Definitions of singularities of the MMP

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1 Some local properties of rings

All rings will be assumed commutative with unit. In this section we list some properties of local rings. For a property $\mathcal{P}$, we say that a ring/scheme has the property $\mathcal{P}$ if its local rings have $\mathcal{P}$.

Definition 1.1. A domain R is normal if integrally closed in its field of fractions.

Example 1.2. Unique factorization domains are an example of normal domains. Nonsingular varieties are examples of normal varieties. $\operatorname{Spec} k[x,y,u]/(xy-u^2)$ is also normal, although not UFD.

The following is Serre's criterion for normality:

Fact 1.3. *A noetherian ring is normal if and only if it is regular in codimension* 1 *and* S_2 *(that is, functions defined outside codimension* 2 *extend to the whole space).*

We will use the following theorem:

Theorem 1.4 ([Sta16], TAG 0EBJ)**.** *Let X be a noetherian integral scheme and $\iota\colon U \hookrightarrow X$ an open such that $X \setminus U$ has codimension ≥ 2. Then ι^* and ι_* induce an equivalence between the categories of reflexive coherent sheaves on U and X.*

Definition 1.5. A local ring is called *factorial* if it is a unique factorizations domain.

Fact 1.6. *Let X be a noetherian normal scheme; let $Div(X)$ be the group of Cartier divisors, and $Z^1(X)$ the group of Weil divisors. There is a natural injective group homomorphism*

$$Div(X) \to Z^1(X) \tag{1}$$

which is an isomorphism if X is factorial.

The homomorphism associates to a local representative f of a Cartier divisor (hence a rational function) its multiplicity at codimension 1 points. The surjectivity part of the above fact is a consequence of the fact that in a UFD, every prime of height one is principal.

Definition 1.7. Let R be a noetherian local ring, M an R-module. The *depth* of M is the supremum of the lengths of sequences $f_1, \dots, f_r \in \mathfrak{m}$ such that f_i is a non-zero divisor in $R/(f_1, \dots, f_{i-1})$.

We have the inequality $\mathrm{depth}(M) \leq \dim(\mathrm{Supp}(M))$. Moreover, the depth of M is the smallest integer for which $\mathrm{Ext}^i_R(k, M) \neq 0$.

Definition 1.8. Let R be a noetherian local ring; we say that R is *Cohen-Macaulay* (CM) if $\mathrm{depth}(R) = \dim R$.

Fact 1.9.

- *A CM ring locally of finite type is equidimensional (for every minimal prime* $\mathfrak{p}$*,* $\dim A/\mathfrak{p} = \dim A$*).*
- *if a CM k-scheme is generically reduced, then it is reduced. In particular it does not have embedded components.*
- *Any* 1*-dimensional reduced scheme is CM.*
- *Any* 2*-dimensional normal scheme is CM.*

Example 1.10. Let k be a field.

- the ring $k[x, y]/(x^2, xy)$ is not CM, as there is an embedded point.
- the ring $k[x, y, z]/xy, xz$ is not CM, as it is not equidimensional (its Spec is the union of a line and a plane).

Definition 1.11. A local ring $(R, \mathfrak{m}, k)$ is *Gorenstein* if $\mathrm{Ext}^i_R(k, R) = 0$ for $i >> 0$.

- We have a list of inclusions regular $\subset$ Gorenstein $\subset$ Cohen Macaulay;
- let $f \in R$ be a non-zero divisor. Then R/fR is Gorenstein if and only if R is Gorenstein;
- the subring $k[t^2, t^3] \subset k[t]$ is Gorenstein;
- the subring $k[t^3, t^4, t^5]$ is CM but not Gorenstein;
- the ring $k[x, y]/(x^2, xy, y^2)$ is Cohen Macaulay but not Gorenstein.

2 Dualizing sheaf and canonical divisor

From now on we work over a perfect field k. By *variety* we mean a quasi-projective k-scheme of finite type which is geometrically irreducible and geometrically connected.

Let X be a smooth (i.e. nonsingular, i.e. regular) variety of dimension n. Then the sheaf $\Omega^1_{X/k}$ of 1-differentials is locally free, of rank n. We define the *canonical sheaf* to be the top exterior product $\omega_{X/k} := \Lambda^n \Omega^1_{X/k}$, which is locally free of rank 1.

Let now X be normal of dimension n and let $\iota \colon U \to X$ be the inclusion of the smooth locus. Since the codimension of $X \setminus U$ is at least 2, $\omega_X := \iota_* \omega_U$ is a reflexive sheaf. Moreover, when X is projective it is a dualizing sheaf in the sense of Serre's duality, that is, for every coherent sheaf $\mathcal{F}$ it induces a natural isomorphism

$$H^i(X, \mathcal{F}) \to H^{n-i}(X, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)).$$

On a normal variety, there is a bijection between Weil divisors up to linear equivalence and reflexive sheaves generically of rank 1, given by $D \mapsto \mathcal{O}_X(D)$. In particular, ω_X corresponds to a divisor class K_X. This is called the *canonical divisor*. In fact, it is the same divisor that one obtains by seeing K_U as a divisor on X.

Lemma 2.1 ([Sta16] TAG 0BFQ)**.** *The dualizing sheaf is invertible if and only if X is Gorenstein.*

Lemma 2.2 (Adjunction formula, [KM], 5.73)**.** *Let X be a projective CM scheme of pure dimension n and $D \subset X$ an effective Cartier divisor. Then $\omega_D \cong \omega_X(D)_{|D}$.*

3 Singularities of the MMP

From now on we work over $k = \mathbb{C}$.

Definition 3.1. We define the groups of $\mathbb{Q}$-Cartier and $\mathbb{Q}$-Weil divisors simply as $Div(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Z^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 3.2. A normal scheme X is $\mathbb{Q}$-*factorial* if the injective homomorphism

$$Div(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to Z^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. That is, every Weil divisor has a multiple which is Cartier.

Remark 3.3. $\mathbb{Q}$-factoriality is not an analytically local condition. Indeed, $X = \{(x, y, z, w) \in \mathbb{C}^4 | xy + zw + z^3 + w^3 = 0\}$ is factorial; however analytic locally at $(0, 0, 0, 0)$, X is given by $\{xy - uv = 0\}$, which is not $\mathbb{Q}$-factorial since the two divisors $x = 0 = u$ and $y = 0 = v$ intersect in a point.

Proposition 3.4 ([Cor07], 3.2.7)**.** *Let X be a normal $\mathbb{Q}$-factorial variety; $Y \to X$ a birational morphism from a normal variety. Then the exceptional locus $E \subset Y$ is of pure codimension* 1*.*

Definition 3.5. Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier and let $f \colon Y \to X$ a birational morphism from a normal variety. Let m be a integer for which mK_X is Cartier. Then mK_Y is linearly equivalent to

$$f^*(mK_X) + \sum_i ma(E_i, X)E_i$$

for some uniquely determined $a(E_i, X) \in (1/m)\mathbb{Z}$, where $\{E_i\}_i$ is the set of f-exceptional divisors. The rational numbers $a(E_i, X)$ are called *discrepancies.*

Proof of the definition. Let E be an f-exceptional divisor and $e \in E$ a general point. Write z for function defining the divisor E at e, and let dy be a generator for the invertible sheaf $\omega_Y^{\otimes m}$, and dx a generator for $f^*\omega_X^{\otimes m}$. Then dy and dx differ by a unit on the complement of E. It follows that

$$dy = z^b u dx$$

for some function u invertible at e, and some $b \in \mathbb{Z}$ (not depending of choices of m, z, dy, dx!). Hence

$$\omega_Y^{\otimes m} \cong f^*\omega_X^{\otimes m}(-bE).$$

This shows that mK_Y is linearly equivalent to

$$f^*(mK_X) + \sum_i b_i E_i.$$

The b_i's are uniquely determined because no principal divisor is supported on the exceptional locus. Then the $a(E_i, X)$'s are the rational number b_i/m's. □

Notice that the discrepancies are well defined, since no divisor supported only on the E_i's can be principal. The local ring at the generic point of an irreducible divisor E is a discrete valuation ring, hence corresponds to a valuation v on the field of fractions $K(X)$. The discrepancy $a(E, X)$ only depends on v. Therefore, if $Y' \to X$ is another birational morphism and $E' \subset Y$ a divisor inducing the valuation v on $K(X)$, then $a(E, X) = a(E', Y)$

Here is a basic property of discrepancies:

Proposition 3.6. *Let X be a smooth variety; then $a(E, X) \geq 1$ for any valuation E of $K(X)$.*

Proof. Choose some birational morphism $Y \to X$ and E an exceptional divisor. Write $y_1, \ldots, y_n$ for local coordinates at a point $e \in E$ and $x_1, \ldots, x_n$ for local coordinates at $f(e) \in X$. Then

$$dy_1 \wedge dy_2 \wedge \ldots \wedge dy_n = \det(\partial y_i / \partial x_j) f^*(dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n).$$

So the discrepancy $a(E, X)$ is the order of vanishing of the determinant of the jacobian matrix along E. Then $a(E, X) \geq 0$, with equality if f is étale at the generic point of E. Since this is not the case, we have $a(E, X) \geq 1$. □

Definition 3.7. A normal variety X such that K_X is $\mathbb{Q}$-Cartier has *terminal singularities* (resp. *canonical singularities*) if there exists a resolution of singularities $f\colon Y \to X$ such that the ramification formula

$$f^*K(X) = K(Y) + \sum_i a_i E_i$$

satisfies

$$a_i > 0 \text{ (resp. } a_i \geq 0) \tag{2}$$

The fact that condition (2) is satisfied does not depend on the choice of resolution of singularities.

It can be seen that the notions of terminal and canonical singularities are in fact analytically local on X.

Theorem 3.8 (Characterization of the catagory of terminal and $\mathbb{Q}$-factorial singularities.)**.** *Assume existence and termination of flips. The category of normal projective varieties that are $\mathbb{Q}$-factorial and with terminal singularities is the smallest category $\mathcal{C}$ such that:*

- *it contains the category of nonsingular projective varieties;*
- *is closed under the operations of the minimal model program;*

- *every object $X \in \mathcal{C}$ can be obtained via the operations of MMP starting from some nonsingular projective variety.*

Theorem 3.9 (Characterization of canonical singularities)**.** *A projective variety has canonical singularities with ample canonical divisor if and only if it is the canonical model $Proj R$ of a nonsingular projective variety V of general type with canonical ring*

$$R = \bigoplus_{m \geq 0} H^0(V, \mathcal{O}_V(mK_V))$$

that is finitely generated as a $\mathbb{C}$-algebra.

Notice that since global sections of K_X are a birational invariant, the canonical ring depends only on the birational class of X. In fact, for X of general type, the birational class of X is determined by the canonical model.

Example 3.10 (2-dimensional case)**.** Normal surfaces with terminal singularities are nonsingular; normal singularities with canonical singularities have Du Val singularities.

Definition 3.11. Let X be a normal variety. We say that X has *rational singularities* if for some resolution of singularities $f\colon Y \to X$ and all $i > 0$ we have

$$R^i f_* \mathcal{O}_Y = 0.$$

If the property above is satisfied, then it is satisfied for all resolutions of singularities.

Fact 3.12. *If X has rational singularities, then for any resolution of singularities $f\colon Y \to X$ and vector bundle $\mathcal{E}$ on X, we have*

$$H^i(X, \mathcal{E}) = H^i(Y, f^*\mathcal{E}).$$

In particular, cohomology can be computed on a resolution of singularities.

Fact 3.13. *Let X be a variety and $f\colon Y \to X$ a resolution of singularities. Then X has rational singularities if and only if:*

- *X is CM;*

- *$f_*\omega_Y \cong \omega_X$.*

4 Log pairs

Definition 4.1. A Weil divisor D on a nonsingular variety X is called *simple normal crossing (snc)* if at any $x \in X$, the divisor is given Zariski-locally by a product $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ where the x_i form a subset of a system of regular parameters. We say that D is normal crossing if étale-locally on X it is snc.

Definition 4.2. Let X be a normal variety and D a $\mathbb{Q}$-Weil divisor. A *log resolution* for the pair (X, D) is a resolution of singularities $f\colon Y \to X$ such that $f^{-1}D \cup Exc(f)$ is a simple normal crossing divisor.

Definition 4.3. Let (X, Δ) be a pair with X normal and $\Delta = \sum d_i D_i$ a $\mathbb{Q}$-Weil divisor. Assume that $m(K_X + \Delta)$ is Cartier. Let $f\colon Y \to X$ be a log resolution of the pair (X, Δ). Then mK_Y is linearly equivalent to

$$f^*(m(K_Y + \Delta)) + \sum_i ma(E_i, X, \Delta)E_i$$

where the E_i are either f-exceptional divisors or strict transforms of the D_i. We call the numbers $a(E_i, X, \Delta)$ *discrepancies* of the pair (X, Δ). We define

$$\mathit{discrep}(X, \Delta) = \inf\{a(E_i, X, \Delta) | E_i \text{ is an } f\text{-exceptional divisors for some log resolution}\}$$

Suppose that for some E, $a(E, X, \Delta) < -1$. Then one can show that there are arbitrarily negative discrepancies, so that $\mathit{discrep}(X, \Delta) = -\infty$. This motivates the following definition:

Definition 4.4. We say that (X, Δ) is

$$\left.\begin{array}{r} \mathit{terminal} \\ \mathit{canonical} \\ \mathit{klt} \\ \mathit{plt} \\ \mathit{lc} \end{array}\right\} \text{ if } \mathit{discrep}(X, \Delta) \begin{cases} > 0 \\ \geq 0 \\ > -1 \text{ and } d_i < 1 \\ > -1 \\ \geq -1 \end{cases}$$

Here *klt* stands for *Kamawata log terminal*, *plt* for *purely log terminal*, *lc* for *log canonical*. When $D = 0$, klt and plt are simply called *log terminal*. All of the above notions are analytic local.

Klt singularities are the class of singularities were proof work better; but they are not suitable for inductive proofs. Klt singularities are rational (hence CM). The class of plt singularities was invented to make inductive arguments work. The class of lc singularities is the largest where the discrepancy still makes sense.

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