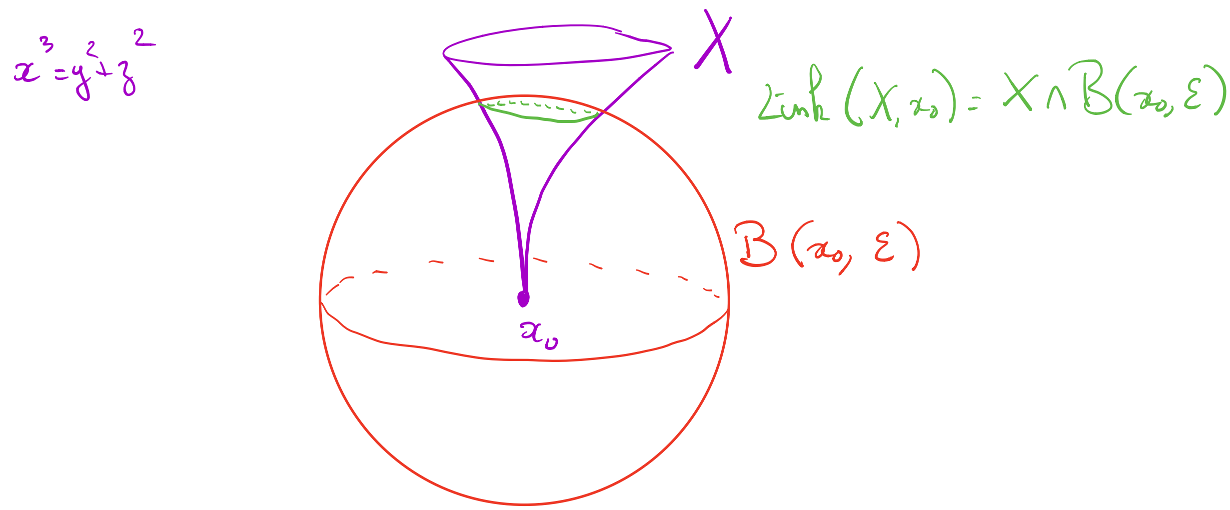
LINKS OF SEMIALGEBRAIC SETS

<u>Idea</u> • the link is a local invariant of the singularities
• it is simpler than the initial singularities (and even smooth if the singularity is isolated).



<u>Goal</u> Sullivan Theorem (1971)

Let $X \subseteq R^n$ be a real algebraic set. For all $x \in X$, the Euler Characteristics $\chi(\mathrm{Link}(X, x))$ is even.

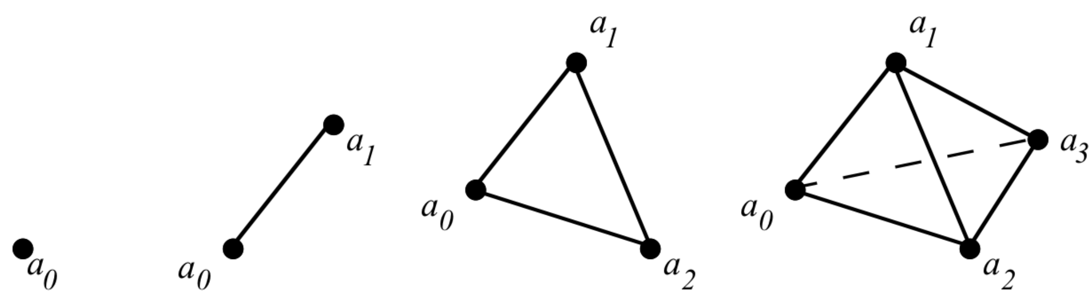
<u>Remark</u> we will use $R = \mathbb{R}$, even if it can be done with real closed fields.

The reason is that we are interested in topology.

<u>Reference</u> : M. Coste, Real Algebraic Sets, in "Arc spaces and additive invariants in real algebraic and analytic geometry".

I TRIANGULATION OF SEMIALGEBRAIC SETS

We first recall some definitions concerning simplicial complexes that we shall need. Let $a_0, \ldots, a_d$ be points of $\mathbb{R}^n$ which are affine independent (i.e. not contained in an affine subspace of dimension $d-1$).



0-simplex 1-simplex 2-simplex 3-simplex

The d-simplex with vertices $a_0, \ldots, a_d$ is :

$$[a_0, \ldots, a_d] = \{x \in \mathbb{R}^n \ ; \ \exists \lambda_0, \ldots, \lambda_d \in [0,1] \ \sum_{i=0}^{d} \lambda_i = 1 \ \text{ and } \ x = \sum_{i=0}^{d} \lambda_i a_i\}$$

The corresponding *open simplex* is

$$(a_0, \ldots, a_d) = \{x \in \mathbb{R}^n \ ; \ \exists \lambda_0, \ldots, \lambda_d \in (0,1] \ \sum_{i=0}^{d} \lambda_i = 1 \ \text{ and } \ x = \sum_{i=0}^{d} \lambda_i a_i\}$$

We shall denote by $\mathring{\sigma}$ the open simplex corresponding to the simplex σ. A *face* of the simplex $\sigma = [a_0, \ldots, a_d]$ is a simplex $\tau = [b_0, \ldots, b_e]$ such that

$$\{b_0, \ldots, b_e\} \subset \{a_0, \ldots, a_d\} \ .$$

A *finite simplicial complex* in $\mathbb{R}^n$ is a finite collection $K = \{\sigma_1, \ldots, \sigma_p\}$ of simplices $\sigma_i \subset \mathbb{R}^n$ such that, for every $\sigma_i, \sigma_j \in K$, the intersection $\sigma_i \cap \sigma_j$ is a common face of σ_i and σ_j (see Figure 3.3).

not a simplicial complex   a simplicial complex

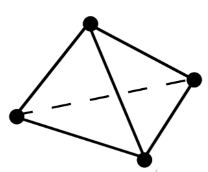
We set $|K| = \bigcup_{\sigma_i \in K} \sigma_i$; this is a semialgebraic subset of $\mathbb{R}^n$. A *polyhedron* in $\mathbb{R}^n$ is a subset P of $\mathbb{R}^n$, such that there exists a finite simplicial complex K in $\mathbb{R}^n$ with $P = |K|$.

Theorem *Let $S \subset \mathbb{R}^n$ be a compact semialgebraic set, and $S_1, \ldots, S_p$, semialgebraic subsets of S. Then there exists a finite simplicial complex K in $\mathbb{R}^n$ and a semialgebraic homeomorphism $h : |K| \to S$, such that each S_k is the image by h of a union of open simplices of K.*

Proof : a consequence of cylindrical decomposition. □

Remark one can even ask $h_{|\mathring{\sigma}}$ to be of class C^∞.

Example $S^2 \simeq$



The semialgebraic homeomorphism $h : |K| \to S$ will be called a semialgebraic triangulation of S (compatible with the S_j).

Continuous semialgebraic functions can also be triangulated, in the following sense.

Theorem 1.11 *Let $A \subset \mathbb{R}^n$ be a compact semialgebraic set, and $B_1, \ldots, B_p$, semialgebraic subsets of A. Let $f : A \to \mathbb{R}$ be a continuous semialgebraic function. Then there exists a semialgebraic triangulation $h : |K| \to A$ compatible with the B_j and such that $f \circ h$ is linear on each simplex of K.*

<u>Idea of proof</u>

The method to prove this theorem is to triangulate the graph of f in $\mathbb{R}^n \times \mathbb{R}$ in a way which is "compatible" with the projection on the last factor.

□

<u>Remark</u>

The fact that f is a function with values in $\mathbb{R}$ and not a map with values in $\mathbb{R}^k$, $k > 1$, is crucial here. Actually, the blowing up map $[-1, 1]^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (x, xy)$ cannot be triangulated (it is not equivalent to a PL map).

<u>Remark</u> A continuous function $f : P \to \mathbb{R}$ on a polyhedron P is said piecewise linear if there exists a simplicial complex K with $|K| = P$ such that $\forall \sigma \in K$ $f|_\sigma$ is linear

<u>Theorem</u> (difficult and important, called Hauptvermutung, Cooke-Shiota)
If two polyhedra P and Q are semialgebraically homeomorphic, then they are PL-homeomorphic.

<u>Consequence</u>: unicity of the triangulation.
$S \subset \mathbb{R}^n$ compact semi-algebraic set.

If
$$\begin{tikzcd} & X & \\ \lvert K_1 \rvert \arrow[ur, dash, "\sim"] & & \lvert K_2 \rvert \arrow[ul, dash, "\sim"'] \end{tikzcd}$$
, then $|K_1| \underset{sa}{\simeq} |K_2|$, so $|K_1| \underset{PL}{\simeq} |K_2|$.

The triangulation theorem can be applied to the non compact case in the following way. Let A be a noncompact semialgebraic subset of $\mathbb{R}^n$. Up to a semialgebraic homeomorphism, we can assume that A is bounded. Indeed, $\mathbb{R}^n$ is semialgebraically homeomorphic to the open ball of radius 1 by $x \mapsto (1+\|x\|^2)^{-1/2}x$. Then one can take a triangulation of the compact semialgebraic set $\operatorname{clos}(A)$ compatible with A. So we obtain:

Proposition 1.13 *Let A be any semialgebraic set and $B_1, \ldots, B_p$ semialgebraic subsets of A. There exist a finite simplicial complex K, a union U of open simplices of K and a semialgebraic homeomorphism $h : U \to A$ such that each B_j is the image by h of a union of open simplices contained in U.*

Addendum on semi-algebraic dimension.

Proposition 3.16 *Let $A \subset \mathbb{R}^n$ be a semialgebraic set. Then*

1. $\dim(\operatorname{clos}(A)) = \dim A$,
2. $\dim(\operatorname{clos}(A) \setminus A) < \dim A$.

The dimension is invariant by a semialgebraic homeomorphism.

Lemma 3.17 *Let $A \subset \mathbb{R}^{n+k}$ be a semialgebraic set, $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ the projection on the space of the first n coordinates. Then* $\dim(\pi(A)) \leq \dim(A)$. *Moreover, if the restriction of π to A is one-to-one, then* $\dim(\pi(A)) = \dim A$.

Theorem 3.18 *Let S be a semialgebraic subset of $\mathbb{R}^n$, and $f : S \to \mathbb{R}^k$ a semialgebraic mapping (not necessarily continuous). Then* $\dim f(S) \leq \dim S$. *If f is one-to-one, then* $\dim f(S) = \dim S$.

II TRIVIALIZATION

Idea: compare general maps to projections.

Interest of projections: the fibers are homeomorphic:

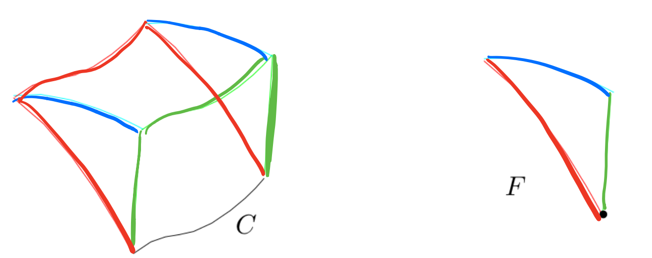
$$p : A \times B \longrightarrow A$$

$\forall a_1, a_2 \in A \qquad p^{-1}(a_1) \simeq p^{-1}(a_2) \simeq \{a_i\} \times B$

Definition

A continuous semi-algebraic mapping $p : A \to \mathbb{R}^k$ is said to be *semialgebraically trivial over a semialgebraic subset C* $\subset \mathbb{R}^k$ is there is a semialgebraic set F and a semialgebraic homeomorphism $h : p^{-1}(C) \to C \times F$, such that the composition of h with the projection $C \times F \to C$ is equal to the restriction of p to $p^{-1}(C)$.

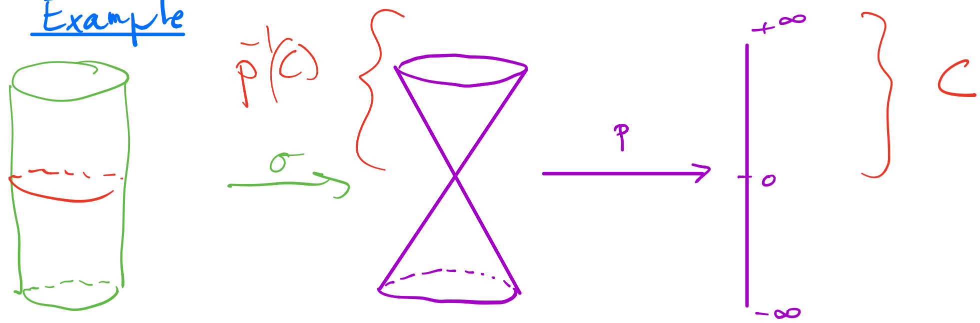
$$\begin{tikzcd}
A \supset p^{-1}(C) \arrow[rr, "h"] \arrow[dr, "p"'] & & C \times F \arrow[dl, "\text{projection}"] \\
& \mathbb{R}^k \supset C &
\end{tikzcd}$$



Local triviality: a neighborhood of C is homeomorphic to $C \times F$.

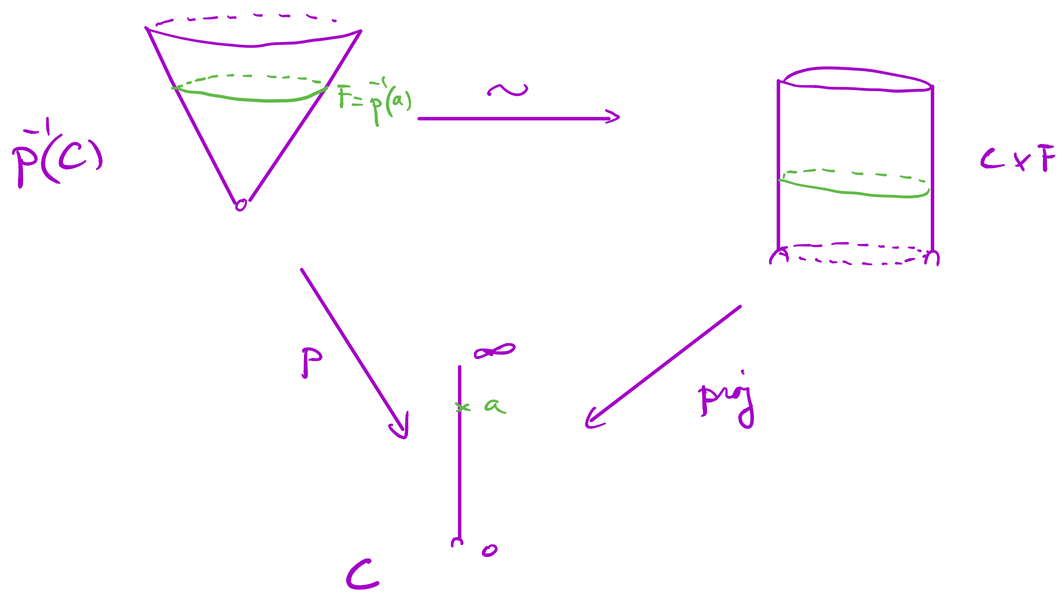
The homeomorphism h is called a semi-algebraic trivialization of p over C. We say that the trivialization h is compatible with a semialgebraic subset $B \subset A$ if there is a semialgebraic subset $G \subset F$ such that $h(B \cap p^{-1}(C)) = C \times G$.

Example



p is semi-algebraically trivial over $(-\infty, 0)$, $\{0\}$ and $(0, +\infty)$

Over $(0, +\infty)$:



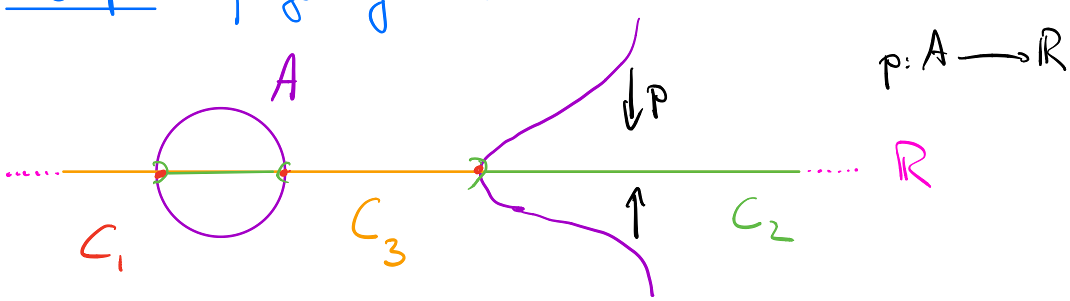
Next result states that any semialgebraic mapping is piecewise trivial

Theorem **(Hardt's semialgebraic triviality)** *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set and $p : A \to \mathbb{R}^k$, a continuous semi-algebraic mapping. There is a finite semialgebraic partition of $\mathbb{R}^k$ into $C_1, \dots, C_m$ such that p is semialgebraically trivial over each C_i.*

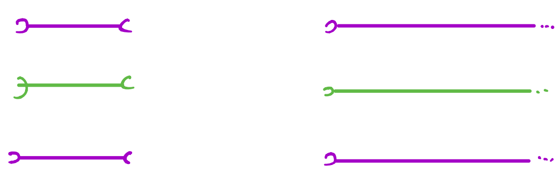
Remark *Moreover, if $B_1, \dots, B_q$ are finitely many semialgebraic subsets of A, we can ask that each trivialization $h_i : p^{-1}(C_i) \to C_i \times F_i$ is compatible with all B_j.*

About the proof: quite involved, via cylindrical decomposition.

Example projecting a cubic on a line



Over C_2:



Technical remark (for future reference)

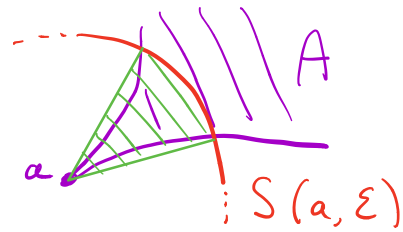
For a point $c \in C_i$ then the dimension of the fiber $p^{-1}(c)$ satisfies:

$$\dim p^{-1}(c) = \dim F_i = \dim p^{-1}(C_i) - \dim C_i \leq \dim A - \dim C_i$$

Application 1 : local conical structure.

Let A be a semialgebraic subset of $\mathbb{R}^n$ and a, a nonisolated point of A: for every $\varepsilon > 0$ there is $x \in A$, $x \neq a$, such that $\|x - a\| < \varepsilon$.

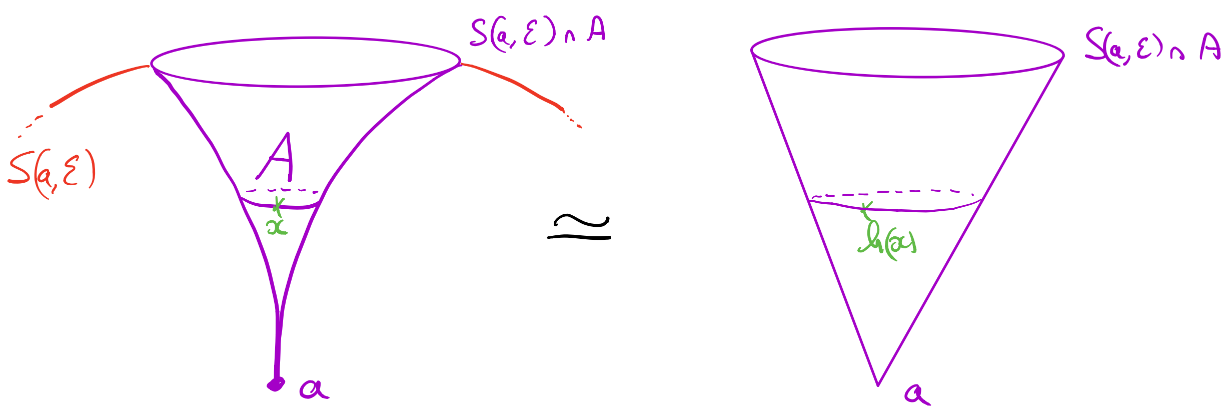
Notation $\quad a * (S(a,\varepsilon) \cap A)$



$a * (S(a,\varepsilon) \cap A)$ the cone with vertex a and basis $S(a,\varepsilon) \cap A$, i.e. the set of points in $\mathbb{R}^n$ of the form $\lambda a + (1-\lambda)x$, where $\lambda \in [0,1]$ and $x \in S(a,\varepsilon) \cap A$.

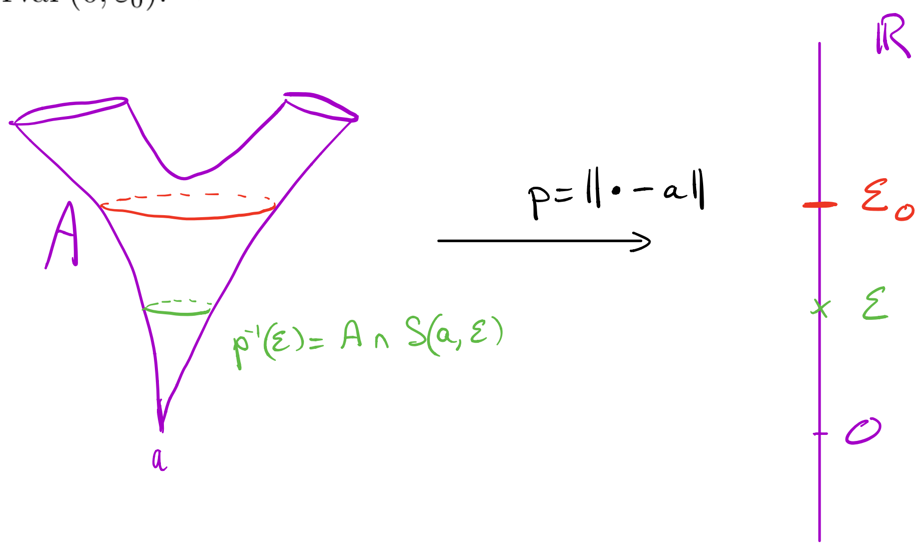
Locally, a semialgebraic set looks like a cone:

Theorem *For $\varepsilon > 0$ sufficiently small, there is a semialgebraic homeomorphism $h : \overline{B}(a,\varepsilon) \cap A \to a * (S(a,\varepsilon) \cap A)$ such that $\|h(x) - a\| = \|x - a\|$ and $h_{|S(a,\varepsilon)\cap A} = \mathrm{Id}$.*



The proof is a consequence of Hardt triviality applied to the distance function.

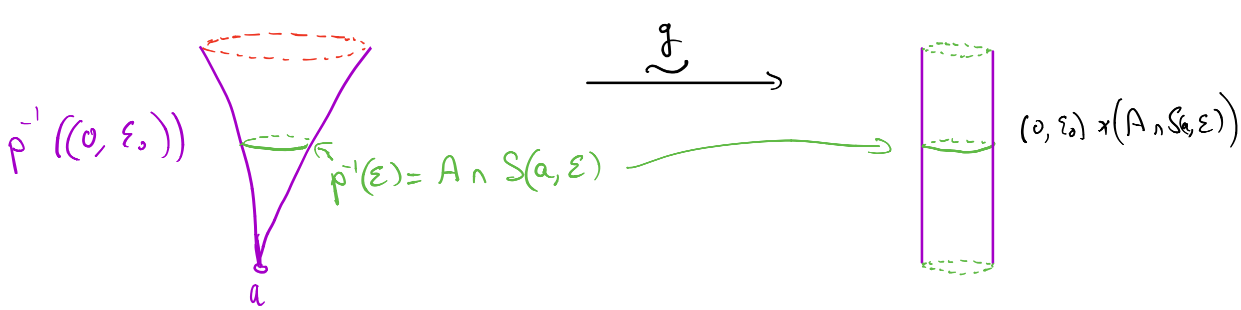
Proof. We apply Hardt's theorem to the mapping $p : A \to \mathbb{R}$ defined by $p(x) = \|x - a\|$. We obtain semialgebraic trivializations of p over a finite semialgebraic partition of $\mathbb{R}$. We can assume that this partition has as member an interval $(0, \varepsilon_0)$.



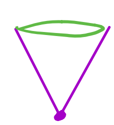
Choose ε such that $0 < \varepsilon < \varepsilon_0$. Since $p^{-1}(\varepsilon) = (A \cap S(a, \varepsilon))$, we have a semialgebraic homeomorphism

$$g : p^{-1}((0, \varepsilon_0)) \to (0, \varepsilon_0) \times (A \cap S(a, \varepsilon))$$

such that $g(x) = (\|x - a\|, g_1(x))$, where the restriction of g_1 to $S(a, \varepsilon) \cap A$ is the identity.



Denote $C_\varepsilon := a * (S(a,\varepsilon) \cap A)$



Now define $h : \overline{B}(a, \varepsilon) \cap A \to C_\varepsilon$ by

$$\begin{cases} h(x) &= \left(1 - \dfrac{\|x-a\|}{\varepsilon}\right) a + \dfrac{\|x-a\|}{\varepsilon} g_1(x) \quad \text{if } x \neq a\,, \\ h(a) &= a\,. \end{cases}$$

We can check that h has the properties of the theorem. The inverse mapping of h is defined by

$$\begin{cases} h^{-1}(\lambda a + (1-\lambda)x) &= g^{-1}((1-\lambda)\varepsilon, x) \text{ for } \lambda \in [0,1),\ x \in S(a,\varepsilon) \cap A\,, \\ h^{-1}(a) &= a\,. \end{cases}$$

□

For instance if $x \in S(x, \varepsilon)$ $h(x) = 0 + g(x) = x$

Application 2 : definition of the link.

Let A be a locally compact semialgebraic subset of $\mathbb{R}^n$ and let a be a point of A. Then we define *the link of a in A* as $\operatorname{lk}(a, A) = A \cap S(a, \epsilon)$ for $\epsilon > 0$ small enough.

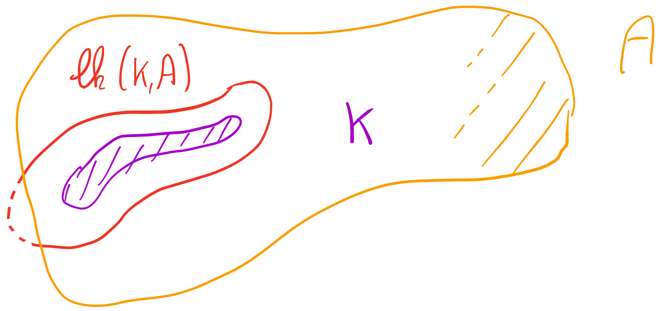
Does it depend on ε ?

Of course, the link depends on ϵ, but the semialgebraic topological type of the link does not depend on ϵ, if it is sufficiently small: there is ϵ_1 such that, for every $\epsilon \le \epsilon_1$, there is a semialgebraic homeomorphism $A \cap S(a, \epsilon) \simeq A \cap S(a, \epsilon_1)$. This is a consequence of the local conic structure theorem.

$A \subseteq \mathbb{R}^n$

More generally, let K be a compact semialgebraic subset of A. We define $\operatorname{lk}(K, A)$, the link of K in A, as follows. Choose a proper continuous semialgebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = K$ and $f(x) \ge 0$ for every $x \in \mathbb{R}^n$. We can take for instance for f the distance to K. Now set $\operatorname{lk}(K, A) = f^{-1}(\epsilon) \cap A$ for $\epsilon > 0$ sufficiently small.

One need to prove it does not depend on f nor on ε.



Proposition 1.20 *The semialgebraic topological type of the link* $\mathrm{lk}(K, A)$ *does not depend on* ϵ *nor on* f*.*

<u>Proof</u> Hardt triviality theorem + triangulation of s.a. functions
+ uniqueness of s.a. triangulation. □

The preceding result shows that the semialgebraic topological type of the link is a semialgebraic invariant of the pair (A, K): if h is a semialgebraic homeomorphism from A onto B, then $\mathrm{lk}(K, A)$ and $\mathrm{lk}(h(K), B)$ are semialgebraically homeomorphic.

We will now be interested in s.a. topological invariants.

III EULER CHARACTERISTICS.

To define it, two options:

- via homology theory for locally compact semialgebraic sets
- via cell decomposition (for any semialgebraic sets)

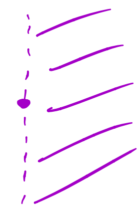
Let us explain
- an idea of construction
- its properties
- how to compute.

We denote by $H_i^{\mathrm{BM}}(A;\mathbb{Z}/2)$ the Borel-Moore homology of a locally compact semialgebraic set (with coefficients in $\mathbb{Z}/2$). We shall not need the definition of this homology. The following properties explain how we can compute it from the ordinary homology.

- If A is compact, $H_i^{\mathrm{BM}}(A;\mathbb{Z}/2)$ coincides with the usual homology $H_i(A;\mathbb{Z}/2)$.
- If A is not compact, we can take an open semialgebraic embedding of A into a compact semialgebraic set B, and this embedding induces an isomorphism of $H_i^{\mathrm{BM}}(A;\mathbb{Z}/2)$ onto the relative homology group $H_i(B, B \setminus A;\mathbb{Z}/2)$.

here we use "locally compact" as follows.

- locally compact means $A = U \cap C$ with U open and C closed



not locally compact

- in particular $A = \underbrace{\left[(\mathbb{R}^n \setminus \overline{A}) \cup A \right]}_{U} \cap \overline{A}$

- Note that $\mathbb{R}^n \setminus U \neq \emptyset$ if A is not closed

- Consider the distance function to $\mathbb{R}^n \setminus U$:

$$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x \longmapsto \mathrm{dist}(x, \mathbb{R}^n \setminus U)$$

- The function φ is then continuous semialgebraic.

- Then $\begin{array}{rcl} A & \longrightarrow & \mathbb{R}^n \times \mathbb{R} \\ x & \longmapsto & \left(x, \dfrac{1}{\varphi(x)}\right) \end{array}$ is a semialgebraic homeomorphism onto the closed subset:

$$A' = \{ (x,y) \in \mathbb{R}^{n+1} : x \in \overline{A},\ y\varphi(x) = 1 \}$$

- It remains to consider the Alexandrof compactification B of A' (it makes sense for semi-algebraic sets too).

* Coming back to Borel-Moore homology:

$$H_i^{BM}(A, \mathbb{Z}/2\mathbb{Z}) = H_i(B, B \setminus A, \mathbb{Z}/2\mathbb{Z}).$$

Long exact sequence in homology :

If F is a closed semialgebraic subset of A, then both F and $A \setminus F$ are locally compact and we have a long exact sequence

$$\ldots \to H_{i+1}^{\mathrm{BM}}(A \setminus F; \mathbb{Z}/2) \to H_i^{\mathrm{BM}}(F; \mathbb{Z}/2) \to H_i^{\mathrm{BM}}(A; \mathbb{Z}/2) \to H_i^{\mathrm{BM}}(A \setminus F; \mathbb{Z}/2) \to \ldots$$

Example $\quad H_i^{BM}\left((-1,1)^d, \mathbb{Z}/_2\right) \simeq \begin{cases} \mathbb{Z}/_2 & \text{if } i=d \\ 0 & \text{otherwise} \end{cases}$

with $A = S^d$ and $F = \{\text{point}\}$

so $A \setminus F \simeq (-1,1)^d$

Usual definition of Euler characteristics

- if A is a locally compact semialgebraic set:

$$\chi^{top}(A) = \sum_i (-1)^i \dim H_i(A, \mathbb{Z}/_2)$$

or another field, it will give the same.

→ good definition if A is compact, or comes from a complex algebraic variety

→ for general semialgebraic sets we use Borel-Moore homology :

$$\chi^{BM}(A) = \sum_i (-1)^i \dim H_i^{BM}(A, \mathbb{Z}/_2)$$

Why doing so?

- If A compact, $\quad \chi^{top}(A) = \chi^{BM}(A)$
- χ^{BM} will satisfy nice $\begin{cases} \text{additive} \\ \text{multiplicative} \end{cases}$ properties.

Example $\quad \chi^{BM}\left((-1,1)^d\right) = (-1)^d = \chi^{BM}\left((-1,1)\right)^d$

From now on we use the notation χ rather than χ^{BM}.

<u>Lemma</u> If $F \subseteq A$ are locally compact, then

$$\chi(A) = \chi(F) + \chi(A \setminus F)$$

<u>Proof</u>: long exact sequence in Borel-Moore homology. $\square$

The Euler characteristic with compact support can be computed using a stratification into cells.

Lemma 1.21 *Let A be a locally compact semialgebraic set and let $A = \sqcup_k C_k$ be a finite stratification into sa-cells C_k sa-homeomorphic to $(-1,1)^{d_k}$. Then $\chi(A) = \sum_k (-1)^{d_k}$.* a triangulation

Proof. Let d be the dimension of A, and let $A^{<d}$ be the union of the cells of dimension $< d$. By the properties of a stratification, $A^{<d}$ is closed in A. The cells of dimension d are the connected components of the complement $A \setminus A^{<d}$. Using this fact and the additivity property mentioned just above, we obtain

$$\chi(A) = (-1)^d \mathrm{card}(\{k\,;\; d_k = d\}) + \chi(A^{<d}) \,.$$

Hence, by induction, the lemma is proved. $\square$

We use this idea to extend χ to all semialgebraic sets (even when the Borel-Moore homology does not make sense).

Theorem 1.22 *The Euler characteristic with compact support on locally compact semialgebraic sets can be extended uniquely to a semialgebraic invariant (still denoted χ) on all semialgebraic sets satisfying*

$$\begin{array}{rcll} \chi(A \sqcup B) & = & \chi(A) + \chi(B) & \textit{disjoint union} \\ \chi(A \times B) & = & \chi(A) \times \chi(B) & \textit{product.} \end{array}$$

<u>Idea of proof</u>:

If such an extension χ exists, it must satisfy the following property. Let $A = \sqcup_k C_k$ be a finite semialgebraic partition of a semialgebraic set into pieces C_k semialgebraically homeomorphic to $(-1,1)^{d_k}$; then $\chi(A) = \sum_k (-1)^{d_k}$. Define $\tilde{\chi}(A) = \sum_k (-1)^{d_k}$, and show that this alternating sum does not depend on the semialgebraic partition of A.

For this: • if we have two different partitions, find a common refinement.

• it suffices to consider the case of the refinement of a partition.

• a cell C is locally compact: we can use the lemma: if $C = \coprod_i D_i$, then

$$\underset{\underset{(-1)^{\dim C}}{\parallel}}{\chi(C)} = \sum_i \chi(D_i) = \sum_i (-1)^{\dim D_i}$$

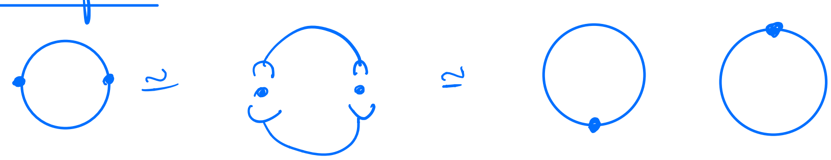
This proves the existence and additivity.
Multiplicativity is an exercise.

$\square$

Here is a first application of Euler characteristics.

__Theorem__ Let A and B be semialgebraic set. Then A is in semialgebraic bijection with B if and only if $\dim A = \dim B$ and $\chi(A) = \chi(B)$.

__Example__ S^1 and $S^1 \cup S^1$ are in semialgebraic bijection.



Idea of proof

$\Rightarrow$ It is possible to stratify A and B in $A=\coprod_{i\in I} A_i$, $B=\coprod_{i\in I} B_i$ such that $A_i \xrightarrow{\sim} B_i$ is a semialgebraic homeomorphism.

Then $\chi(A_i)=\chi(B_i)$ and $\dim A_i = \dim B_i$.

As a consequence

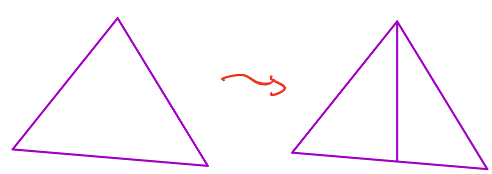
- $$\chi(A) = \sum_{i\in I} \chi(A_i) = \sum_{i\in I} \chi(B_i) = \chi(B)$$
- $$\dim A = \max_{i\in I} \dim A_i = \max_{i\in I} \dim B_i = \dim B.$$

$\Leftarrow$ Consider a triangulation for A and B:

$$A \simeq |K_A| \quad \text{and} \quad B \simeq |K_B|.$$

- One can assume K_A and K_B have the same number k of simplices of dimension $d=\dim A=\dim B$.

If not



- Therefore the unions of the top-dimensional simplices of K_A and K_B are in semialgebraic bijection.
- An open simplex σ of dimension d satisfies $\chi(\sigma)=(-1)^d$.
- Then

$$\chi(A) = \chi(|K_A|) = \sum_{\substack{\sigma\in K_A \\ \dim\sigma = d}} \chi(\sigma) + \sum_{\substack{\sigma\in K_A \\ \dim\sigma < d}} \chi(\sigma) = k(-1)^d + \sum_{\substack{\sigma\in K_A \\ \dim\sigma < d}} \chi(\sigma)$$

• Similarly

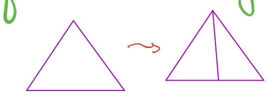
$$\chi(B) = k(-1)^d + \sum_{\substack{\sigma \in K_B \\ \dim \sigma < d}} \chi(\sigma) = k(-1)^d + \chi(|K'_B|)$$

where K'_B is K_B minus the k simplices of dimension d

(and similarly for A : K'_A).

• Finally • $\chi(|K'_A|) = \chi(A) - k(-1)^d = \chi(B) - k(-1)^d = \chi(|K'_B|)$

• $\dim |K'_A| = \dim |K'_B|$ (comes from the cutting)



• We conclude by an induction on dimension $\square$

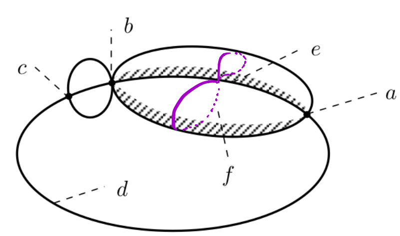
IV FUTUR OBJECTIVES

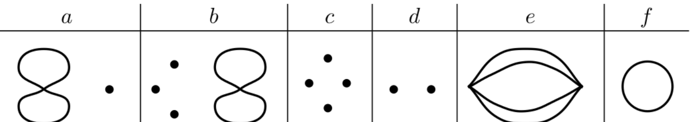
Theorem **(Sullivan)** *Let X be a real algebraic set. For every $x \in X$, the Euler characteristic $\chi(\mathrm{lk}(x, X))$ of the link of x in X is even*

Example

X : $(z+\rho)^2 \rho^5 = x y^2 (x^2+y^2+2x\rho)^2$

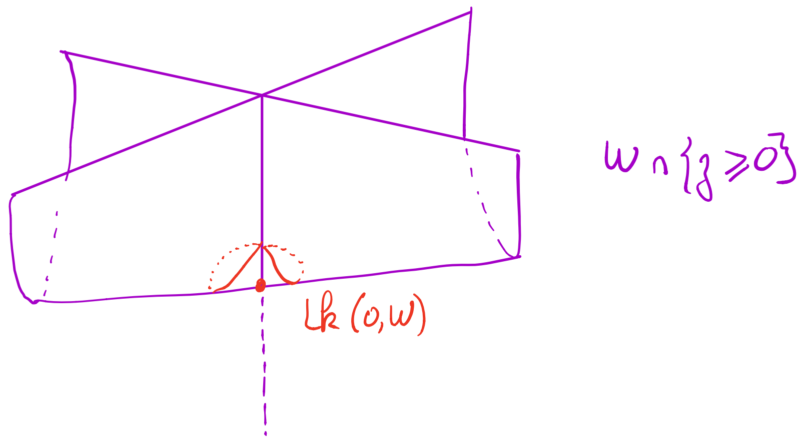
where $\rho = x^2+y^2+z^2$





	a	b	c	d	e	f
lk						
χ	0	2	4	2	−2	0

Counter-example via Whitney Umbrella W : $x^2 = zy^2$



Here the link at 0 is s.a homeomorphic to a figure eight:

Its Euler characteristics is:

$$\chi(\infty) = \chi(\infty) + \chi(\text{point})$$

$$= \chi(S^1 \setminus \{2 \text{ points}\}) + 1$$

$$= \chi(S^1) - 1$$

$$= -1 .$$

So $W \cap \{z \geq 0\}$ is not sa-homeomorphic to a real algebraic set.

Remark $\quad \text{OO} = S^1 \cup S^1$, $\quad S^1 \ni a \in S^1$

$$\chi(\text{OO}) = \chi(\infty)$$

This result leads to:

Definition 2.9 *Let A be a locally compact semialgebraic set. Then A is said to be* Euler *if, for every $x \in A$, the Euler characteristic of the link of x in A is even.*

In dimension ≤ 2, the Euler condition suffices to characterize topologically the real algebraic sets.

Theorem 2.10 (Akbulut-King, Benedetti-Dedo) *Let A be an Euler set of dimension at most 2. Then A is homeomorphic to a real algebraic set.*

In dimension 3, one need more conditions:

Theorem 3.22 (Akbulut-King) *A compact semialgebraic set S of dimension 3 is homeomorphic to a real algebraic set if and only if it is Euler and the four local obstructions*

$$\left(\int_{\mathrm{lk}(x,S)} \varphi_i \, d\chi \bmod 2\right)_{i=1,\dots,4}$$

defined above vanish everywhere on S.

It can be understood via algebraically constructible functions. (goal of the end of the lectures).

The analysis of the local obstructions given by the theory of algebraically constructible functions can be pushed further. In dimension 4, they give a total of $2^{43} - 43$ independent local obstructions!!!

Moreover, it is not known in this case whether the vanishing of these obstructions suffices to characterize topologically real algebraic sets of dimension 4...