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Real Algebraic Geometry

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REAL CLOSED FIELDS

Goal:

- study algebraically fields admitting an ordering
- consider the analog of algebraically closed fields.

I ORDERED FIELDS

transitive : $x \leq y \leq z \Rightarrow x \leq z$
anti-symmetric : $x \leq y$ and $y \leq x \Rightarrow x = y$
reflexive : $x \leq x$
total : $x \leq y$ or $y \leq x$

Definition 1.1.1. *An* ordering *of a field F is a total order relation $\leq$ satisfying:*

(i) $x \leq y \Rightarrow x + z \leq y + z$,

(ii) $0 \leq x,\ 0 \leq y \Rightarrow 0 \leq xy$.

An ordered field $(F, \leq)$ *is a field F, equipped with an ordering $\leq$.*

Remarks

① A square is positive : let $x \in F$. Then $x \geq 0$ or $-x \geq 0$ (total order). Then by (ii) $x^2 = (-x)^2 \geq 0$.

② -1 is negative. Actually 1 is a square, so $1 > 0$. Adding -1, by (i) we obtain $1 + (-1) > 0 + (-1)$ i.e. $-1 < 0$.

Examples ① $(\mathbb{Q}, \leq)$; $(\mathbb{R}_{alg}, \leq)$; $(\mathbb{R}, \geq)$

② On $\mathbb{Q}(\sqrt{2})$, one can define 2 orderings: one with $\sqrt{2} > 0$, another with $\sqrt{2} < 0$. But on $\mathbb{R}$ $\sqrt{2} = (\sqrt{\sqrt{2}})^2 > 0$.

Example 1.1.2. There is exactly one ordering of $\mathbb{R}(X)$ such that X is positive and smaller than any positive real number. If

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_k X^k \in \mathbb{R}[X] \quad \text{with } a_k \neq 0\,,$$

then $P(X) > 0$ for this ordering if and only if $a_k > 0$, and $P(X)/Q(X) > 0$ if and only if $P(X)Q(X) > 0$.

Check it defines an ordering!

Note that, with this ordering, the field $\mathbb{R}(X)$ is not archimedean. It contains infinitely small elements (i.e. positive and smaller than $1/n$, for every $n \in \mathbb{N}$ with $n \neq 0$), such as X, and also infinitely large elements (i.e. bigger than n, for every $n \in \mathbb{N}$) such as $1/X$.

Orderings on $\mathbb{R}(X)$ are in bijection with cuts in $\mathbb{R}$.

Given any ordering of $\mathbb{R}(X)$, X determines a *cut* (I, J) in $\mathbb{R}$ where $I = \{x \in \mathbb{R} \mid x < X\}$ and $J = \{x \in \mathbb{R} \mid X < x\}$.

List of cuts: $(\emptyset, \mathbb{R})$, $((-\infty, a), [a, \infty))$, $((-\infty, a], (a, +\infty))$ and $(\mathbb{R}, \emptyset)$

Performing, respectively, the change of variables $Y = -1/X$, $Y = a - X$, $Y = X - a$ and $Y = 1/X$, we get an ordering of $\mathbb{R}(Y)$ such that Y is positive and smaller than any positive real number.

There is exactly one such ordering.

$\rightsquigarrow$ Bijection between the set of orderings of $\mathbb{R}(X)$ and the set of cuts of $\mathbb{R}$.

Denote by $a_+, a_-, -\infty, +\infty$ the orderings determined by these cuts. Note that the sign of $f \in \mathbb{R}(X)$ for the ordering a_- is the sign of f on some small open interval $]a - \varepsilon, a[$.

Definition 1.1.3. *A* cone *of a field F is a subset P of F such that:*

(i) $x \in P,\ y \in P \Rightarrow x + y \in P$,

(ii) $x \in P,\ y \in P \Rightarrow xy \in P$,

(iii) $x \in F \Rightarrow x^2 \in P$.

The cone P is said to be proper *if in addition:*

(iv) $-1 \notin P$.

Examples

① $P = \sum F^2$ is always a cone.

$\sum F^2$ is not proper

It is included in any cone by (i) and (iii).

② If F admits an ordering $\leq$, then

$F_+ = \{x \in F : x \geq 0\}$

is a cone, called the positive cone of F.

Remark In general $\sum F^2 \neq F_+$, cf H17.

Proposition 1.1.5. *Let $(F, \leq)$ be an ordered field. The positive cone P of $(F, \leq)$ is a proper cone satisfying:*

(v) $P \cup -P = F$ *(where $-P = \{x \in F \mid -x \in P\}$).*

Conversely, if P is a proper cone of a field F satisfying (v)*, then F is ordered by*

$$x \leq y \Leftrightarrow y - x \in P\,.$$

Proof • $-1 < 0$ so F_+ is proper

• $F_+ \cup -F_+ = F$ because $\leq$ is total.

Conversely, the relation $\leq$ defined is

(i) • a total order relation

(ii) • an ordering

$x \leq y \iff y - x \in P$
P proper

(i) reflexive: $x - x = 0 = 0^2 \in P$
so $x \leq x$

transitive: $y - x \in P$ and $z - y \in P$
$\Rightarrow (y-x) + (z-y) \in P$ so $x \leq z$

anti-symmetric: assume $x \leq y \leq x$ but $x \neq y$.

Then $\left.\begin{array}{l} z = y - x \in P \\ -z = x - y \in P \end{array}\right\}$ so $-z^2 \in P$

But $\frac{1}{z^2} = \left(\frac{1}{z}\right)^2 \in P$ Hence $-1 = \frac{1}{z^2} \cdot (-z^2) \in P$

total: because P satisfies (v).

(ii) to be checked, but routine ...

□

<u>Remark</u> Under preceding assumptions "P proper and $P \cup -P = K$", then $P \cap -P = \{0\}$.

Actually if $x \in P \cap -P$, then $-x^2 \in P$
so $-1 = -x^2 \times \left(\frac{1}{x}\right)^2 \in P$ if $x \neq 0$.

Now we state a key lemma.

Key **Lemma 1.1.7.** *Let P be a proper cone of F.*

(i) *If $-a \notin P$ then $P[a] = \{x + ay \mid x, y \in P\}$ is a proper cone of F.*

(ii) *The cone P is contained in the positive cone of an ordering of F.*

$P[a]$ is a cone.

Proof. (i) Let us show that $-1 \notin P[a]$: if $-1 = x + ay$, with $x, y \in P$, then either $y = 0$ and $-1 \in P$, or $-a = \underbrace{(1/y)^2 y}_{1/y}(1 + x) \in P$. Both cases lead to a contradiction.

(ii) Using Zorn's lemma, there exists a maximal proper cone Q containing P. It is enough to show that $Q \cup -Q = F$. Let $a \notin Q$. By (i), $Q[-a]$ is a proper cone and, hence, $Q = Q[-a]$, since Q is maximal. This implies that $-a \in Q$. $\square$

using the previous proposition: • Q induces an ordering
• and $Q = F_+$ for this ordering
• $P \subseteq Q = F_+$

Now we characterize fields admitting an ordering.

Theorem 1.1.8. *Let F be a field. Then the following properties are equivalent:*

(i) *F can be ordered.*

(ii) *The field F has a proper cone.* P

(iii) $-1 \notin \sum F^2$.

(iv) *For every $x_1, \ldots, x_n$ in F*

$$\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_1 = \ldots = x_n = 0 .$$

Take F_+

$\sum F^2 \subseteq P$ and $-1 \notin P$

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) are easy. We show that (iii) $\Rightarrow$ (i) : if $-1 \notin \sum F^2$, it follows that $\sum F^2$ is a proper cone. Then use condition (ii) of Lemma 1.1.7. (Key Lemma). $\square$

Definition 1.1.9. *A field satisfying the properties of the preceding theorem is called a* real field.

ex $\mathbb{Q}(\sqrt{2})$.

It is worth noting that a real field always has characteristic 0.

$$0 = p = \underbrace{1 + \cdots + 1}_{p \text{ times}} = \underbrace{1^2 + \cdots + 1^2}_{p \text{ times}} \Rightarrow p = 0$$

Next result will be useful for H17

(and if no orderings $\Leftrightarrow$ $\sum_i F^2 = F$)

Proposition

Let F be a field of characteristic zero. Then $\sum_i F^2$ is the intersection of the positive cones for all orderings on F.

Proof
- We know $\sum_i F^2 \subseteq P$ for P any cone.
- Let $a \notin \sum F^2$; we are going to exhibit a positive cone not containing a.
- Note that $\sum_i F^2$ is proper, namely $-1 \notin \sum_i F^2$, otherwise

$$a = \frac{1}{4}\left(\underset{\substack{\cap\\ F^2}}{(1+a)^2} + \underset{\substack{\cap\\ \sum F^2}}{(-1)}\ \underset{\substack{\cap\\ F^2}}{(1-a)^2}\right) \in \sum_i F^2 .$$

- Then $\sum_i F^2[-a]$ is a proper cone by Key Lemma.
- By Key Lemma again, $\sum_i F^2[-a]$ is included in the positive cone F_+ for some ordering. So $a \notin F_+$.

$\square$

Remarks

① In other words, if F is real:

$x \notin \sum F^2 \Rightarrow$ there exists an ordering such that $x < 0$

② Assume F admits a unique ordering. Then

$F_+ = \sum F^2$

II REAL CLOSED FIELDS

Definition 1.2.1. *A* real closed field *F is a real field that has no nontrivial real algebraic extension $F_1 \supset F$, $F_1 \neq F$.*

Counter-example: $\mathbb{Q} \hookrightarrow \mathbb{Q}[\sqrt{2}]$ is algebraic, and $\mathbb{Q}[\sqrt{2}]$ admits orderings so is a real field.

So $\mathbb{Q}$ is not real closed.

Real closed fields admits a unique ordering.

Proposition Let R be a real closed field. Then:

- $R_+ = \sum R^2 = R^2$
- there exists a unique ordering on R.

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Proof

FIRST STEP: $R \setminus R^2 \subseteq -\sum_i R^2$ (to be proved)

SECOND STEP: $R^2 = \sum_i R^2$

Let $x \in \sum_i R^2 \setminus R^2 \subseteq -\sum_i R^2$. Then $x = -\sum_i z_i^2 = +\sum_j y_j^2$

so that

$$\sum_j y_j^2 + \sum_i z_i^2 = 0$$

and therefore $y_j = z_i = 0 \quad \forall i,j$ since R is real.

THIRD STEP: existence of a unique ordering.

We know
- $R^2 \cup -R^2 = R$ (since $R \setminus R^2 \subseteq -\sum_i R^2 = -R^2$)
- $-1 \notin R^2 = \sum_i R^2$ since R is real

So R^2 is a proper cone

By Key Lemma, a proper cone can be enlarge to become a positive cone (for some ordering).

- Recall how to enlarge: $R^2 \subseteq R^2[a]$ where $-a \notin R^2$. But $-a \notin R^2 \Rightarrow a \in R^2$, so $R^2[a] = R^2$.

In particular, there is a unique ordering on R, and its positive cone is R^2.

TO FINISH: proof of the first step $R \setminus R^2 \subseteq -\sum R^2$

Let $a \in R \setminus R^2$. Consider the algebraic extension:

$$R \hookrightarrow \frac{R[X]}{(X^2 - a)} = R[\sqrt{a}]$$

Then $R(\sqrt{a})$ cannot be real since R is real closed, therefore $-1 \in \sum R(\sqrt{a})^2$:

$$\underset{\substack{\cap \\ R}}{-1} = \sum_i (\underset{\substack{\cap \\ R}}{x_i} + \sqrt{a}\, \underset{\substack{\cap \\ R}}{y_i})^2$$ with $x_i, y_i \in R$ and at least one $y_i \neq 0$.

$$= \underbrace{\sum_i x_i^2 + a \sum_i y_i^2}_{\in R} + 2\sqrt{a} \underbrace{\sum_i x_i y_i}_{\in R}$$

Thus
$$\begin{cases} -1 = \sum_i x_i^2 + a \sum_i y_i^2 & (*) \\ \sum_i x_i y_i = 0 & \\ \sum_i y_i^2 \neq 0 & (\text{at least one } y_i \neq 0 \text{ and } R \text{ real}) \end{cases}$$

Then

$$(*) \qquad -a = \frac{1 + \sum_i x_i^2}{\sum_i y_i^2} = \frac{(1 + \sum_i x_i^2)(\sum_i y_i^2)}{(\sum_i y_i^2)^2} \in \sum_i R^2.$$

We have just proved:

$$a \in R \setminus R^2 \implies -a \in \sum_i R^2$$

$\square$

Next result will be important for the uniqueness of the real closure of a real field.

<u>Lemma</u> Let F be a real field.
Let $F \longrightarrow R$ be an real algebraic extension with R real closed (meaning the ordering on R extends that on F).
Then any F-automorphism of R is trivial.

<u>Proof</u>: • Let $\phi : R \longrightarrow R$ be a F automorphism.
Then ϕ respects the ordering: $\phi(R^2) = \phi(R)^2$ so $\phi(R_+) = R_+$ since R is real closed.

• Let $a \in R$. Then $F \longrightarrow F(a)$ is algebraic by assumption. Let $P \in F[X]$ be the minimal polynomial of a.

Then ϕ * sends a root of P on a root of P
* respects the ordering
so ϕ stabilizes the root of P. Thus $\phi(a) = a$. □

We are in position to give a characterization of real closed fields:

Theorem 1.2.2. *Let F be a field. Then the following properties are equivalent:*

(i) *The field F is real closed.*

(ii) *There is a unique ordering of F (whose positive cone is the set of squares of F) and every polynomial of $F[X]$, of odd degree, has a root in F.*

(iii) *The ring $F[i] = F[X]/(X^2+1)$ is an algebraically closed field.*

Examples ① $\mathbb{R}$!!!

② $\mathbb{R}_{alg} = \mathbb{Q}_{alg} \cap \mathbb{R}$: adding i gives the algebraic closure $\mathbb{Q}_{alg}$ of $\mathbb{Q}$.

③ $\mathbb{R}((t^{\frac{1}{\mathbb{N}}}))$ real Puiseux series.

$= \left\{ \sum_{k \geq p} a_k t^{k/q} , \; p \in \mathbb{Z}, q \in \mathbb{N}, a_k \in \mathbb{R} \right\}$

Actually $\mathbb{C}((t^{\frac{1}{\mathbb{N}}}))$ is the algebraic closure of $\mathbb{C}((t))$

and $\mathbb{C}((t^{\frac{1}{\mathbb{N}}})) = \mathbb{R}((t^{\frac{1}{\mathbb{N}}}))[i]$

④ Replacing $\mathbb{C}((t))$ by $\mathbb{C}(t)$ gives rise to $\mathbb{C}((t^{\frac{1}{\mathbb{N}}}))_{alg}$ and then $\mathbb{R}((t^{\frac{1}{\mathbb{N}}}))_{alg}$: Puiseux series satisfying a polynomial equation.

Remark that • there exist several orderings on $\mathbb{R}(t)$

• but in $\mathbb{R}((t^{\frac{1}{\mathbb{N}}}))$: • $t > 0$ since $t = (t^{\frac{1}{2}})^2$

• $t^{\frac{1}{2}} > 0$ since $t^{\frac{1}{2}} = (t^{1/4})^2$

etc ...

Note $\mathbb{R}_{alg} \subseteq \mathbb{R} \subseteq \mathbb{R}((t^{\frac{1}{\mathbb{N}}}))_{alg} \subseteq \mathbb{R}((t^{\frac{1}{\mathbb{N}}}))$

not connected archimedean ... not connected, not archimedean

Theorem 1.2.2. *Let F be a field. Then the following properties are equivalent:*

(i) *The field F is real closed.*

(ii) *There is a unique ordering of F whose positive cone is the set of squares of F and every polynomial of $F[X]$, of odd degree, has a root in F.*

(iii) *The ring $F[i] = F[X]/(X^2+1)$ is an algebraically closed field.*

Proof of (i) $\Rightarrow$ (ii) The uniqueness comes from a proposition.

- It remains to show that, if $f \in F[X]$ has odd degree, then f has a root in F. If this is not the case, let f be a polynomial of odd degree $d > 1$ such that every polynomial of odd degree $< d$ has a root in F. Since a polynomial of odd degree has at least one odd irreducible factor, f is irreducible since it has not root in F.

- $F \longrightarrow \frac{F[X]}{(f)} = K$ is a non trivial algebraic extension of the real closed field F, so it cannot be real. Therefore $-1 \in \Sigma K^2$

i.e.

$$-1 = \sum_{i=1}^{n} h_i^2 + fg \quad \text{with} \quad \deg(h_i) < d\,.$$

Since the term of highest degree in the expansion of $\sum_{i=1}^{n} h_i^2$ has a coefficient which is a sum of squares and F is real, $\sum_{i=1}^{n} h_i^2$ is a polynomial of even degree $\leq 2d-2$. The polynomial g is, hence, of odd degree $\leq d-2$ and has a root x in F. But then $-1 = \sum_{i=1}^{n} h_i(x)^2$, which contradicts the fact that F is real.

Theorem 1.2.2. *Let F be a field. Then the following properties are equivalent:*

(i) *The field F is real closed.*

(ii) *There is a unique ordering of F whose positive cone is the set of squares of F and every polynomial of $F[X]$, of odd degree, has a root in F.*

(iii) *The ring $F[i] = F[X]/(X^2+1)$ is an algebraically closed field.*

Proof of (ii) $\Rightarrow$ (iii)

- X^2+1 is irreducible on F so $F[i]$ is a field.
- First we deal with polynomials in $F[X]$, then in $F[i][X]$.

Let $f \in F[X]$ of degree $d = 2^m n$ with n odd. Let us show by induction on m that f has a root in $F[i]$. For $m = 0$, we know that f has a root in F. Let us suppose, now, that the result is true for $m-1$.

Let $y_1, \dots, y_d$ be the roots of f in an algebraic closure of F and define

$$g_h = \prod_{\lambda<\mu} (X - y_\lambda - y_\mu - h y_\lambda y_\mu) \,, \quad \text{for } h \in \mathbb{Z} \,.$$

$n' = n(2^m n - 1)$

The polynomial g_h is symmetric in $y_1, \dots, y_d$ and, hence, $g_h \in F[X]$. The degree of g_h is $d(d-1)/2 = 2^{m-1} n'$ with n' odd. By induction, g_h has a root in $F[i]$ and, hence, there exist λ and μ with $y_\lambda + y_\mu + h y_\lambda y_\mu \in F[i]$.

Letting h range over $\mathbb{Z}$, we see that there exist λ and μ with $y_\lambda + y_\mu \in F[i]$ and $y_\lambda y_\mu \in F[i]$. These elements y_λ and y_μ are the solutions of a quadratic equation with coefficients in $F[i]$, which has its two solutions in $F[i]$ (proceed as for $\mathbb{C}$). The polynomial f has, thus, a root in $F[i]$.

Suppose now that $f \in F[i][X]$. Let $\overline{f}$ be the polynomial obtained by replacing the coefficients of f with their conjugates. Since $f\overline{f} \in F[X]$, $f\overline{f}$ has a root x in $F[i]$. Then either x is a root of f, or it is a root of $\overline{f}$, and in this case, its conjugate $\overline{x}$ is a root of f.

Theorem 1.2.2. *Let F be a field. Then the following properties are equivalent:*

(i) *The field F is real closed.*

(ii) *There is a unique ordering of F whose positive cone is the set of squares of F and every polynomial of $F[X]$, of odd degree, has a root in F.*

(iii) *The ring $F[i] = F[X]/(X^2+1)$ is an algebraically closed field.*

Proof of (iii) $\Rightarrow$ (i)

- F is real : let prove $-1 \notin \sum F^2$.
  - first $-1 \notin F^2$ otherwise $F[i]$ would not be a field.
  - second $F^2 = \sum F^2$. Actually if $a, b \in F$, then $a^2+b^2 = (a+ib)(a-ib)$

  Since $F[i]$ is algebraically closed, $a+ib$ is a square in $F[i]$ so $a+ib = (c+id)^2$

  and then $a-ib = (c-id)^2$.

  Finally $a^2+b^2 = (c^2+d^2)^2 \in F^2$. Then induction...

- Let $F \to K$ be an algebraic extension of F.

If -1 is a square in K, then $K = F[i]$ (since $F[i]$ is alg. closed). If not

$$\begin{tikzcd} F \arrow[r] \arrow[d] & K \arrow[d] \\ F[i] \arrow[r] & K[i] \end{tikzcd}$$

alg. extension, thus trivial

so $F \to K$ is trivial.

$\square$

III ANALYSIS ON A REAL CLOSED FIELD

One can review several classical results of analysis.

- Intermediate value theorem

Proposition 1.2.4. *Let R be a real closed field, $f \in R[X]$, $a, b \in R$ with $a < b$. If $f(a)f(b) < 0$, then there exists x in $]a, b[$ such that $f(x) = 0$.*

Proof. Since $R[i]$ is algebraically closed, the irreducible factors of f are linear, or have the form $(X-c)^2 + d^2 = (X - c - id)(X - c + id)$. If $f(a)$ and $f(b)$ have opposite signs, then $g(a)$ and $g(b)$ have opposite signs for some linear factor g of f. Hence the root of g is in $]a, b[$. $\square$

- Rolle theorem

Proposition 1.2.5. *Let R be a real closed field, $f \in R[X]$, $a, b \in R$ with $a < b$ and $f(a) = f(b) = 0$. Then the derivative polynomial f' has a root in $]a, b[$.*

Proof. We can suppose that a and b are two consecutive roots of f, i.e. that f never vanishes in $]a, b[$. Then

$$f = (X-a)^m (X-b)^n g \ ,$$

where g never vanishes in $[a, b]$. Hence, by Proposition above, g has constant sign on $[a, b]$. Then

$$f' = (X-a)^{m-1}(X-b)^{n-1} g_1$$

where

$$g_1 = m(X-b)g + n(X-a)g + (X-a)(X-b)g' \ .$$

Thus, $g_1(a) = m(a-b)g(a)$ and $g_1(b) = n(b-a)g(b)$, hence $g_1(a)$ and $g_1(b)$ have opposite signs. By Proposition above, g_1 has a root in $]a, b[$ and so does f'. $\square$

Mean value theorem

Corollary 1.2.6. *Let R be a real closed field, $f \in R[X]$, $a, b \in R$ with $a < b$. There exists $c \in]a, b[$ such that $f(b) - f(a) = (b-a)f'(c)$.*

Corollary 1.2.7. *Let R be a real closed field, $f \in R[X]$, $a, b \in R$ with $a < b$. If the derivative f' is positive (resp. negative) on $]a, b[$, then f is strictly increasing (resp. strictly decreasing) on $[a, b]$.*

III REAL CLOSURE

Definition Let $(F, \leq)$ be an ordered field. An algebraic extension $F \to R$ is a real closure if:

- R is a real closed field
- the (unique) ordering on R extends the (given) ordering on F.

Theorem Any ordered field F admits a real closure. The real closure is unique up to a unique F-automorphism.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & R' \\ R \arrow[ur, dashed] & \end{tikzcd}$$

$\rightsquigarrow$ there exists an unique F-automorphism

Remark: the uniqueness is stronger than the uniqueness of the algebraic closure.

Idea of the proof : close to the case of algebraic closure

→ existence via Zorn Lemma : take a maximal extension between real extensions included in an algebraic closure

→ check this maximal extension is real closed.

→ uniqueness : a bit more work because of the order.

□

Examples

① $\overline{\mathbb{Q}} = \mathbb{R}_{alg}$

② $\overline{(\mathbb{R}(X), 0^+)} = \mathbb{R}((X^{\frac{1}{\mathbb{N}}}))_{alg}$

$X > 0$ and $X < r$ $\forall r \in \mathbb{R}^*_+$

Actually . $X > 0$ since $X = (X^{\frac{1}{2}})^2$

$(r>0)$. $r - X = \left(\sqrt{r - X}\right)^2 = \left(\sqrt{r} - \frac{\sqrt{r}}{2}X + \dots\right)^2 > 0$ so $X < r$

so the ordering on $\mathbb{R}((X^{\frac{1}{\mathbb{N}}}))_{alg}$ extends 0^+.

IV COMING BACK TO SEMI-ALGEBRAIC SETS

- (essentially) All what we discussed about semi-algebraic sets in $\mathbb{R}^n$ is valid on any real closed field.
- Exceptions: • connexity • compactness (more details soon)

We have a useful version of Tarski-Seidenberg principle in this context :

↳ H17

Theorem

- F a real field
- $f_1(T,X), \dots, f_\ell(T,X) \in F[T, X_1, \dots, X_n]$

$$S(T,X) = \begin{cases} f_1(T,X) \ ?\ 0 \\ \vdots \\ f_\ell(T,X) \ ?\ 0 \end{cases}$$

There exists a boolean combination $\mathcal{C}(X)$ of polynomial equalities and inequalities in $F[X_1, \dots, X_n]$ such that for any real closed field R containing F, and every $x \in R^n$:

"the system $S(T,x)$ has a solution in R"

$\iff$ " $\mathcal{C}(x)$ holds true in R "

- A few more remarks about topology : 1. Connexity

- $\mathbb{R}_{alg}$ is not connected : $(-\infty, \pi)$ is open and closed
- $\mathbb{R}((t^{\frac{1}{\mathbb{N}}}))$ idem : the set of infinitely small elements is open and closed.

BUT: these "strange" sets are not semi-algebraic!

$\rightsquigarrow$ One can define a semi-algebraic version of connexity.

Definition $A \subseteq R^n$ semi-alg. A is sa-connected if for any $F_1, F_2 \subseteq A$ closed semi-alg. sets and disjoint, then

$$A = F_1 \cup F_2 \implies A = F_1 \text{ or } A = F_2$$

Example: the sa-connected semi-alg. subsets of R are the intervals.

Lemma $(0,1)^d \subseteq R^d$ is sa-connected.

Proof Otherwise $(0,1)^d = F_1 \cup F_2$ with $F_i \subseteq (0,1)^d$ closed sa with $(0,1)^d \neq F_i$ for $i = 1, 2$.

Take $x_1 \in F_1 \setminus F_2$ and $x_2 \in F_2 \setminus F_1$ and consider the segment $S = [x_1, x_2]$.

Then $S = (S \cap F_1) \cup (S \cap F_2)$ is not sa connected, Contradiction.

□

Using this:

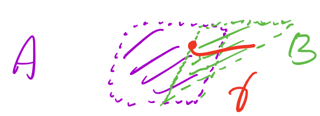
- define the notion of sa-connected components
- decompose a s.a set into union of sa-connected components.
- if $R = \mathbb{R}$, sa-connexity = Euclidean connexity

Remark: one can even talk about path connected sa connexity. Using Curve Selection Lemma, one can show that it is the same notion as sa-connexity (for any R).

Idea: * path sa connected $\Rightarrow$ sa connected : as usual.

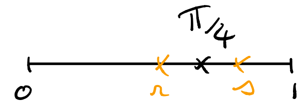
* converse : • $(0,1)^d$ is sa path connected
  • if $\underset{\subseteq (0,1)^d}{A}$, $\underset{\subseteq (0,1)^e}{B}$ sa with $A \cap \overline{B} \neq \emptyset$

then $A \cup B$ path sa-connexe via Curve selection lemma



② Compactness

- $[0\ 1]$ not compact in $\mathbb{R}_{alg}$



$$[0,1] = \bigcup_{0<r<\frac{\pi}{4}} [0,r) \cup \bigcup_{\frac{\pi}{4}<s<1} (s,1]$$ infinite covering

- $[0\ 1]$ not compact in $\mathbb{R}((t^{\frac{1}{\mathbb{N}}}))$: similar phenomenon around the cut.

So that a closed and bounded semi-alg set is not necessarily compact. The useful notion will be that of "closed and bounded" semi-alg. set.

For instance:

- The image of a closed and bounded sa-set by a continuous sa-application is closed and bounded.