Université de Rennes 1 Master 2, 2020

Real algebraic geometry ; homework

For february 12th

Exercice 1.

(1) Is the exponential function $exp : \mathbb{R} \to \mathbb{R}$ semi-algebraic ?

Consider the ring $\mathcal{F}$ of germs at the origin $0 \in \mathbb{R}^n$ of C^∞ and semialgebraic functions with values in $\mathbb{R}$ (namely these functions are defined on some semialgebraic neighborhood, depending of the function, of 0 in $\mathbb{R}^n$, have values in $\mathbb{R}$, and are of class C^∞ and semialgebraic).

(2) Let $f \in \mathcal{F}$. Prove that the function $g : [0, \infty) \to \mathbb{R}$ defined by

$$g(r) = \sup\{|f(x)|;\ ||x|| \leq r\}$$

is semi-algebraic.

Consider the morphism $\phi : \mathcal{F} \to \mathbb{R}[[X_1, \ldots, X_n]]$ assigning to a function in $\mathcal{F}$ its Taylor series at the origin.

(3) Prove that ϕ is injective.

Exercice 2.

Let F be an ordered field and K be a field extension of F.

(1) Let P be a cone in K. Show that P is the intersection of the positive cones K_+ of all orderings in K with $P \subset K_+$ (the intersection being K if there exist no such ordering).

(2) Show that

$$C = \{a_1 b_1^2 + \cdots + a_r b_r^2; a_i \in F_+, b_i \in K\}$$

is a cone in K.

(3) Show that K has an ordering which extends the ordering on F if and only if $-1 \notin C$.

(4) Show there exists an ordering on $F(X_1, \ldots, X_n)$ extending the ordering on F.

(5) How many orderings do exist on $\mathbb{Q}$?