# Properness and log-smoothness of LogPic

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**Abstract.** These are notes for a talk at a seminar organized by David Holmes, Adrien Sauvaget and Arne Smeets in the spring of 2019, over the preprint *The logarithmic Picard group and its tropicalization* by Molcho and Wise.

Let $X \to S$ be a proper vertical log curve. We have defined the stack $\mathbf{LogPic}(X/S)$ over $\underline{\text{LogSch}}/S$ with the étale topology (or log étale), classifying logarithimic line bundles with bounded monodromy on $X$. We denote by $\text{LogPic}(X/S)$ the associated sheaf on $\underline{\text{LogSch}}/S$ of isomorphism classes.

We will prove that these objects are proper and log smooth over $S$

## 1 Properness of LogPic

Before trying to prove that $\mathbf{LogPic}^0(X/S)$ is proper, we should say what properness means for a stack over $\underline{\text{LogSch}}/S$. We cannot really ask that the underlying stack over the big étale site $\underline{\text{Sch}}/S_{et}$ be proper, since $\mathbf{LogPic}^0(X/S)$ it is not algebraic. What is proven in [MW18] is that $\mathbf{LogPic}^0(X/S)$ satisfies the following four properties:

i) it is locally of finite presentation; that is, for every cofiltered diagram of affine log $S$-schemes $T_i$, the natural map

$$\operatorname{colim} \mathbf{LogPic}(X_{T_i}/T_i) \to \mathbf{LogPic}(\lim X_{T_i}/T_i)$$

is an equivalence of categories;

ii) It is bounded. That is, there exists a log scheme of finite type $Z$ and a morphism $Z \to \mathbf{LogPic}^0(X/S)$ which is surjective on valuative geometric point.

iii) it has diagonal representable by proper (in fact, finite) morphisms of log schemes;

iv) it satisfies the valuative criterion for properness (see thm 1.4 of these notes).

We will focus on the valuative criterion for properness, for which we need a few lemmas.

**Lemma 1.1** ([MW18] 4.6.1)**.** *Let $\pi\colon X \to S$ be a proper, vertical log curve. Then*

$$M_S^{gp} \to \pi_* M_X^{gp}$$

*is an isomorphism.*

*Proof.* We may reduce to $S$ atomic, and to $X$ having constant tropicalization $\mathfrak{X}$ on the closed stratum. Consider the exact sequence

$$H^0(X, M_X^{gp}) \xrightarrow{\alpha} H^0(X, \bar{M}_X^{gp})(= H^0(\mathfrak{X}, \mathfrak{P})) \xrightarrow{\beta} H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X).$$

and let $\varphi \in H^0(X, M_X^{gp})$.

We have an exact sequence of sheaves on the tropical curve $\mathfrak{X}$:

$$0 \to \mathfrak{L} \to \mathfrak{P} \to \mathfrak{V} \to 0.$$

In order of appearance, these are the sheaves of linear functions, piecewise linear functions, and the free abelian sheaf generated by vertices of $\mathfrak{X}$. While the first map is the obvious inclusion, the second map sends a piecewise linear function $f$ to the vertex labelling that to a vertex $v$ assigns the sum of the slopes of the edges having root $v$.

The induced map $H^0(\mathfrak{X}, \mathfrak{P}) \to H^0(\mathfrak{X}, \mathfrak{V}) = \mathbb{Z}^V$ factors as $\beta$ followed by the multidegree map. It follows that $\alpha(\varphi)$is in the image of $H^0(\mathfrak{X}, \mathfrak{L}) \to H^0(\mathfrak{X}, \mathfrak{P})$. As $\mathfrak{X}$ is compact, $H^0(\mathfrak{X}, \mathfrak{L}) = H^0(\mathfrak{X}, \bar{M}_S^{gp})$.

Finally, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_S^\times & \longrightarrow & M_S^{gp} & \longrightarrow & \bar{M}_S^{gp} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \nearrow & \downarrow & & \\ 0 & \longrightarrow & \pi_*\mathcal{O}_X^\times & \longrightarrow & \pi_*M_X^{gp} & \longrightarrow & \pi_*\bar{M}_X^{gp} & \longrightarrow & R^1\pi_*\mathcal{O}_X^\times \end{array}$$

The leftmost vertical arrow is an isomorphism because $X/S$ is a nodal curve, hence has geometric fibres connected and reduced. On the other hand, by the previous argument, there is a diagonal dashed arrow making the diagram commute, and the statement follows.

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**Definition 1.2.** Let $Y$ be a scheme, $j\colon U \subset Y$ an open immersion, and $M_U$ a log structure on $U$. The *maximal log structure* on $Y$ extending $M_U$ is the fibre product $M_Y = j_*M_U \times_{j_*\mathcal{O}_U} \mathcal{O}_Y$.

**Lemma 1.3** ([MW18], 4.4.3)**.** *Let $S$ be the spectrum of a valuation ring, $j\colon \eta \hookrightarrow S$ the inclusion of the generic point; let $M_S$ be the maximal log structure on $S$ extending a given valuative log structure $M_\eta$ on $\eta$. Let $X \to S$ be a proper vertical log curve. Then*

$$R^1 j_* M_{X_\eta}^{gp} = 0.$$

*Proof.* The sheaf in question is the sheafification of the presheaf

$$\begin{aligned} Et/X &\to Sets \\ U &\mapsto H^1(U_\eta, M_{U_\eta}^{gp}) \end{aligned}$$

So what we're trying to prove is that every $M_{X_\eta}^{gp}$-torsor becomes trivial étale locally on $X$.

The hypotheses on the lemma make it possible to reduce to the case where the generic fibre $X_\eta$ is smooth, see the proof of [MW18] 4.4.3 for details. In this case a

$M^{gp}_{X_\eta}$-torsor is an old-fashioned line bundle $L_\eta$ on $X_\eta$ (since $H^1(X, \bar{M}^{gp}_{X_\eta})$ vanishes). If $S$ is noetherian, (hence a discrete valuation ring) we can take a regular nodal model $Y \to X$; since $Y$ is factorial, the line bundle $L_\eta$ extends to a line bundle $L$ on $Y$, which is in particular a $M^{gp}_Y$-torsor. Now we use that $H^1(X, M^{gp}_X) = H^1(Y, M^{gp}_Y)$ ([MW18] 4.4.1 or Adrien's talk ). This tells us that $L_\eta$ extends to a $M^{gp}_X$-torsor on $X$ (which, by [MW18], 3.5.1., has bounded monodromy). In particular, it is trivial étale locally on $X$. □

**Theorem 1.4** ([MW18] 4.10,1)**.** *Let $X \to S$ be a proper, vertical, log curve. Then **LogPic**$(X/S)$ satisfies the valuative criterion for properness. That is, for every valuation ring $R$ having the maximal log structure $M_R$ that extends a given valuative log structure $M_K$ on the fraction field $K$, and every commutative diagram of solid arrows*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \boldsymbol{LogPic}(X/S) \\ \downarrow j & \nearrow & \downarrow \\ \operatorname{Spec} R & \longrightarrow & S \end{array}$$

*there exists up to isomorphism only one dashed arrow making the diagram 2-commute.*

*Proof.* The situation is the following: we have a diagram

$$\begin{array}{ccc} X_K & \xrightarrow{j'} & X_R \\ \downarrow \pi' & & \downarrow \pi \\ \operatorname{Spec} K & \xrightarrow{j} & \operatorname{Spec} R \end{array}$$

and a log line bundle $L$ on $X_K$, which we would like to extend uniquely to $X_R$. We will therefore show that the natural morphism of stacks

$$\alpha \colon BM^{gp\dagger}_{X_R} \to j_*j^*BM^{gp\dagger}_{X_R} = j_*BM^{gp\dagger}_{X_K}$$

is an equivalence. Taking $X_R$-valued points, this will gives us the desired equivalence between $M^{gp}_{X_R}$-torsors of bounded monodromy and $M^{gp}_{X_K}$-torsors of bounded monodromy.

To prove that $\alpha$ is an equivalence, we show that:

i) it induces an isomorphism on sheaves of automorphisms;

ii) it induces an isomorphism on sheaves of isomorphism classes of objects.

By [MW18] theorem 2.4.2.1, the log structure $M_{X_R}$ is the maximal one extending $M_{X_K}$. We get

$$M^{gp}_{X_R} = (j_*M_{X_K} \times_{j_*\mathcal{O}_{X_K}} \mathcal{O}_{X_R})^{gp} = j_*M^{gp}_{X_K}.$$

Hence part i) is fine.

For part ii), the sheaf of isomorphism classes of $BM^{gp\dagger}_{X_R}$ is zero, since every $M^{gp}_{X_R}$-torsor is trivial étale locally on $X$. On the other hand, the sheaf of isomorphism classes of $j_*BM^{gp\dagger}_{X_K}$ is the sheafification of $(T \to X) \mapsto H^1(T_K, M^{gp}_{T_K})^\dagger$; the latter vanishes by lemma 1.3. So part ii) is fine as well. □

**Corollary 1.5** ([MW18] 4.10.2)**.** *Let $X \to S$ be a proper, vertical log curve. Then* $\mathrm{LogPic}(X/S)$ *satisfies the valuative criterion for properness.*

*Proof.* $\mathbf{LogPic}(X/S) \to \mathrm{LogPic}(X/S)$ is a gerbe banded by $\pi_* \mathbb{G}_{m,log,X} = \mathbb{G}_{m,log,S}$ (lemma 1.1); locally on $S$, it is the trivial gerbe, which means that there exists a section $\mathrm{LogPic}(X/S) \to \mathbf{LogPic}(X/S) = \mathrm{LogPic}(X/S)/\mathbb{G}_{m,log,S}$. The latter is a $\mathbb{G}_{m,log,S}$-torsor. We are done if we show that $\mathbb{G}_{m,log}$ satisfies the valuative criterion for properness; but this is exactly the equality $M_X^{gp} = j'_* M_{X_K}^{gp}$. □

**Theorem 1.6** ([MW18] 4.12.5)**.** *For every integer $d$, the sheaf* $\mathrm{LogPic}^d(X/S)$ *and the stack* $\boldsymbol{LogPic}^d(X/S)$ *are proper over $S$, that is, they satisfy the four properties at the beginning of these notes.*

*Sketch of proof.* We only need to prove that $\mathrm{LogPic}^d(X/S)$ is proper, since étale locally on $S$, $\mathbf{LogPic}^d(X/S)$ is equivalent to $\mathrm{LogPic}^d(X/S) \times B\mathbb{G}_{m,log,S}$ and $B\mathbb{G}_{m,log,S}$ is proper. Moreover, we may assume $d = 0$, since every $\mathrm{LogPic}^d(X/S)$ is a $\mathrm{LogPic}^0(X/S)$-torsor.

Local finite presentation is formal and based on the fact that the étale sheaf $M_X^{gp}$ is finitely generated.

Boundedness can be reduced to proving that the tropical jacobian is bounded, since $\mathrm{LogPic}^0(X/S)$ is an extension of $\mathrm{TropJac}(X/S)$ by $\mathrm{Pic}^0(X/S)$. Recall that, letting $\mathfrak{X}$ be the tropicalization of $X$, we have an exact sequence

$$0 \to H_1(\mathfrak{X}, \mathbb{Z}) \to \mathrm{Hom}(H_1(\mathfrak{X}, \mathbb{Z}), \bar{\mathbb{G}}_{m,log})^\dagger \to \mathrm{TropJac}(X/S) \to 0.$$

We let $g = \mathrm{rk}\, H_1(\mathfrak{X})$ and choose a basis $e_1, \dots, e_g$ for $H_1(\mathfrak{X})$. Let $l(e_i) \in \bar{M}_S^{gp}$ be the lengths of these cycles. Take $Z \subset \mathrm{Hom}(H_1(\mathfrak{X}, \mathbb{Z}), \bar{\mathbb{G}}_{m,log})^\dagger$ consisting of those $\mu$ such that for all $i = 1, \dots, g$

$$-r(g+1)l(e_i) \leq \mu(e_i) \leq r(g+1)l(e_i)$$

where $r$ is the rank of $\bar{M}_S^{gp}$. Then $Z$ is bounded and the map $Z \to \mathrm{TropJac}(\mathfrak{X}/S)$ is surjective on valuative geometric points. (Theorem 3.10.2). Finiteness of the diagonal is theorem 4.12.1, and we have proven the valuative criterion for properness. □

# 2 Log smoothness

**Theorem 2.1** ([MW18], 4.13.1)**.** *Let $X \to S$ be a proper vertical log curve. Then* $\boldsymbol{LogPic}(X/S)$ *is log smooth, that is, it is locally of finite presentation, and satisfies the infinitesimal lifting criterion: for every affine log $S$-scheme $T = \mathrm{Spec}\, A$ and strict closed immersion $\iota \colon T_0 \to T$ defined by a square-zero ideal $J \subset A$, and commutative diagram*

$$\begin{array}{ccc} T_0 & \longrightarrow & \boldsymbol{LogPic}(X/S) \\ \downarrow{\scriptstyle \iota} & \nearrow & \downarrow \\ T & \longrightarrow & S \end{array}$$

*there exists a dashed arrow making the diagram 2-commute.*

*Proof.* We have already seen local finite presentation. For the infinitesimal lifting property, we may assume $T = S$ and write $S_0 = T_0$, $X_0 = X \times_S S_0$. We may also assume that the log scheme $S$ is locally of finite type, that is, $\underline{S}$ is locally of finite type and $\bar{M}_S$ has charts by finitely generated monoids.

Every logarithmic scheme locally of finite type $(Y, M)$ comes equipped with a stratification induced by the log structure $M$: if $M$ has a global chart given by a monoid $P$, each generator of $P$ determines an ideal of $\mathcal{O}_Y$, hence a closed subscheme. Intersections and complements give rise to a stratification; and the characteristic monoid $\bar{M} = M/\mathcal{O}_Y^\times$ is locally constant on each stratum, and so is $\bar{M}^{gp}$. In particular $\bar{M}^{gp}$ is a constructible sheaf on $Y_{et}$, hence representable by an étale algebraic space.

We apply these considerations to $(X, M_X)$. As the sheaf $\bar{M}_X^{gp}$ is representable by an étale algebraic space, and because the universal homeomorphism $X_0 \to X$ induces an equivalence $Et/X_0 \to Et/X$ of étale sites, every $\bar{M}_{X_0}^{gp}$-torsor on $X_0$ lifts uniquely to a $\bar{M}_X^{gp}$-torsor on $X$. This is expressed by saying that the restriction map

$$B\bar{M}_X^{gp} \to \iota_* B\bar{M}_{X_0}^{gp}$$

is an equivalence. In particular, it induces an equivalence on the substacks of torsors with bounded monodromy.

Even though I am not sure how to make sense of exact sequence of stacks, we can keep the following picture in mind for reference:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\mathcal{O}_X^\times & \longrightarrow & BM_X^{gp} & \longrightarrow & B\bar{M}_X^{gp} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & \iota_* B\mathcal{O}_{X_0}^\times & \longrightarrow & \iota_* BM_{X_0}^{gp} & \longrightarrow & \iota_* B\bar{M}_{X_0}^{gp} & \longrightarrow & 0 \end{array}$$

Take $\mathcal{L}_0$ an $M_{X_0}^{gp}$-torsor with bounded monodromy. We want to show that there exists a $M_X^{gp}$-torsor $\mathcal{L}$ with bounded monodromy on $X$ lifting $\mathcal{L}_0$. Actually, any lift $\mathcal{L}$ will automatically have bounded monodromy, as the condition depends only on its image in $B\bar{M}_X^{gp}$. Therefore we may forget about the bounded monodromy condition.

Liftings of $\mathcal{L}_0$ to $M_X^{gp}$-torsors can be encoded into a category fibered in groupoids $\mathbf{Lifts}_{\mathcal{L}_0}$ over $Log/X$. Given a morphism of log schemes $U \to X$, and letting $U_0 = U \times_X X_0$, the category $\mathbf{Lifts}_{\mathcal{L}_0}(U)$ is defined as follows:

- its objects are pairs $(\mathcal{L}, \varphi)$ with $\mathcal{L}$ a $\mathbb{G}_{m,log}$-torsor on $X_U$ and $\varphi \colon \mathcal{L} \times_U U_0 \to \mathcal{L}_0 \times_{X_0} U_0$ an isomorphism;
- its morphisms $(\mathcal{L}, \varphi) \to (\mathcal{L}', \varphi')$ are (iso)morphisms $f \colon \mathcal{L} \to \mathcal{L}'$ of $\mathbb{G}_{m,log}$-torsors such that $\varphi' \circ f_{|U} = \varphi$.

First of all, notice that étale locally on $X$, liftings exist, so there exists $U \to X$ an étale atlas with $\mathbf{Lifts}_{\mathcal{L}_0}(U)$ non empty. Moreover, given two objects $(\mathcal{L}, \varphi)$, $(\mathcal{L}', \varphi')$ of $\mathbf{Lifts}_{\mathcal{L}_0}(U)$, there exists an atlas $V \to U$ where $\mathcal{L}_V, \mathcal{L}'_V$ become trivial torsors. By surjectivity of

$$\mathrm{Aut}(\mathbb{G}_{m,log})(U) = \mathbb{G}_{m,log}(U) \to \mathrm{Aut}(\mathbb{G}_{m,log})(U_0) = \mathbb{G}_{m,log}(U_0)$$

there exists an isomorphism $f\colon \mathcal{L}_U \to \mathcal{L}'_U$ lifting $\varphi'^{-1}\varphi\colon \mathcal{L}_{U_0} \to \mathcal{L}'_{U_0}$. This shows that the two lifts are isomorphic locally on $X$.

By the above considerations, $\mathbf{Lifts}_{\mathcal{L}_0}$ is a gerbe on $X$. In fact, the automorphism sheaf of an object $(\mathcal{L}, \varphi) \in \mathbf{Lifts}_{\mathcal{L}_0}(U)$ is $\mathcal{O}_{U_0} \otimes_{\mathcal{O}_{S_0}} J$, and therefore $\mathbf{Lifts}_{\mathcal{L}_0}$ is banded by $\mathcal{O}_{X_0} \otimes J$.

Equivalence classes of gerbes banded by $\mathcal{O}_{X_0} \otimes J$ are classified by $H^2(X_0, \mathcal{O}_{X_0} \otimes J)$ which vanishes since $X_0$ is a curve. Then $\mathbf{Lifts}_{\mathcal{L}_0}$ is equivalent to the gerbe $X/(\mathcal{O}_{X_0} \otimes J)$ classifying $\mathcal{O}_{X_0} \otimes J$-torsors on $X$. The trivial torsor gives us a global lift of $\mathcal{L}_0$.

□

# References

[MW18] Samouil Molcho, Jonathan Wise. The logarithmic Picard group and its tropicalization. *arXiv e-prints*, page arXiv:1807.11364, Jul 2018.