A problem, perhaps non-impossible

Conjecture

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz and subanalytic. Then there exists a (uniformly Lipschitz) sequence $f_1, f_2, \ldots$ of PL functions converging to f locally uniformly such that

$$\mathrm{mass}(\Gamma_{df_i} \cap \pi^{-1}(K)) \leq C(K) < \infty$$

for every compact $K \subset \mathbb{R}^2$.

This is true if $f \in C^2$, but the local mass bounds depend on the local C^2 norms of f.

Two geometric applications

Main goals today:

- The Gauss-Bonnet theorem for complete asymptotically conic subsets of $\mathbb{R}^n$ (Dillen-Kühnel, Dutertre)
- Langevin's formula for the total curvature of a complex analytic hypersurface in the neighborhood of an isolated singularity

Ancillary goals:

- Valuations and Integral geometry of S^n
- The normal cycle of a transverse intersection
- Decomposition of the normal cycle of a complex analytic variety

Theorem (Dutertre 2008)

Suppose $X \subset \mathbb{R}^{n+1}$ is semialgebraic. Put $\mathrm{Lk}^\infty(X)$ for the constructible function on S^n given by specializing the family $\{S^n \cap tX\}_{t>0}$ at $t = 0$. Then

$$\int_{N(X)} \kappa_0 = \chi(X) - \frac{1}{2}\chi(\mathrm{Lk}^\infty(X)) - \frac{1}{2}\int_{\mathrm{Gr}_n} \chi(\mathrm{Lk}^\infty(X \cap H))\, dH \qquad (1)$$

where dH is the probability measure on the Grassmannian Gr_n.

Theorem (Langevin 1979)

If $V \subset \mathbb{C}^n$ has an isolated singularity at 0 and V_ϵ is a smoothing of V then

$$\lim_{r\downarrow 0}\lim_{\epsilon\downarrow 0}\int_{V_\epsilon \cap B(0,r)} K = (-1)^{n-1}(\mu_n + \mu_{n-1}) \qquad (2)$$

Integral geometry of S^n

Recall $\exp : S\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \ \exp(x, v, t) := x + tv$

$$\exp^*(d\,\mathrm{vol}) \equiv dt \wedge \sum_{i=0}^{n-1} \kappa_i \tag{3}$$

In fact

$$\frac{\Omega^{n-1}(S\mathbb{R}^n)^{\overline{SO(n)}}}{(\alpha, d\alpha)} = \langle \kappa_0, \dots, \kappa_n \rangle \simeq \frac{\Omega^{n-1}(SS^n)^{SO(n+1)}}{(\alpha, d\alpha)} \tag{4}$$

(The actions of the isotropy subgroup $SO(n-1)$ on $T_{x,v}S\mathbb{R}^n$, $T_{x,v}SS^n$ are isomorphic.) For $A \subset \mathbb{R}^n$ or S^n define the invariant valuations

$$\mu_n := c_n \,\mathrm{vol}, \quad \mu_i(A) := c_i \int_{N(A)} \kappa_i, \ i = 0, \dots, n-1 \tag{5}$$

for appropriate constants c_i.

If A is a smooth domain bounded by M^{n-1} then

$$\mu_i(A) = c_i \int_M K_{n-i-1}$$

where K_j is the jth elementary symmetric function of the principal curvatures of M.
In the case of the ambient space S^n, approximating by small tubes yields (after renormalizing)

$$\mu_i(S^j) = \delta^i_j, \quad i, j = 0, \ldots, n$$

$$\text{and} \quad \chi = 2 \sum_{0 \le 2i \le n} \mu_{2i}$$

by Chern-Gauss-Bonnet.

Theorem (Blaschke's kinematic formula)

For nice subsets $A, B \subset S^n$ and $\ell = 0, \ldots, n$

$$\int_{SO(n+1)} \mu_\ell(A \cap gB)\, dg = \sum_{i+j=n+\ell} \mu_i(A)\mu_j(B) \quad \square \tag{6}$$

Corollary

For nice $A \subset S^n$

$$\mu_j(A) = \int \mu_0(A \cap gS^{n-j})\, dg \tag{7}$$

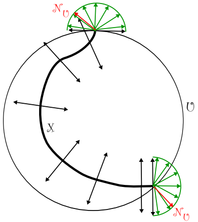
The transverse intersection formula

Suppose $X \subset \mathbb{R}^n$ is a nice set, $U \subset \mathbb{R}^n$ is a smooth compact domain and X meets ∂U transversely. Then

$$N(X \cap U) = N(X) \llcorner U + (N(X) \cap \partial U) \divideontimes n_U$$

If X meets ∂U orthogonally then this can also be expressed

$$N(X \cap U) = N(X) \llcorner U + (N_{\partial U}(X \cap \partial U)) \divideontimes n_U$$



The Dillen-Kühnel-Dutertre formula

Theorem (Dutertre version)

Suppose $X \subset \mathbb{R}^{n+1}$ is semialgebraic. Put $\mathrm{Lk}^\infty(X)$ for the constructible function on S^n given by specializing the family $\{S^n \cap tX\}_{t>0}$ at $t = 0$. Then

$$\mu_0(X) = \int_{N(X)} \kappa_0 = \chi(X) - \frac{1}{2}\chi(\mathrm{Lk}^\infty(X)) - \frac{1}{2}\int_{\mathrm{Gr}_n} \chi(\mathrm{Lk}^\infty(X \cap H))\, dH \quad (8)$$

where dH is the probability measure on the Grassmannian Gr_n.

Theorem (Dillen-Kühnel version)

Suppose $M^n \subset \mathbb{R}^{n+1}$ is a smooth hypersurface with finitely many asymptotically conic ends, and put $M^\infty \subset S^n$ for the link at ∞. Then

$$\mu_0(M) = \int_{N(M)} \kappa_0 = \chi(M) + \sum_{0 \le 2i \le n} c_i \int_{M^\infty} K_{2i} \tag{9}$$

The gradient of μ_0 with respect to a vector field of $\mathbb{R}^{n+1}$ that is asymptotic to a vector field ξ on S^n_∞ is

$$\delta_\xi \mu_0(M) = c \int_{M^\infty} \langle \xi, n \rangle K_n \tag{10}$$

Conjecture (D-K)

If M is stationary with respect to μ_0 then $\mu_0(M) \in \mathbb{Z}$.

Proof:

As $t \downarrow 0$, the family $X_t := tX$ specializes to ϕ with

$$\phi(tx) \equiv \phi(x) \text{ for } t > 0, \quad \phi|_{S^n} = \mathrm{Lk}^\infty(X)$$

By the transverse intersection formula, if $B \subset \mathbb{R}^n$ is the open unit ball then for small $t > 0$

$$\begin{aligned} N(\phi \cdot 1_{\overline{B}}) &= N(\phi) \llcorner \pi^{-1} B + N_{S^n}(\phi|_{S^n}) \divideontimes n_B \\ \implies \chi(X) = \chi(X_t \cap \overline{B}) &= N(\phi \cdot 1_{\overline{B}})(\kappa_0) \\ &= \left(N(\phi) \llcorner \pi^{-1} B + N_{S^n}(\phi|_{S^n}) \divideontimes n_B \right) (\kappa_0) \\ &= \mu_0(X) + \sum_{i=0}^{n} c_i \mu_i(\phi|_{S^n}) \end{aligned}$$

for some constants c_i. In fact the $c_i \equiv 1$: if $X = \mathbb{R}^{k+1}$ then $\phi = 1_{S^k}$.

- To get the Dillen-Kühnel variation formula (10) observe that only the last term in (8) can change in the course of a smooth variation. Furthermore the variation in this integral occurs around the hyperspheres H that are tangent to M_∞. Hence this may be identified with the (signed) $(n-1)$-dimensional measure of the set of such hyperplanes (spherical Gauss map), which corresponds to $\mu_0(M_\infty) = c \int_{M_\infty} K_n$.
- Finally, we can prove the Dillen-Kühnel conjecture: if M is stationary then the closed set of hyperspheres tangent to M_∞ has $(n-1)$-dimensional measure zero. Such a set cannot separate the space of all hyperspheres. So $H \mapsto \chi(M_\infty \cap H)$ is constant (and even) a.e.

The (co)normal cycle of a complex variety

Proposition

Let M be a smooth $\mathbb{C}$-analytic manifold, $\pi : S^*M \to \mathbb{P}T^*M$ the Hopf fibration from its cosphere bundle to its projectivized cotangent bundle, $V \subset M$ a $\mathbb{C}$-analytic subvariety. Then π gives a fibration of $N^*_M(V)$ over a cycle $\mathbb{P}N^*M(V)$, supported on an analytic subvariety of $\mathbb{P}T^*M$. The irreducible components of this subvariety have the form $\mathbb{P}N^*_M(W)$, where the W are open strata of a stratification $\mathcal{S}$ of V. Thus

$$\mathbb{P}N^*_M(V) = \sum_{W \in \mathcal{S}} d^V_W [\![\mathbb{P}N^*_M(W)]\!] \tag{11}$$

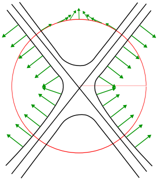
for some $d^V_W \in \mathbb{Z}$.

Theorem

Suppose $V = \bigcap_{i=1}^N f_i^{-1}(0) \subset \mathbb{C}^n$ *is a cone with vertex* 0*. Put* $g := \sum_{i=1}^N |f_i|^2$*. Then the map*

$$\frac{\nabla g}{|\nabla g|} : S^{2n-1} \to S^{2n-1}$$

a well-defined degree. This degree is $d^V_{\{0\}}$*.*



This is true in a limiting sense even if V is not a cone. Taking normal sections of strata at generic points, it gives (in principle) a recipe for computing all the coefficients d^V_W.

$d^V_{\{0\}}$ as a Milnor number

Now suppose that $V \subset \mathbb{C}^n$ has an isolated singularity at 0. If V_t is a smoothing of V then in the neighborhood of 0 the family $\{V_t\}$ specializes to

$$1_V + (-1)^{n-1}\mu \cdot 1_{\{0\}}$$

where $\mu = \mu_n =$ the **Milnor number** of the singularity.
For generic $\mathbb{C}$-subspaces P^k the section $V \cap P$ again has an isolated singularity at 0. Put μ_k for their common Milnor number.

Proposition

$$d^V_{\{0\}} = (-1)^n \mu_{n-1}$$

Proof:

By the Morse-theory of height functions, for generic $v \in S^{2n-1}$

$$d^V_{\{0\}} = \chi((V \cap J_\epsilon \cap B(0, \delta)) - 1 \quad (12)$$

for $0 < \epsilon \ll \delta \ll 1$, where $J_\epsilon := h_v^{-1}(\epsilon)$.
On the other hand, for generic linear functions $\lambda : \mathbb{C}^n \to \mathbb{C}$ the family $H_\epsilon \cap V$ is a smoothing of $H_0 \cap V$, where $H_\epsilon := \lambda^{-1}(\epsilon)$, so

$$\mu_{n-1} = (-1)^{n-1} (\chi(V \cap H_\epsilon \cap B(0, \delta)) - 1) \quad (13)$$

Claim: if $h_v = \operatorname{Re} \lambda$ then

$$\chi(V \cap H_\epsilon \cap B(0, \delta)) = \chi(V \cap J_\epsilon \cap B(0, \delta)) \quad (14)$$

To prove the Claim it is enough to show that $\operatorname{Im}\lambda$ has no critical points in $V \cap J_\epsilon$ near 0— then we can flow without obstruction to $V \cap H_\epsilon$. Otherwise for some p near 0

$$\begin{aligned} & T_p(V \cap J_\epsilon) \subset \ker \operatorname{Im}\lambda \\ \implies & T_p(V \cap J_\epsilon) \subset \ker \lambda \quad (\text{since } T_p J_\epsilon = \ker \operatorname{Re}\lambda) \\ \implies & T_p V = \operatorname{span}_{\mathbb{C}} T_p(V \cap J_\epsilon) \subset \ker \lambda \end{aligned}$$

i.e. $p \in \operatorname{crit}(\lambda|_V)$. But there are no such points p near 0. $\square$

Langevin's formula

Theorem (Langevin 1979)

If $V \subset \mathbb{C}^n$ has an isolated singularity at 0 and V_ϵ is a smoothing of V then

$$L := \lim_{r \downarrow 0} \lim_{\epsilon \downarrow 0} \int_{V_\epsilon \cap B(0,r)} K = (-1)^{n-1}(\mu_n + \mu_{n-1}) \tag{15}$$

Proof.

$$\begin{aligned} & N(V_\epsilon \cap \overline{B}(0,r)) \llcorner \partial B(0,r) \rightharpoonup N(V \cap \overline{B}(0,r)) \llcorner \partial B(0,r) \\ \Longrightarrow \ 1 - (1 + (-1)^{n-1})\mu_n &= \left[N(V \cap \overline{B}(0,r)) - N(V_\epsilon \cap \overline{B}(0,r)) \right] (\kappa_0) \\ &\sim [(N(V) - N(V_\epsilon)) \llcorner \pi^{-1}(B(0,r))](\kappa_0) \\ &\sim d^V_{\{0\}} - L = (-1)^{n-1}\mu_{n-1} - L \end{aligned}$$

□