CONSTRUCTIBLE FUNCTIONS

I RING OF CONSTRUCTIBLE FUNCTIONS

Let S be a semialgebraic set.

Definition *A* constructible function *on S is a function $\varphi : S \to \mathbb{Z}$ which takes finitely many values and such that, for every $n \in \mathbb{Z}$, $\varphi^{-1}(n)$ is a semialgebraic subset of S.*

In other words, a constructible function φ is a function that can be written as a finite sum

$$\varphi = \sum_{i \in I} m_i \mathbf{1}_{X_i} \ , \qquad (*)$$

where, for each $i \in I$, m_i is an integer and $\mathbf{1}_{X_i}$ is the characteristic function of a semialgebraic subset X_i of S.

Remarks ① $\bigcup_{m \in \mathbb{Z}} \varphi^{-1}(m)$ is a finite partition of S

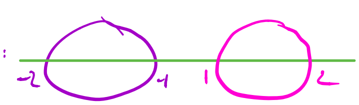
② One can construct a triangulation $|K| \xrightarrow{\sim} S$ adapted to φ, in the sense:

$\forall \sigma \in K$, $\varphi_{|\mathring{\sigma}}$ is constant

The sum and the product of two constructible functions on S are again constructible. The constructible functions on S form a commutative ring that we shall denote by $F(S)$. If $\varphi : S \to \mathbb{Z}$ is a constructible function, we define its *support* to be $\{x \in S \ ; \ \varphi(x) \neq 0\}$.

Examples ① $\mathbb{1}_{[0,1]}(x) \times \mathbb{1}_{[0,1]}(y) = \mathbb{1}_{[0,1]^2}(x,y)$

② $C : \ y^2 + (x^2-1)(x^2-4) = 0$ $\qquad C$:



$\mathbb{1}_C \cdot \mathbb{1}_{x>0} = \mathbb{1}_{\bigcirc}$

Definition *Let φ be a constructible function on S. The* Euler integral *of φ over a semialgebraic subset X of S is*

$$\int_X \varphi \, d\chi = \sum_{n\in\mathbb{Z}} n\chi(\varphi^{-1}(n) \cap X) \ .$$

$\varphi = \sum_i m_i \mathbb{1}_{X_i}$

If we have a representation φ as in $(*)$, then by additivity of χ we obtain

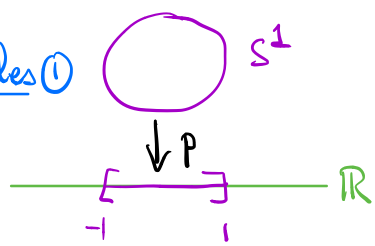
$$\int_S \varphi \, d\chi = \sum_{i\in I} m_i \chi(X_i) \ .$$

We are going to define push-forward and pull-back of constructible functions.

Definition *Let $f : S \to T$ be a continuous semialgebraic map and φ a constructible function on S. The* pushforward $f_*\varphi$ *of φ along f is the function from T to $\mathbb{Z}$ defined by*

$$y \longmapsto f_*\varphi(y) = \int_{f^{-1}(y)} \varphi \, d\chi \ .$$

<u>Examples</u> ①



$p_*(\mathbb{1}_{S^1}) = \mathbb{1}_{\{-1\}} + 2 \cdot \mathbb{1}_{(-1,1)} + \mathbb{1}_{\{1\}}$

② For $P \in \mathbb{R}[X_1, \dots, X_n]$

$$\begin{array}{c} B = \{ y^2 P(x) = 1 \} \subseteq \mathbb{R}^{n+1} \\ \Big\downarrow p \\ A = \{ P > 0 \} \subseteq \mathbb{R}^n \end{array}$$

$p_*(\mathbb{1}_B) = 2 \cdot \mathbb{1}_A$

Let us prove that the push-forward preserves constructible functions.

Let $f : S \longrightarrow T$ be a s.a. continuous map.

Proposition *The pushforward of a constructible function is constructible.*

Proof. Assume $\varphi = \sum_{i \in I} m_i \mathbf{1}_{X_i}$. By Hardt's theorem there is a finite semialgebraic partition $T = \bigcup_{j \in J} Y_j$ such that, over every Y_j, there is a trivialization of f compatible with all X_i. Then $f_* \varphi$ is constant on each Y_j.

- Indeed by linearity it suffices to deal with $\varphi = \mathbb{1}_X$, $X \subseteq S$.
- For any j, pick $y_j \in Y_j$. Then by the trivialization:

$$\begin{tikzcd}
f^{-1}(Y_j) \cap X \arrow[r, "\sim"] \arrow[dr, "f"'] & (f^{-1}(y_j) \cap X) \times Y_j \arrow[d, "\text{proj.}"] \\
& Y_j
\end{tikzcd}$$

$f^{-1}(y) \cap X$ $\qquad (f^{-1}(y_j) \cap X) \times \{y\}$ $\qquad y \in Y_j$

- Therefore

$$f_* \varphi(y) = \int_{f^{-1}(y)} \mathbb{1}_X \, d\chi = \chi(f^{-1}(y) \cap X)$$

is constant on each Y_j. $\square$

Remarks

(1) A continuous semialgebraic map $f : S \to T$ induces a morphism of additive groups $f_* : F(S) \to F(T)$.

(2) One defines more easily a ring morphism

$$f^* : F(T) \longrightarrow F(S), \quad \varphi \longmapsto \varphi \circ f$$

- finite image because φ constructible.
- $(\varphi \circ f)^{-1}(n) = f^{-1}(\varphi^{-1}(n))$ is semialgebraic.

One justification to use the name "integral" is :

Theorem **(Fubini's theorem)** *Let $f : S \to T$ be a semialgebraic map and φ a constructible function on S. Then*

$$\int_T f_*\varphi \, d\chi = \int_S \varphi \, d\chi \, .$$

$\varphi = \sum_j' m_i \mathbb{1}_{X_i}$

Proof. We keep the notation of the proof of the preceding proposition. Choose $y_j \in Y_j$ for each $j \in J$. Then, for every $i \in I$, $f^{-1}(Y_j) \cap X_i$ is semialgebraically homeomorphic to $Y_j \times (f^{-1}(y_j) \cap X_i)$. Hence,

since $f_*\varphi = \sum_i' m_i f_* \mathbb{1}_{X_i}$

triviality

additivity

$$\begin{aligned}
\int_T f_*\varphi \, d\chi &= \sum_{j\in J} \chi(Y_j)\, f_*\varphi(y_j) \\
&= \sum_{j\in J} \left[\chi(Y_j) \sum_{i \in I} \left(m_i\, \overbrace{\chi(f^{-1}(y_j) \cap X_i)}^{f_*(\mathbb{1}_{X_i})(y_j)}\right) \right] \\
&= \sum_{i\in I} \left(m_i \sum_{j\in J} \chi(f^{-1}(Y_j) \cap X_i) \right) \\
&= \sum_{i\in I} (m_i\, \chi(X_i)) = \int_S \varphi \, d\chi \, .
\end{aligned}$$

$\square$

Corollary *Let $f : S \to T$ and $g : T \to U$ be semialgebraic maps. Then $(g \circ f)_* = g_* \circ f_*$.*

Proof. Just apply Fubini to $\int_{g^{-1}(z)} f_*(\varphi)\, d\chi$. Check as exercise! $\square$

II THE OPERATOR LINK.

To avoid technicities, from now we suppose S to be a closed semialgebraic set (even algebraic set if we want: it is the ultimate goal).

Definition *Let φ be a constructible function on the semialgebraic set S. The* link *of φ is the function $\Lambda\varphi : S \to \mathbb{Z}$ defined by*

$$\Lambda\varphi(x) = \int_{\mathrm{lk}(x,S)} \varphi \, d\chi \, .$$

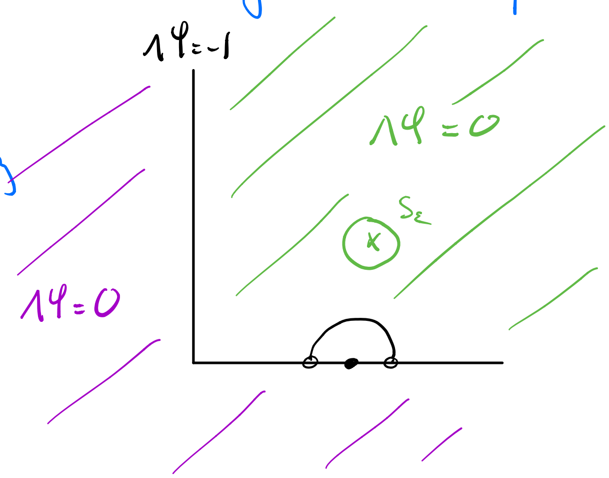
Remark: it is well-defined since, for $\varphi = \sum_i m_i \mathbb{1}_{X_i}$:

- $\mathrm{lk}(x, S)$ together with
- $\mathrm{lk}(x, X_i)$

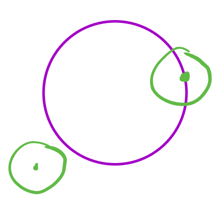
are well-defined up to a semi-algebraic homeomorphism

Examples

① $\varphi = \mathbb{1}_{\{x>0,\, y>0\}}$



② If $\varphi = \mathbb{1}_{S^1}$, then $\Lambda\varphi = 2 \cdot \mathbb{1}_{S^1}$



Remark S is Euler $\iff$ $\Lambda \mathbb{1}_S$ takes only even values

Definition $\varphi \in F(S)$ is called Euler if $\Lambda\varphi$ takes only even values.

Proposition *The link of a constructible function is a constructible function. The link operator $\varphi \mapsto \Lambda\varphi$ is a homomorphism from the additive group of $F(S)$ to itself.*

Proof • We need to prove that $\Lambda\varphi$ is constructible if so is φ. The homomorphism property will be automatic.

• One can assume $\varphi = \mathbb{1}_X$

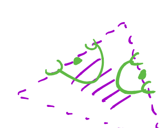
• Choosing a triangulation of S compatible with X, one can assume S and X being polyhedra $|K|$ and $|K'|$ with $K' \subseteq K$.

$$X = \Big(\coprod_{\substack{p \in K' \\ p \text{ vertex}}} p \Big) \coprod \Big(\coprod_{\sigma \in K'} \mathring{\sigma} \Big)$$

so that
$$\mathbb{1}_X = \sum_{p \in K'} \mathbb{1}_p + \sum_{\sigma \in K'} \mathbb{1}_{\mathring{\sigma}}$$

• Consider a vertex $p \in K'$:
$$\Lambda \mathbb{1}_p = 0$$

• Consider $\sigma \in K'$:
$$\Lambda \mathbb{1}_{\mathring{\sigma}}(x) = \begin{cases} 0 & \text{if } x \notin \bar{\sigma} \\ \text{constant} & \text{if } x \in \mathring{\sigma} \\ \text{(another) constant} & \text{on any open face of } \bar{\sigma} \setminus \mathring{\sigma} \end{cases}$$



· As a consequence :

$\Lambda \mathbb{1}_X$ is constant on each open simplex in K'

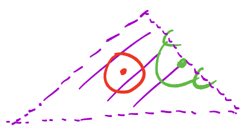
so it takes a finite number of values

· $(\Lambda \mathbb{1}_X)^{-1}(m)$ is semialgebraic $\forall m$

$\square$

<u>Computations</u> (for future use)

Let $\begin{cases} \sigma & \text{be an open simplex of dimension } d \\ \bar\sigma & \text{its closure} \end{cases}$



$\chi(S^n) = 1 + (-1)^n$ since $S^n \cong \mathbb{R}^n \sqcup \{p\}$

$$\Lambda \mathbb{1}_\sigma (x) = \begin{cases} 1 + (-1)^{d-1} & \text{if } x \in \sigma \\ (-1)^{d-1} & \text{if } x \in \bar\sigma \setminus \sigma \\ 0 & \text{if } x \notin \bar\sigma \end{cases}$$

$$= (-1)^{d-1} \mathbb{1}_{\bar\sigma} + \mathbb{1}_\sigma$$

Similarly

$$\Lambda \mathbb{1}_{\bar\sigma} (x) = \begin{cases} 1 + (-1)^{d-1} & \text{if } x \in \sigma \\ 1 & \text{if } x \in \bar\sigma \setminus \sigma \\ 0 & \text{if } x \notin \bar\sigma \end{cases}$$

$$= \mathbb{1}_{\bar\sigma} + (-1)^{d-1} \mathbb{1}_\sigma$$

The link operator commute with proper push-forward. It will be important in future applications.

Proposition *The link operator commutes with proper pushforward. If $f : S \to T$ is a continuous proper semialgebraic map and $\varphi : S \to \mathbb{Z}$ a constructible function, then $\Lambda(f_*\varphi) = f_*(\Lambda\varphi)$.*

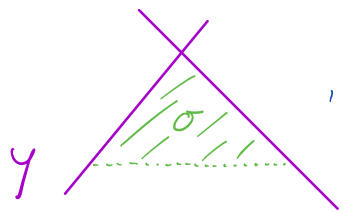
We need an auxilliary result:

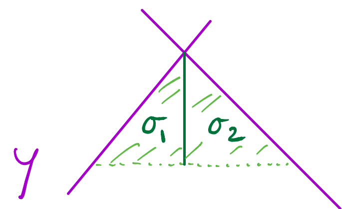
Lemma *Let Y be a compact semialgebraic subset of a semialgebraic set S, and $\varphi : S \to \mathbb{Z}$ a constructible function. Then*

$$\int_{\mathrm{lk}(Y,S)} \varphi \, d\chi = \int_Y \Lambda\varphi \, d\chi \,. \qquad (*)$$

Proof. Using a semialgebraic triangulation of S compatible with Y and φ, we can assume that S is a union of open simplices of a finite simplicial complex K, Y a union of closed simplices and φ is constant on open simplices. By additivity it suffices to prove the formula $(*)$ for $\varphi = \mathbf{1}_\sigma$, where σ is an open simplex of K.

By subdivision of K, we can assume that for every open simplex σ the intersection $\overline{\sigma} \cap Y$ is a closed face $\overline{\tau}$ (possibly empty) of $\overline{\sigma}$.

If 

, then 

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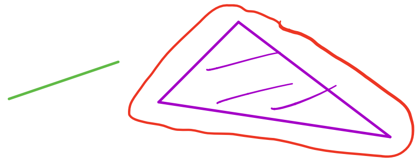
It follows that $\mathrm{lk}(Y,S) \cap \sigma$ is semialgebraically homeomorphic to an open $(d-1)$-cell if $\sigma \cap Y = \emptyset$ and $\overline{\sigma} \cap Y \neq \emptyset$, and empty otherwise.

Indeed $\mathrm{lk}(Y,S) = f^{-1}(\varepsilon)$ where $\begin{cases} f : S \to [0,\infty) \\ f^{-1}(0) = K \end{cases}$

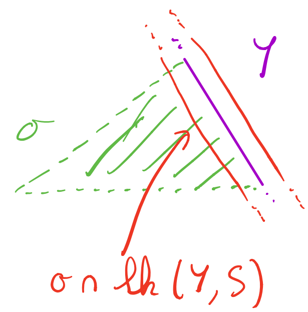
so that:

(a) • if $\sigma \subseteq Y$, then $\sigma \cap \mathrm{lk}(Y,S) = \emptyset$:

(b) • if $\sigma \cap Y = \emptyset$ and $\overline{\sigma} \cap Y = \emptyset$, again $\sigma \cap \mathrm{lk}(Y,S) = \emptyset$:



(c) • if $\sigma \cap Y = \emptyset$ and $\underbrace{\bar{\sigma} \cap Y}_{\substack{\text{closed face } \bar{\tau} \\ \text{of } \bar{\sigma}}} \neq \emptyset$



Then :

$$\int_{\mathrm{lk}(Y,S)} \mathbf{1}_\sigma \, d\chi = \chi(\mathrm{lk}(Y,S) \cap \sigma) = \begin{cases} (-1)^{d-1} & \text{if } \sigma \cap Y = \emptyset \text{ and } \overline{\sigma} \cap Y \neq \emptyset, \ (c) \\ 0 & \text{otherwise.} \ (a) \text{ and } (b) \end{cases}$$

On the other hand:

$$\Lambda \mathbb{1}_\sigma (x) = (-1)^{d-1} \mathbb{1}_{\bar{\sigma}} + \mathbb{1}_\sigma$$

$$\int_Y \Lambda \mathbb{1}_\sigma \, d\chi = (-1)^{d-1} \underbrace{\chi(\bar{\sigma} \cap Y)}_{\chi(\bar{\tau}) = \begin{cases} 1 & \text{if not empty} \\ 0 & \text{if } \emptyset \end{cases}} + \chi(\sigma \cap Y)$$

$$= \begin{cases} (-1)^{d-1} & \text{if } \sigma \cap Y = \emptyset \text{ and } \bar{\sigma} \cap Y \neq \emptyset & (c) \\ (-1)^{d-1} - (-1)^{d-1} = 0 & \text{if } \sigma \subseteq Y & (a) \\ 0 & \text{if } \sigma \cap Y = \emptyset \text{ and } \bar{\sigma} \cap Y = \emptyset & (b) \end{cases}$$

Therefore

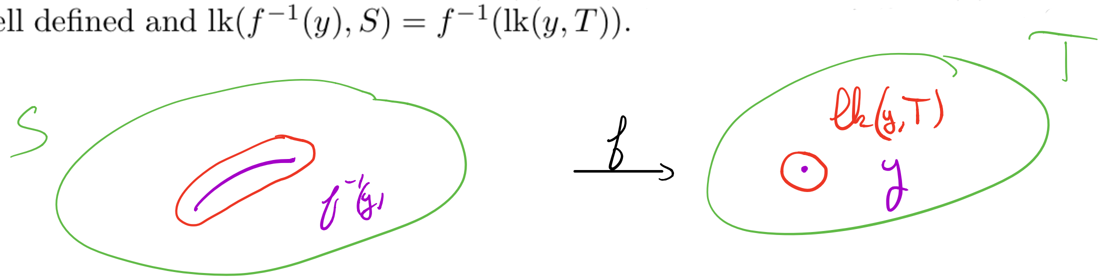
$$\int_{\mathrm{lk}(Y,S)} \mathbf{1}_\sigma \, d\chi = \int_Y \Lambda \mathbf{1}_\sigma \, d\chi \, ,$$

which completes the proof of the lemma □

Now we can prove the proposition :

Proposition *The link operator commutes with proper pushforward. If $f : S \to T$ is a continuous proper semialgebraic map and $\varphi : S \to \mathbb{Z}$ a constructible function, then $\Lambda(f_*\varphi) = f_*(\Lambda\varphi)$.*

Proof. If y is a point of T, then $f^{-1}(y)$ is compact, the link of $f^{-1}(y)$ in S is well defined and $\mathrm{lk}(f^{-1}(y), S) = f^{-1}(\mathrm{lk}(y, T))$.



As a consequence:

$$\Lambda(f_*\varphi)(y) \underset{def.}{=} \int_{\ell k(y,T)} (f_*\varphi)\, d\chi \underset{Fubini}{=} \int_{f^{-1}(\ell k(y,T))} \varphi\, d\chi = \int_{\ell k(f^{-1}(y), S)} \varphi\, d\chi .$$

Finally using the lemma ($f^{-1}(y)$ is compact)

$$\int_{\ell k(f^{-1}(y), S)} \varphi\, d\chi = \int_{f^{-1}(y)} \Lambda\varphi\, d\chi = f_*(\Lambda\varphi)(y)$$

$\square$.

Corollary 3.11 *Let $\varphi : S \to \mathbb{Z}$ be a constructible function on a compact semialgebraic set S. Then $\int_S \Lambda\varphi\, d\chi = 0$.*

<u>Proof</u> Consider $f : S \longrightarrow \{0\}$ a constant function.

Then $f_*\varphi \in F(\{0\})$ so $f_*\varphi(0) = \int_{f^{-1}(0)} \varphi\, d\chi = \int_S \varphi\, d\chi \in \mathbb{Z}$

In particular $\Lambda(f_*\varphi) = 0$

On the other hand

$$\Lambda(f_*\varphi) \underset{\substack{\uparrow \\ \text{proposition}}}{=} f_*(\Lambda\varphi) = \int_S \Lambda\varphi \, d\chi$$

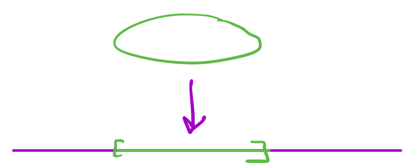
$\square$

III ALGEBRAICALLY CONSTRUCTIBLE FUNCTIONS

From now on, $S = V$ is a real algebraic set.

Idea: consider $\varphi = \sum_i m_i \mathbb{1}_{X_i}$ with X_i real algebraic subset of V.

But not stable by push forward:



Definition. $\varphi \in F(V)$ is said algebraically constructible if

$$\varphi = \sum_i m_i \, (f_i)_*(\mathbb{1}_{W_i})$$

where $f_i : W_i \longrightarrow V$ is a regular function on an algebraic set W_i.

. Denote by $A(V)$ the set of algebraically constructible functions on V.

Lemma $A(V)$ is a subring of $F(V)$.

Proof For the product:

$$\begin{tikzcd} W_1 \times_V W_2 \arrow[r] \arrow[d] \arrow[dr, "g"] & W_2 \arrow[d, "f_2"] \\ W_1 \arrow[r, "f_1"] & V \end{tikzcd}$$

$$f_{1*}(\mathbb{1}_{W_1}) \cdot f_{2*}(\mathbb{1}_{W_2}) = g_*(\mathbb{1}_{W_1 \times_V W_2})$$

Indeed $\chi(g^{-1}(v)) = \chi(f_1^{-1}(v) \times f_2^{-1}(v)) = \chi(f_1^{-1}(v)) \, \chi(f_2^{-1}(v))$

so $g_*(\mathbb{1}_{W_1 \times_V W_2})(v) = \int_{g^{-1}(v)} \mathbb{1}_{W_1 \times_V W_2} \, d\chi = \chi(g^{-1}(v)) = \left(\int_{f_1^{-1}(v)} \mathbb{1}_{W_1} \, d\chi\right)\left(\int_{f_2^{-1}(v)} \mathbb{1}_{W_2} \, d\chi\right)$ $\square$

Remarks ① If $f: V_1 \to V_2$ is regular, then
$$f_*: A(V_1) \longrightarrow A(V_2)$$
since $(f \circ g)_* = f_* \circ g_*$.

② The algebraically constructible functions form the smallest class of constructible functions on algebraic sets containing the constant functions and stable by pushforward along regular mappings. Phrased differently, the algebraically constructible functions are the functions $V \ni x \mapsto \chi(f^{-1}(x))$, where $f: W \to V$ is a regular mapping.

Via resolution of singularities:

Proposition *Let φ be an algebraically constructible function on a real algebraic set V. Then there exists a representation of φ where*

$$\varphi = \sum_i m_i \, f_{i*}(\mathbb{1}_{W_i})$$

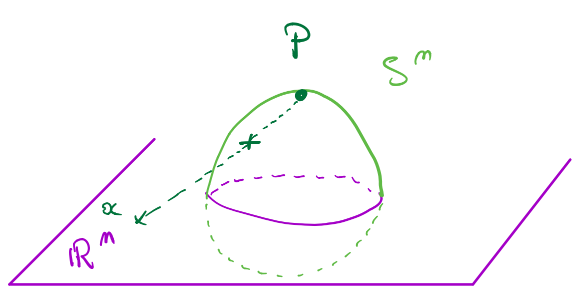
1. *all f_i are proper regular mappings*
2. *all W_i nonsingular.*

Proof. We begin with condition 1. Consider a regular map $f: W \to V$. Replacing W with the graph of f, one can assume that W is a real algebraic subset of $\mathbb{R}^n \times V$ and that f is the projection on V.

$$\begin{tikzcd} \operatorname{graph} f \subseteq \mathbb{R}^n \times V \arrow[r, "\text{proj.}"] \arrow[d, "\wr"'] & V \\ W \arrow[u, bend right, "{(x, f(x))}"'] & \end{tikzcd}$$

x

Now embed $\mathbb{R}^n \times V$ in $S^n \times V$



using the inverse stereographic projection which is a biregular isomorphism between $\mathbb{R}^n$ and the sphere $S^n \subset \mathbb{R}^{n+1}$ minus its north pole $P = (0, \ldots, 0, 1)$.

Then set $W' = W \cup (\{P\} \times V)$ and denote by $p : W' \to V$ the projection, which is proper by construction: we have added $\{P\}$ to $\mathbb{R}^n$ to compactify the fibers.

Then $f_*(\mathbf{1}_W) = p_*(\mathbf{1}_{W'}) - \mathbf{1}_V$ and 1. is achieved.

Actually, for $v \in V$:

* $f_*(\mathbb{1}_W)(v) = \int_{f^{-1}(v)} \mathbb{1}_W \, d\chi = \chi(f^{-1}(v))$
* $p_*(\mathbb{1}_{W'})(v) = \chi(p^{-1}(v)) = \chi(f^{-1}(v) \amalg \{P\} \times \{v\}) = \chi(f^{-1}(v)) + 1$

Now we realize condition 2. We use the resolution of singularities and an induction on dimension. We start with a representation

$$\varphi = \sum_{i \in I} m_i (f_i)_*(\mathbf{1}_{W_i}) \ ,$$

where all f_i are proper regular maps. Let d be the maximum of the dimensions of those among the W_i which are singular (assume that these are $W_1, \ldots, W_k$).

Then take a resolution of singularities $\pi_i : \widetilde{W_i} \to W_i$ for $i = 1, \ldots, k$. There are algebraic subsets $Z_i \subset W_i$ such that π_i is a biregular isomorphism from $\widetilde{W_i} \setminus \pi_i^{-1}(Z_i)$ onto $W_i \setminus Z_i$, and $\pi_i^{-1}(Z_i)$ and Z_i are of dimension $< d$.

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\pi} & W \\ \cup & & \cup \\ \pi^{-1}(Z) & \longrightarrow & Z \end{array} \qquad \widetilde{W} \setminus \pi^{-1}(Z) \xrightarrow{\sim} W \setminus Z$$

so that $\quad f_*(\mathbb{1}_{W \setminus Z}) = (f \circ \pi)_*(\mathbb{1}_{\widetilde{W} \setminus \pi^{-1}(Z)})$

hence $\quad f_*(\mathbb{1}_W) - f_*(\mathbb{1}_Z) = (f \circ \pi)_*(\mathbb{1}_{\widetilde{W}}) - (f \circ \pi)_*(\mathbb{1}_{\pi^{-1}(Z)})$

and therefore for any $i \in \{1, \ldots, k\}$:

non singular

dim $< d$

$$(f_i)_*(\mathbf{1}_{W_i}) = (f_i \circ \pi_i)_*(\mathbf{1}_{\widetilde{W_i}}) - (f_i \circ \pi_i)_*(\mathbf{1}_{\pi_i^{-1}(Z_i)}) + (f_i)_*(\mathbf{1}_{Z_i}) \ ,$$

proper since π proper.

and in this way we have decreased the maximum dimension of singular algebraic sets appearing in the representation of φ. Notice that $(f_i \circ \pi_i)$ is proper. $\square$

The behavior of algebraically constructible functions under the link operator is particularly interesting. As we shall see, it encompasses many local topological properties of real algebraic sets.

(McCrory and Parusiński, 1997)

Theorem (*) *Let φ be an algebraically constructible function on a real algebraic set V. Then $\Lambda\varphi$ takes only even values, and $\frac{1}{2}\Lambda\varphi$ is again algebraically constructible.*

Proof. By previous proposition, one may assume:

$$\varphi = \sum_{i \in I} m_i (f_i)_* (\mathbf{1}_{W_i}) \ ,$$

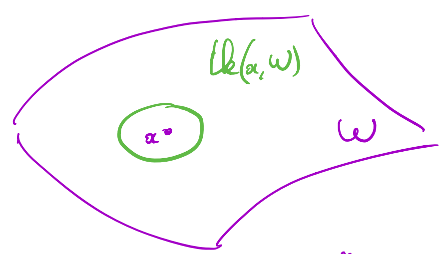
where all W_i are non singular of dimension d_i and all f_i are proper regular maps. Then we have, by commutation of push-forward along proper maps and link operator:

$$\Lambda\varphi = \Lambda\Big(\sum_{i \in I} m_i (f_i)_* (\mathbf{1}_{W_i})\Big) = \sum_{i \in I} m_i (f_i)_* (\Lambda \mathbf{1}_{W_i}) \ .$$

Since W_i is nonsingular of dimension d_i, $\Lambda \mathbf{1}_{W_i}$ is the constant 2 if d_i is odd and 0 if d_i is even.

Therefore

$$\Lambda \mathbb{1}_{W_i} = \begin{cases} 2\,\mathbb{1}_{W_i} & \text{if } d_i \text{ odd} \\ 0 & \text{if } d_i \text{ even} \end{cases}$$



$\chi(S^n) = 1 + (-1)^n$

so that

$$\Lambda\varphi = \sum_{i:\, d_i \text{ odd}} m_i (f_i)_* (2 \cdot \mathbb{1}_{W_i})$$

and

$$\frac{1}{2}\Lambda\varphi = \sum_{i \in I,\ d_i \text{ odd}} m_i (f_i)_* (\mathbf{1}_{W_i}) \in A(V)$$

which proves the theorem. $\square$

As a corollary, one obtain:

<u>Sullivan Theorem</u>

$$\forall x \in V \qquad \chi(Lk(x,V)) \equiv 0 \, [2]$$

<u>Proof</u>. $\varphi = \mathbb{1}_V$ is algebraically constructible

. Therefore $\Lambda\varphi$ takes only even values by the Theorem.

. In particular V is Euler.

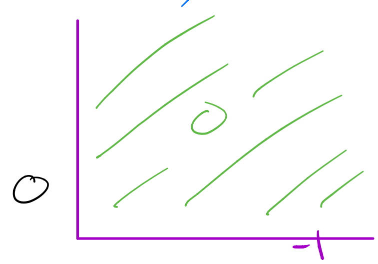
$\square$

<u>Remark</u> In general, we have the inclusions:

$$A(V) \subsetneq E(V) \subsetneq F(V)$$

The inclusions are strict:

- consider the open quadrant $Q \subseteq \mathbb{R}^2$, $Q = \{x > 0, y > 0\}$
- Then $\Lambda \mathbb{1}_Q$ is as follows:

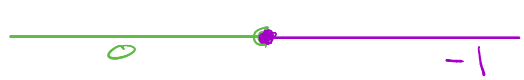


so $\mathbb{1}_Q$ is not Euler

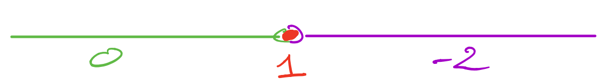
- $2 \cdot \mathbb{1}_Q$ is Euler, but not algebraically constructible.

Otherwise $\frac{1}{2} \Lambda (2 \cdot \mathbb{1}_Q)$ would be algebraically constructible.

By product $\varphi := \left(\frac{1}{2} \Lambda (2 \cdot \mathbb{1}_Q) \right) \mathbb{1}_{\{y=0\}} \in A(\mathbb{R}^2)$



But $\Lambda\varphi$ is like this:



In particular φ is not Euler, a contradiction.

• Finally $4 \cdot \mathbb{1}_Q$ is algebraically constructible

Indeed for $W = \{(x,y,s,t) \in Q \times \mathbb{R}^2 : s^2 x = 1, \ t^2 y = 1\}$

$$\begin{array}{ccc} W & & (x,y,s,t) \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbb{R}^2 & & (x,y) \end{array}$$

we have $p_*(\mathbb{1}_W) = 4 \cdot \mathbb{1}_Q$

More generally:

Proposition *Let V be a real algebraic set of dimension d. For every constructible function $\varphi : V \to \mathbb{Z}$, the function $2^d \varphi$ is algebraically constructible.*

Proof. Every semialgebraic subset of V can be represented as a finite disjoint union of subsets of the form

$$S = \{x \in V \ ; \ f(x) = 0, \ g_1(x) > 0, \dots, g_d(x) > 0\} \ ,$$

where $f, g_1, \dots, g_d$ are polynomials on V. It is important that we can take the number of inequalities to be the dimension of V: this is the celebrated Broecker-Scheiderer theorem (6.5.1 in [BCR]).

↳ proof via quadratic forms.

So every constructible function on V is a linear combination with integer coefficients of characteristic functions of semialgebraic sets S. Set

$$W = \{(x, t_1, \dots, t_d) \in V \times \mathbb{R}^d \ ; \ f(x) = t_1^2 g_1(x) - 1 = \dots = t_d^2 g_d(x) - 1 = 0\} \ ,$$

and let $p : W \to V$ be the projection. Then $2^d \mathbf{1}_S = p_*(\mathbf{1}_W)$, and the proposition follows. $\square$

Let's go further and signal some more involved results.

Theorem 3.16 *Let V be a real algebraic set. A function $\varphi : V \to \mathbb{Z}$ is algebraically constructible if and only if there are finitely many polynomials $f_1, , \ldots, f_p$ on V such that*

$$\varphi = \sum_{i=1}^{p} \operatorname{sign}(f_i) \ .$$

$\operatorname{sign} f : V \longrightarrow \{-1, 0, 1\}$
$x \mapsto \begin{cases} 1 & \text{if } f(x) > 0 \\ 0 & \cdots = 0 \\ -1 & \cdots < 0 \end{cases}$

Proof of $\Leftarrow$ For $W = \{(x,t) \in V \times \mathbb{R} : t^2 = f(x)\}$

$$\begin{array}{c} W \\ \downarrow p \\ V \end{array}$$

we have $p_*(\mathbb{1}_W) - \mathbb{1}_V = \begin{cases} 2-1=1 & \text{if } f>0 \\ 1-1=0 & \text{if } f=0 \\ 0-1=-1 & \text{if } f<0 \end{cases}$

so it is equal to $\operatorname{sign} f$

$\square$

Remark The proof of $\Rightarrow$ is more involved.

Corollary V irreducible algebraic set.
For any $\varphi \in A(V)$, there exists an algebraic subset $W \subsetneq V$ such that φ is constant modulo 2 on $V \setminus W$.

Proof: This is the case of $\operatorname{sign} f$, for f polynomial. $\square$

Sullivan Theorem illustrates the fact that algebraically constructible functions inherit properties of real algebraic sets.

An important point becomes to be able to produce such functions.

For instance, the link operator Λ is a tool.
Another is:

Corollary *Let φ be an algebraically constructible function on a real algebraic set φ. Then $\frac{1}{2}(\varphi^4 - \varphi^2)$ is again algebraically constructible.*

Proof. We start with a representation $\varphi = \sum_{i=1}^{p} \operatorname{sign} f_i$, where the f_i are polynomial functions on V. Set $\sigma_i = \operatorname{sign} f_i$. Each σ_i is algebraically constructible, and $\sigma_i^4 = \sigma_i^2$. Then

$$\varphi^2 = \sum_i \sigma_i^2 + 2\sum_{i<j} \sigma_i\sigma_j = \sum_i \sigma_i^2 + 2\psi_2 \ ,$$

where ψ_2 is algebraically constructible. Taking the square we obtain

$$\varphi^4 = \sum_i \sigma_i^4 + 2\sum_{i<j} \sigma_i^2\sigma_j^2 + 4\psi_2 \sum_i \sigma_i^2 + 4\psi_2^2 = \sum_i \sigma_i^2 + 2\psi_4 \ ,$$

where ψ_4 is algebraically constructible. The corollary follows immediately. □

Remarks ① There exists no geometric interpretation of this corollary.
②

The preceding corollary is not the only result of this kind:
we know the characterization of all polynomials P with rational coefficients such that $P(\varphi)$ is algebraically constructible for every algebraically constructible φ.

Let S be a locally compact semialgebraic set. Denote by $\widetilde{\Lambda}(S)$ the smallest subring of $F(S)\left[\frac{1}{2}\right]$ containing $\mathbf{1}_S$ and stable by the half-link operator $\widetilde{\Lambda} = \frac{1}{2}\Lambda$.

Theorem *If S is homeomorphic to a real algebraic set, then all functions in $\widetilde{\Lambda}(S)$ have values in $\mathbb{Z}$.*

Proof. Let $h : S \to V$ be a homeomorphism from S to a real algebraic set. Since h preserves the link operator, it induces an isomorphism $h^* : \varphi \mapsto \varphi \circ h$ from $\widetilde{\Lambda}(V)$ to $\widetilde{\Lambda}(S)$. Since $\widetilde{\Lambda}(V)$ is contained in $A(V)$ (a consequence of Theorem (✱)), it consists of functions with values in $\mathbb{Z}$. □

Remarks ① $\tilde{\Lambda}(S)$ is a semialgebraic invariant of S:

$$S_1 \underset{\text{sa homeo}}{\simeq} S_2 \implies \tilde{\Lambda}(S_1) \underset{\text{ring morphism}}{\simeq} \tilde{\Lambda}(S_2)$$

② This theorem produces necessary conditions for a semialgebraic set to be sa-homeomorphic to an algebraic set.

In particular, in dimension 3, there exist $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathcal{F}(S)$ constructed using Λ such that:

Theorem **(Akbulut-King)** *A compact semialgebraic set S of dimension 3 is homeomorphic to a real algebraic set if and only if it is Euler and the four local obstructions*

$$\left(\int_{\mathrm{lk}(x,S)} \varphi_i \, d\chi \bmod 2\right)_{i=1,\dots,4}$$

vanish everywhere on S.