tilde{\tilde{X}} & C \arrow[l, hook'] \end{tikzcd}$$

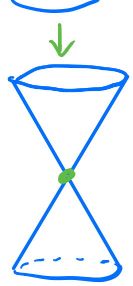
It is true by the lemma.

□

Some examples of computation

• 

$\mathcal{B}(\text{cylinder}) = \mathcal{B}(\mathbb{R} \times S^1) = u(u+1)$

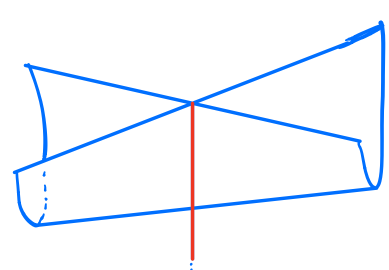
• 

$\mathcal{B}(\text{cone}) - 1 = \mathcal{B}(\text{cylinder}) - \mathcal{B}(S^1)$

so $\mathcal{B}(\text{cône}) = u^2$

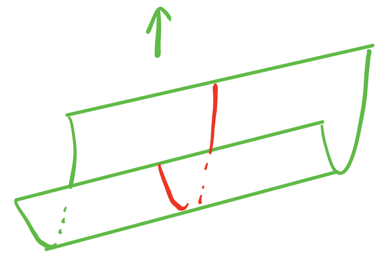
• Whitney umbrella

W



$\beta(W) - \beta(\mathbb{R}) = \beta(\mathbb{R} \times P) - \beta(P)$

where P is a parabola



However $\beta(P) = u$ (compactify in an ellipse)

so $\beta(W) = u^2$.

More generally, for quadratic polynomials:

PROPOSITION $(p,q) \neq (0,0)$

Let $Q(x,y) = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 \in \mathbb{R}[x_1, \dots, x_p, y_1, \dots, y_q]$

Then $\beta(Q=0) = u^{p+q-1} - u^{\max(p,q)-1} + u^{\min(p,q)}$

and

$$\beta(Q=1) = \begin{cases} u^{q-1}(u^p - 1) & \text{if } p \leq q \\ u^q(u^{p-1}+1) & \text{if } p > q \end{cases}$$

<u>Proof</u> Assume $p \leq q$. Then

$$Q(x,y) = \sum_{i=1}^{p} \underbrace{(x_i - y_i)}_{u_i}\underbrace{(x_i + y_i)}_{v_i} - \sum_{j=p+1}^{q} y_j^2$$

• <u>If $u_1 \neq 0$</u>: we have a linegular isomorphism

$$\mathbb{R}^* \times \mathbb{R}^{p+q-2} \longrightarrow \{Q=0\} \cap \{u_1 \neq 0\}$$

$$(u_1, u_2, \dots, u_p, v_2, \dots, v_p, y_{p+1}, \dots, y_q) \longmapsto (u, v, y) \text{ with } v_1 = -\frac{1}{u_1}\left(\sum_{i=2}^{p} u_i v_i - \sum_{p+1}^{q} y_j^2\right)$$

so that $\mathcal{B}\left(\{Q=0\}\cap u_1\neq 0\right) = (u-1)\, u^{p+q-2}$

- $\underline{\text{If } u_1=0}$:
  - no condition on v_1
  - either $u_2\neq 0$ or $u_2=0$

as we continue the process until $u_1=\dots=u_p=0$, the remaining equation being

$$-\sum_{j=p+1}^{q} y_j^2 = 0$$

with no condition on $v_1, \dots, v_p$:

$$\mathcal{B}\left(\{Q=0\}\cap\{u_1=\dots=u_p=0\}\right) = u^p$$

- Finally

$$\mathcal{B}\{Q=0\} = \sum_{i=1}^{p} (u-1)\, u^{p+q-1-i} + u^p$$

$$= u^{p+q-1} - u^{q-1} + u^p.$$

- The case of $\{Q=1\}$ is similar:

$$\mathcal{B}(\{Q=1\}) = \sum_{i=1}^{p} (u-1)\, u^{p+q-1-i} + u^p\mathcal{B}\Big(\overbrace{-\sum_{j=p+1}^{q} y_j^2 = 1}^{\emptyset}\Big)$$

$$= u^{p+q-1} - u^{q-1}$$

$$= u^{q-1}(u^p-1)$$

- Finally if $p>q$

$$\mathcal{B}(\{Q=-1\}) = \sum_{i=1}^{q} (u-1)\, u^{p+q-1-i} + u^q\mathcal{B}\Big(\overbrace{+\sum_{j=q+1}^{p} y_j^2 = 1}^{S^{p-q-1}}\Big)$$

$$= u^{p+q-1} \cancel{- u^{p-1}} + u^q \cancel{+ u^{p-1}}$$

$$= u^q (u^{p-1} + 1)$$

$\square$

<u>Remark</u> Let $Q \in \mathbb{R}[x,y]$ be a quadratic polynomial.

Then it is clear that the value of $\beta(Q=1)$ completely determines p and q.

Similarly, one can show the value of $\beta(Q=1)$ completely determines p and q.

$\rightsquigarrow$ the virtual Poincaré polynomial is a complete invariant for quadratic varieties.

IV FURTHER RESULTS

- X real algebraic variety, $a \in X$. Then $Lk(X,a)$ is well-defined up to a semi-algebraic homeomorphism; in particular $\chi^{BM}(Lk(X,a))$ makes sense.

THEOREM

$\beta(Lk(X,a))$ is well-defined

Idea of proof:

- resolve the singularities of X;
- compute $\mathcal{B}(Lk(X,a))$ after the resolution.
- in a normal crossing situation, one can prove that the link is well-defined under Nash diffeomorphisms.
- actually $\mathcal{B}$ is invariant under Nash diffeomorphisms.

$\square$

- What is the good notion of invariance for $\mathcal{B}$?

THEOREM

. X and Y real algebraic varieties.

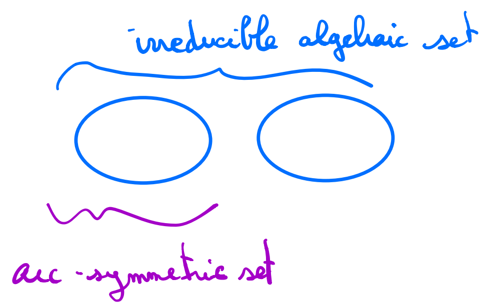
. $f : X \to Y$ regular morphism.

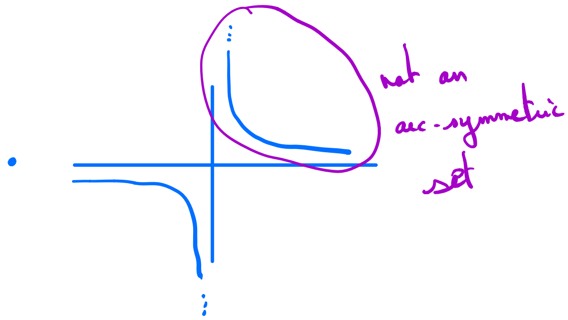
If f is a homeomorphism with arc-symmetric graph, then $\mathcal{B}(X) = \mathcal{B}(Y)$

- Arc-symmetric sets (Kurdyka, 1988)

  $\rightsquigarrow$ look like connected components of real algebraic varieties

- $y^2 + (x-2)(x-1)(x+1)(x+2) = 0$





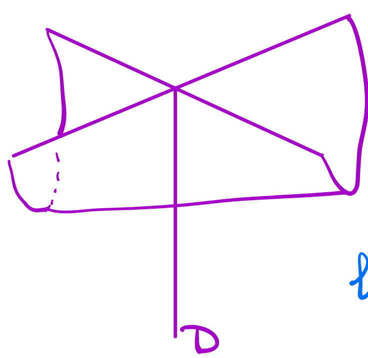
DEFINITION

- $X \subseteq \mathbb{R}^n \hookrightarrow \mathbb{P}^n(\mathbb{R})$ semi-algebraic
- X is arc-symmetric if: for any analytic arc $\gamma : (-1,1) \to \mathbb{P}^n$, if $\gamma(-1,0) \subseteq X$, then $\gamma(0,1) \subseteq X$.
- we say $X \in AS$ if X is in the boolean category generated by arc-symmetric sets.

<u>Remark</u> • Another way to say $X \in AS$: for any analytic arc $\gamma : (-1,1) \to \mathbb{P}^n$, if $\gamma(-1,0) \subseteq X$, there exists $\varepsilon > 0$ such that $\gamma(0,\varepsilon) \subseteq X$

• Again another: $\exists\, s_1, \dots, s_n \in (0,1)$ such that $\gamma\big((0,1) \setminus \{s_1, \dots, s_n\}\big) \subseteq X$.

<u>Example</u>



Whitney umbrella X

Then $X \setminus D \in AS$, but not $\overline{\mathrm{Reg}\, X}^{\text{eucl.}}$.

Indeed, let $\gamma : (-1,1) \longrightarrow \mathbb{P}^3(\mathbb{R})$ analytic such that $\gamma(-1,0) \subseteq X \setminus D$. By analytic continuation, $\gamma(0,1)$ meets D in a finite number of points: otherwise $\operatorname{Im} \gamma \subseteq D$.

<u>Remarks</u> • similarly to semi-algebraic sets & algebraic ones, one can define:
* arc-symmetric closure $\overline{A}^{AS}$ of a semi-algebraic set $A \in SA$
* decomposition in irreducible components
* AS-maps (via the graph).
* β for AS-category.

THEOREM

• $\beta : K^0(AS) \xrightarrow{\sim} \mathbb{Z}[u]$ is an isomorphism.

<u>Remark</u> Larsen-Lunts question in algebraic geometry:

If $[X] = [Y]$ in $K_0(\mathrm{Var}_k)$, what can we say about X and Y?

If X and Y are piecewise isomorphic, then $[X] = [Y]$.

$\rightsquigarrow$ what about the converse?

→ Yes for curves (Liu, Sebag)
→ Yes for some surfaces (Liu, Sebag,)
→ No in general (Borisov, 2015)

THEOREM Let X and Y be arc-symmetric sets.
If $\beta(X) = \beta(Y)$, then there exist stratifications $X = \coprod_{i\in I} X_i$ and
$Y = \coprod_{i \in I} Y_i$ such that $X_i \xrightarrow{\sim} Y_i$ in AS-category.

<u>Idea of proof</u>

- By induction on dimension.

- For simplicity assume X and Y are nonsingular algebraic varieties. Choose non-singular compactifications:

$$X \hookrightarrow \bar{X} \quad , \quad Y \hookrightarrow \bar{Y}$$

- $\beta(X) = \beta(Y) \Rightarrow \bar{X}$ and $\bar{Y}$ have the same number of connected components

- Consider $C \subseteq \bar{X}$ and $D \subseteq \bar{Y}$ connected components of the same dimension.

→ By results of Mikhalkin and Nash, there is a "strong factorization theorem" in the Nash category

$\shortparallel$

analytic + semi-algebraic $\Longrightarrow$ arc-symmetric

In our setting:

$$\begin{tikzcd} \tilde{C} \arrow[r, "\sim"] \arrow[d, "\text{composition of blowings up}"'] & \tilde{D} \arrow[d, "\text{composition of blowings up}"] \\ C & D \end{tikzcd}$$

- As a consequence: $\exists\, C_0 \subseteq C$ $\quad \dim C_0 < \dim C$

  $\exists\, D_0 \subseteq D$ $\quad \dim D_0 < \dim D$

with $C_0, D_0 \in AS$ and

$$C \setminus C_0 \underset{\substack{\uparrow \\ \text{arc-symmetric.}}}{\simeq} D \setminus D_0$$

- Conclusion via the induction.

$\square$

Corollary (Gromov question in real geometry)

$A, B, C \in AS$ with

$$\begin{tikzcd} A \arrow[dr, hook] & \\ & C \\ B \arrow[ur, hook] & \end{tikzcd}$$

Assume $C \setminus A \simeq C \setminus B$. Then A and B are piecewise AS-homeomorphic.