Coupling of finite element and integral representation methods in magnetostatics

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SUMMARY

We present an original method to compute the magnetic field generated by some electromagnetic device through the coupling of an integral representation formula and a finite element method. The unbounded three dimensional magnetostatic problem is formulated in terms of the reduced scalar potential. Through an integral representation formula, an equivalent problem is set in a bounded domain and discretized using a standard finite element method. A complete error analysis of the method is achieved and presented in the paper.

KEY WORDS: Finite element method, integral formulation, artificial boundary, magnetostatics, reduced scalar potential

1. Introduction

Our goal is to present an original method in order to compute the magnetic field generated by an electromagnetic device made of a weak ferromagnetic material. Nowadays a lot of work has been done in magnetostatics and numerous numerical methods are known to solve such problems. The choice of the physical unknown either the magnetic strength $\mathbf{H}$, or the magnetic induction $\mathbf{B}$, or the magnetic potential, the choice of the numerical approximation methods either finite difference (FD) or finite element (FEM) or boundary element (BEM) methods will lead to many different schemes. Each one has its own advantages and drawbacks.

The features of the magnetostatic problem considered in the following are : it is a three dimensional problem set in an unbounded domain. As the problem is set in an unbounded domain, an artificial boundary is generally introduced at a finite distance from the electromagnetic device to use FEM methods [1] and the behaviour of the solution at

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infinity is handled through an approximate boundary condition set on this artificial boundary, see [2], [3]. We also have the boundary element method [4] where an integral representation formula is used to write the problem as an integral equation on the boundary of the device. As a drawback of this method we need to compute nearly singular integrals which require the use of elaborated quadrature schemes. It can also be irrelevant to use the integral representation formulae to compute the solution in a large number of points in the interior or exterior domains.

Here we present a third way to compute the magnetic unknowns through the coupling of a finite element method with an integral representation formula. Namely, an integral representation formula is used to take into account the behaviour of the solution in the exterior domain while a finite element approximation is used in the interior domain. This approach is now classical and there exist different ways to write the coupled problem. The one we use originates from the work of A. Jami and M. Lenoir in the field of hydrodynamics, see [5]. The magnetostatic problem is written using the reduced scalar potential [6]. The basic idea of the method is to bound the exterior domain using an artificial boundary that can be close to the device boundary but always distinct from it. The boundary condition to set on this artificial boundary is obtained using a boundary integral representation formula of the solution, the support for the integral representation being the device boundary. As the two boundaries are distinct, all the involved integrals are regular and standard quadrature schemes are used. The magnetic potential in the interior domain surrounded by the artificial boundary is computed using a standard finite element approximation. Then we use an integral representation formula, deduced from the one for the potential, to compute the magnetic field in any point of the space.

Our method reconciles the advantages of both FEM and BEM. The behaviour of the solution at infinity is handled exactly by an exact boundary condition set on an artificial boundary. As the artificial boundary and the device boundary differ, all the involved integrals are regular. This computational approach is well suited for shape optimisation where the area of interest is often well localised, the air-gap of an electromagnet for instance. The coupling boundary can be chosen close to the electromagnetic device to reduce the size of the interior domain where the finite element method is employed. The magnetic field at the node on the control surface can be computed efficiently using the integral representation formula.

The content of the paper is the following. In section 2 we present the magnetostatic problem and the variational formulation obtained for the problem set in a bounded domain with a integral representation formula set as boundary condition. The finite element discretisation is presented in section 3. A careful error analysis of the method is achieved in section 4.

For $\Omega \subset \mathbb{R}^3$, $\mathbb{L}^2(\Omega)$ denotes the set of square integrable functions over Ω and $\mathbb{H}^m(\Omega)$ $(m \in \mathbb{N}^*)$ denotes the set of functions with derivatives up to the order m in $\mathbb{L}^2(\Omega)$. To handle functions defined over the unbounded domain $\complement\overline{\Omega}$ we will use the standard weighted Sobolev spaces $\mathbb{W}^1(\mathbb{R}^3)$ and $\mathbb{W}^1(\complement\overline{\Omega})$ defined by

$$\mathbb{W}^1(\complement\overline{\Omega}) = \Big\{\psi\ ;\ \frac{\psi}{\sqrt{1+|x|^2}} \in \mathbb{L}^2(\complement\overline{\Omega}), \quad \nabla\psi \in \mathbb{L}^2(\complement\overline{\Omega})^3\Big\}.$$

This set is equipped with the norm $|\,.\,|_{1,\complement\overline{\Omega}}$ defined for $\psi \in \mathbb{W}^1(\complement\overline{\Omega})$ by $|\psi|^2_{1,\complement\overline{\Omega}} = \int_{\complement\overline{\Omega}} \nabla\psi\cdot\nabla\psi \; \mathrm{d}x$.

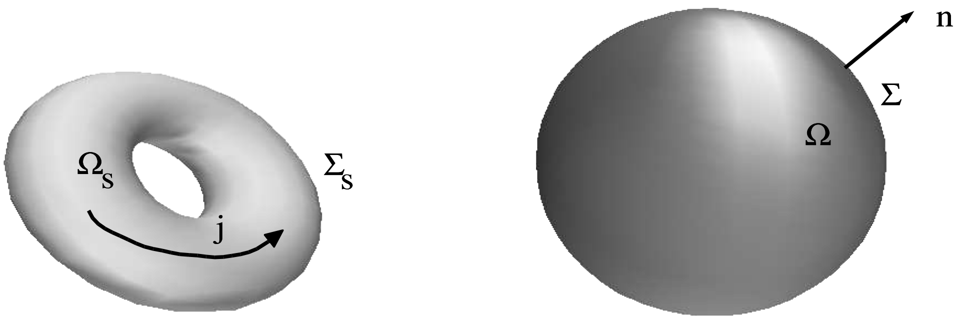

Figure 1. Situation under consideration.

2. The continuous problem

2.1. The magnetostatic problem

The situation under consideration is the computation of the magnetic field generated by an electromagnetic device composed of a weakly ferromagnetic core Ω and an inductor Ω_s characterised by a time independent current density $\mathbf{j}$ in a three dimensional geometry. The domain Ω is a simply connected open set in $\mathbb{R}^3$. We denote by Σ and Σ_s the boundary of Ω and Ω_s respectively and by $\complement\overline{\Omega}$ and $\complement\overline{\Omega}_s$ the complement of their adherence in $\mathbb{R}^3$. We assume for simplicity that the metallic core Ω has a constant permeability. This assumption is not a limitation of the study but avoid cumbersome notations.

The magnetic field $\mathbf{H}$ satisfies the basic equations of magnetostatics

$$\left\{\begin{array}{lcll} \operatorname{div} \mu\mathbf{H} & = & 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{H} & = & 0 & \text{in } \complement\overline{\Omega}, \\ \mu\mathbf{H}\big|_{\Omega}\cdot\mathbf{n} & = & \mathbf{H}\big|_{\complement\overline{\Omega}}\cdot\mathbf{n} & \text{across } \Sigma, \\ \operatorname{rot} \mathbf{H} & = & \mathbf{j} & \text{in } \Omega_s, \\ \operatorname{rot} \mathbf{H} & = & \mathbf{0} & \text{in } \complement\overline{\Omega_s}, \\ [\mathbf{H}\wedge\mathbf{n}] & = & 0 & \text{across } \Sigma_s, \end{array}\right. \tag{1}$$

where $\mathbf{n}$ is the outward unit normal to Σ or Σ_s, $\mathbf{H}\big|_{\Omega}$ (resp. $\mathbf{H}\big|_{\complement\overline{\Omega}}$,) denotes the restriction of $\mathbf{H}$ to the domain Ω (resp. $\complement\overline{\Omega}$) and [ ] indicates the jump across the interface. It is classical, see [6], to express the total magnetic field $\mathbf{H}$ as $\mathbf{H} = \mathbf{H_s} + \mathbf{H_m}$ where $\mathbf{H_s}$, the field due to the source currents, satisfies

$$\left\{\begin{array}{rcll} \operatorname{div} \mathbf{H_s} & = & 0 & \text{in } \mathbb{R}^3, \\ \operatorname{rot} \mathbf{H_s} & = & \mathbf{j} & \text{in } \Omega_s, \\ \operatorname{rot} \mathbf{H_s} & = & \mathbf{0} & \text{in } \complement\overline{\Omega}_s, \\ \left[\mathbf{H_s}\wedge\mathbf{n}\right] & = & \mathbf{0} & \text{across } \Sigma_s, \end{array}\right. \tag{2}$$

and $\mathbf{H_m}$, the reaction of the ferromagnetic piece, satisfies

$$\left\{\begin{array}{rcll} \operatorname{rot}\mathbf{H_m} & = & \mathbf{0} & \text{in } \mathbb{R}^3, \\ \operatorname{div}\mathbf{H_m} & = & 0 & \text{in } \Omega \text{ and } \complement\overline{\Omega}, \\ (\mu-1)\,\mathbf{H_m}\cdot\mathbf{n} & = & (1-\mu)\,\mathbf{H_s}\cdot\mathbf{n} & \text{across } \Sigma. \end{array}\right. \tag{3}$$

On the one hand, $\mathbf{H_s}$ can be efficiently computed, see [7], [8], through the evaluation of the Biot and Savart integral

$$\mathbf{H_s}(x) = \frac{1}{4\pi}\int_{\Omega_s}\left(\mathbf{j}(y)\wedge\frac{\mathbf{r}}{r^3}\right)\,\mathrm{d}y \tag{4}$$

with $\mathbf{r} = \mathbf{x} - \mathbf{y}$. Thus the computation of the total magnetic field $\mathbf{H}$ is reduced to the computation of $\mathbf{H_m}$. One advantage of this approach is that although the conductors lie within the computational area, they do not need to be meshed when using a finite element discretisation to solve problem (3). Moreover all what is needed for the computation of $\mathbf{H_m}$ is the value of the field $\mathbf{H_s}$ over the surface Σ.

On the other hand, as the field $\mathbf{H_m}$ is curl free, we can introduce the so-called *reduced scalar magnetic potential* (RSP) φ such that $\mathbf{H_m} = -\nabla\varphi$. The RSP satisfies the following problem deduced from (3),

$$\left\{\begin{array}{ll} \Delta\varphi = 0 & \text{in } \Omega \text{ and } \complement\overline{\Omega}, \\ \mu\dfrac{\partial\varphi}{\partial n}\big|_{\Omega} - \dfrac{\partial\varphi}{\partial n}\big|_{\complement\overline{\Omega}} = (\mu-1)\,g & \text{across } \Sigma, \end{array}\right. \tag{5}$$

where $g = \mathbf{H_s}\cdot\mathbf{n}$ is considered as a given function since $\mathbf{H_s}$ is computed using (4). One can prove that problem (5) has a unique solution in the Sobolev space $\mathbb{W}^1(\mathbb{R}^3)$ and that this solution is continuous in $\mathbb{R}^3$. The main advantage of introducing the RSP is to transform a problem for a 3 components vector field throughout the space into a problem for a scalar function, thus reducing the number of degrees of freedom in the discretisation.

A numerical method widely used to solve magnetostatic problems is the finite element method (FEM). To deal with the unboundedness of the domain in problem (5), an artificial boundary can be introduced at a finite distance from the electromagnetic device. The behaviour of the solution at infinity is usually handled through an approximate boundary condition set on this artificial boundary deduced from an asymptotic expansion of the solution at infinity [2],[3]. A drawback of this approach is that the artificial boundary has to be introduced at a sufficiently large distance from the electromagnetic device in order the approximate boundary condition gives satisfactory numerical results. An other way to proceed is to set an exact boundary condition on the artificial boundary. It is the approach we develop in this paper in the context of magnetostatics. Our exact boundary condition is obtained using a boundary integral representation formula for the reduced scalar potential. As the boundary condition is an exact one, the artificial boundary can be placed very close to the electromagnetic device.

2.2. Integral representation formulae

The boundary condition on the artificial boundary introduced to bound the domain is obtained by applying potential theory results to the reduced scalar potential φ. Let G denotes the Green kernel associated with the three-dimensional Laplacian,

$$G(x,y) = \frac{1}{4\pi|x-y|} \qquad \text{for } x,y\in\mathbb{R}^3, x\neq y,$$

and $G_n(x,y) = \nabla_x G(x,y).\mathbf{n}, x \in \Sigma, y \in \mathbb{R}^3$, denotes its normal derivative on Σ.

Since the magnetic potential φ is harmonic in the exterior domain $\complement\overline{\Omega}$ we have the Green representation formula for $y \in \complement\overline{\Omega}$, see [4],

$$\varphi(y) \quad = \quad \int_\Sigma \varphi\,|_{\complement\overline{\Omega}}(x)\, G_n(x,y)\ \mathrm{d}\sigma_x - \int_\Sigma \frac{\partial\varphi}{\partial n}\,|_{\complement\overline{\Omega}}(x)\ G(x,y)\ \mathrm{d}\sigma_x. \tag{6}$$

Let's take $x \in \Omega$ and $y \in \complement\overline{\Omega}$. We deduce from Green second identity the relation,

$$\begin{aligned} 0 \quad &= \quad \int_\Omega \left(\Delta_x\varphi(x)\, G(x,y) - \varphi(x)\Delta_x G(x,y)\right)\ \mathrm{d}x \\ &= \quad \int_\Sigma \left(\frac{\partial\varphi}{\partial n}\,|_\Omega(x)\, G(x,y) - \varphi\,|_\Omega(x)\, G_n(x,y)\right)\ \mathrm{d}\sigma_x. \end{aligned} \tag{7}$$

Then we multiply (7) by μ and add it to (6) to get for $y \in \complement\overline{\Omega}$

$$\begin{aligned} \varphi(y) \quad &= \quad \int_\Sigma \left(\varphi\,|_{\complement\overline{\Omega}}(x) - \mu\,\varphi\,|_\Omega(x)\right)\ G_n(x,y)\ \mathrm{d}\sigma_x \\ &\quad - \int_\Sigma \left(\frac{\partial\varphi}{\partial n}\,|_{\complement\overline{\Omega}}(x) - \mu\,\frac{\partial\varphi}{\partial n}\,|_\Omega(x)\right)\ G(x,y)\ \mathrm{d}\sigma_x. \end{aligned} \tag{8}$$

Using the boundary condition on Σ in (5), we obtain the following representation formula for $y \in \complement\overline{\Omega}$,

$$\varphi(y) \quad = \quad (\mu-1)\int_\Sigma g(x)\, G(x,y)\ \mathrm{d}\sigma_x - (\mu-1)\int_\Sigma \varphi(x)\, G_n(x,y)\ \mathrm{d}\sigma_x. \tag{9}$$

As a byproduct we can express for $y \in \complement\overline{\Omega}$ the reaction field $\mathbf{H_m}(y) = -\nabla\varphi(y)$ as

$$\mathbf{H_m}(y) = (1-\mu)\int_\Sigma g(x)\,\nabla_y G(x,y)\ \mathrm{d}\sigma_x - (1-\mu)\int_\Sigma \varphi(x)\,\nabla_y G_n(x,y)\ \mathrm{d}\sigma_x. \tag{10}$$

Suppose we have computed φ on Σ then $\mathbf{H_m}$ can be computed via the formula (10) since g and G are known functions.

Let Γ be an *artificial* surface surrounding Ω. We define a boundary differential operator on Γ by $\mathrm{D}^\Gamma u = \frac{\partial u}{\partial n} + u$. From (9) we have for y in Γ,

$$\mathrm{D}^\Gamma\varphi(y) \quad = \quad (\mu-1)\int_\Sigma g(x)\,\mathrm{D}^\Gamma G(x,y)\ \mathrm{d}\sigma_x - (\mu-1)\int_\Sigma \varphi(x)\,\mathrm{D}^\Gamma G_n(x,y)\ \mathrm{d}\sigma_x. \tag{11}$$

Relation (11) will be used as boundary condition on Γ. In the following we denote by Ω_Γ the open set delimited by the boundary Γ and by $\Omega_{\Sigma\Gamma}$ the open set $\Omega_\Gamma \setminus \overline{\Omega}$, see Fig. 2.2.

Thus, we now consider the following problem set in the bounded domain Ω_Γ : find $\phi \in \mathbb{H}^1(\Omega_\Gamma)$ such that

$$\left\{\begin{array}{l} \Delta\phi = 0 \quad \text{in } \ \Omega, \\ \Delta\phi = 0 \quad \text{in } \ \Omega_{\Sigma\Gamma}, \\ \mu\dfrac{\partial\phi}{\partial n}\,|_\Omega - \dfrac{\partial\phi}{\partial n}\,|_{\Omega_{\Sigma\Gamma}} = (\mu-1)g \quad \text{on } \ \Sigma, \\ \mathrm{D}^\Gamma\phi(y) = (\mu-1)\displaystyle\int_\Sigma g(x)\,\mathrm{D}^\Gamma G(x,y)\ \mathrm{d}\sigma_x \\ \qquad\qquad -(\mu-1)\displaystyle\int_\Sigma \phi(x)\,\mathrm{D}^\Gamma G_n(x,y)\ \mathrm{d}\sigma_x \quad \text{on } \ \Gamma. \end{array}\right. \tag{12}$$

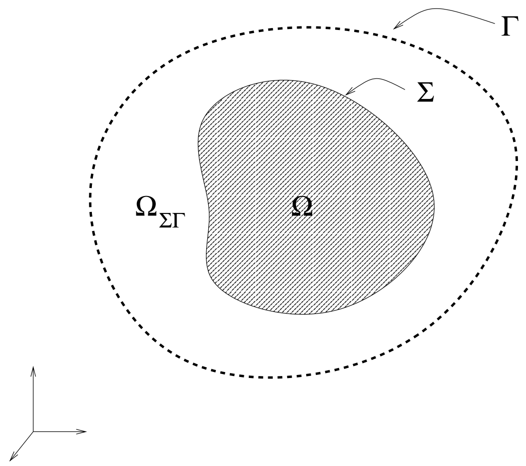

Figure 2. Notations for the problem set in a bounded domain.

Remark More generally we could define a family of boundary differential operators on the artificial boundary Γ by $\mathrm{D}^\lambda u = \frac{\partial u}{\partial n} + \lambda u$ where λ is a non negative real, see [9]. ◇

Proposition 1. *Problem (12) has a unique solution ϕ. This function ϕ is the restriction of the solution φ to problem (5) to the open set Ω_Γ. Moreover, the function ϕ prolongated to the open set $\complement\overline{\Omega_\Gamma}$ by the relation*

$$\phi(y) = (\mu - 1) \int_\Sigma g(x)\, G(x,y)\ d\sigma_x + (1-\mu) \int_\Sigma \phi(x)\, G_n(x,y)\ d\sigma_x \qquad \forall y \in \complement\overline{\Omega_\Gamma} \tag{13}$$

is solution of problem (5).

Proof The existence of a solution to problem (12) is obvious since the solution φ to problem (5) satisfies all the conditions in (12). Let us examine the uniqueness of the solution.

The solution ϕ to problem (12) is a harmonic function in $\Omega_{\Sigma\Gamma}$ and Ω and as a consequence it admits the following integral representation in $\Omega_{\Sigma\Gamma}$:

$$\begin{aligned} \phi(y) \quad = \quad & \int_\Gamma \Big(\phi(x)\, G_n(x,y) - \frac{\partial\phi}{\partial n}(x)\, G(x,y)\Big)\ \mathrm{d}\gamma_x \\ & + + (\mu - 1) \int_\Sigma \Big(g(x)\, G(x,y) - \phi(x)\, G_n(x,y)\Big)\ \mathrm{d}\sigma_x . \end{aligned} \tag{14}$$

Let $\psi \in \mathbb{W}^1(\complement\overline{\Omega})$ be the harmonic function defined by

$$\psi(y) = (\mu - 1) \int_\Sigma \Big(g(x)\, G(x,y) - \phi(x)\, G_n(x,y)\Big)\ \mathrm{d}\sigma_x \qquad \forall y \in \complement\overline{\Omega} \tag{15}$$

and let $w = \phi - \psi$ in $\Omega_{\Sigma\Gamma}$. Relations (12) and (15) imply that

$$\forall y \in \Gamma \qquad \mathrm{D}^\Gamma w(y) = \mathrm{D}^\Gamma \phi(y) - \mathrm{D}^\Gamma \psi(y) = 0$$

and relations (14) and (15) imply that

$$\forall y \in \Omega_{\Sigma\Gamma} \qquad w(y) = \int_\Gamma \Big(\phi(x)\, G_n(x,y) - \frac{\partial \phi}{\partial n}(x)\, G(x,y) \Big)\ \mathrm{d}\gamma_x. \tag{16}$$

We deduce from (16) that w is the restriction to $\Omega_{\Sigma\Gamma}$ of the harmonic function $\widetilde{w}$ defined in Ω_Γ by:

$$\widetilde{w}(y) = \int_\Gamma \Big(\phi(x)\, G_n(x,y) - \frac{\partial \phi}{\partial n}(x)\, G(x,y) \Big)\ \mathrm{d}\gamma_x.$$

Now, it is well known that the following problem

$$\left\{ \begin{array}{l} \text{find } \ \breve{w} \in \mathbb{H}^1(\Omega_\Gamma) \text{ such that:} \\ \begin{array}{lcll} \Delta \widetilde{w} & = & 0 & \text{in } \Omega^\Gamma \\ D^\lambda \widetilde{w} & = & 0 & \text{on } \Gamma \end{array} \end{array} \right.$$

has the zero solution as an unique solution. It follows that $\phi = \psi$ in $\Omega_{\Sigma\Gamma}$.

Let us now consider the function $v \in \mathbb{W}^1(\mathbb{R}^3)$ defined by

$$v = \left\{ \begin{array}{ll} \phi & \text{in } \overline{\Omega}, \\ \phi = \psi & \text{in } \Omega_{\Sigma\Gamma}, \\ \psi & \text{in } \complement\overline{\Omega_\Gamma}. \end{array} \right.$$

This function is harmonic in each of the set Ω and $\complement\overline{\Omega}$ and satisfies

$$\left[\mu \frac{\partial v}{\partial n}\right] = \left[\mu \frac{\partial \phi}{\partial n}\right] = (\mu - 1) g \qquad \text{across } \ \Sigma.$$

This means that v is a solution of problem (5). Thus any solution ϕ to problem (12) is the restriction to the set Ω_Γ of a solution of problem (5). From the uniqueness of the solution of problem (5) we can conclude that problem (12) admits a unique solution. ⋄

2.3. Variational formulation

To dicretise problem (12) it is convenient to write it in variational form. It is not difficult to check that its variational formulation reads :

$$\left\{ \begin{array}{l} \text{find } \ \phi \in \mathbb{H}^1(\Omega_\Gamma) \text{ such that forall } \ \psi \in \mathbb{H}^1(\Omega_\Gamma) \\ \qquad\qquad b(\phi, \psi) + k(\phi, \psi) = f(\psi) \end{array} \right. \tag{17}$$

where we define the bilinear forms $b, k : \mathbb{H}^1(\Omega_\Gamma) \times \mathbb{H}^1(\Omega_\Gamma) \longrightarrow \mathbb{R}$ by

$$b(\phi, \psi) \ = \ \mu \int_\Omega \nabla\phi \cdot \nabla\psi \ \mathrm{d}\omega + \int_{\Omega_{\Sigma\Gamma}} \nabla\phi \cdot \nabla\psi \ \mathrm{d}\omega + \int_\Gamma \phi\psi \ \mathrm{d}\gamma, \tag{18}$$

$$k(\phi, \psi) \ = \ (\mu - 1) \int_\Gamma \psi(y) \Big\{ \int_\Sigma \phi(x)\, \mathrm{D}^\Gamma G_n(x,y) \ \mathrm{d}\sigma_x \Big\} \ \mathrm{d}\gamma_y, \tag{19}$$

and the linear form $f : \mathbb{H}^1(\Omega_\Gamma) \longrightarrow \mathbb{R}$ by

$$f(\psi) \ = \ (\mu - 1) \int_\Sigma g\,\psi \ \mathrm{d}\sigma + (\mu - 1) \int_\Gamma \psi(y) \ \Big\{ \int_\Sigma g(x)\, \mathrm{D}^\Gamma G(x,y) \ \mathrm{d}\sigma_x \Big\} \ \mathrm{d}\gamma_y. \tag{20}$$

The equivalence between problems (12) and (17) can be easily established in the usual way.

We set $a = b + k$ and we denote by A, B and K the linear operators associated with the bilinear forms a, b and k respectively. As the mapping $v \in \mathbb{H}^1(\Omega_\Gamma) \longmapsto \left(|v|^2_{1,\Omega_\Gamma} + \|v\|^2_{0,\Gamma}\right)^{\frac{1}{2}}$ is a norm on $\mathbb{H}^1(\Omega_\Gamma)$ equivalent to the canonical norm $\|\ \|_{1,\Omega_\Gamma}$, the bilinear form b is continuous and elliptic on $\mathbb{H}^1(\Omega_\Gamma)$. Unfortunately, the bilinear form k is not elliptic on $\mathbb{H}^1(\Omega_\Gamma)$. Still it is a compact and continuous linear operator on $\mathbb{H}^1(\Omega_\Gamma)$. Indeed we have for $\phi \in \mathbb{H}^1(\Omega_\Gamma)$,

$$\begin{aligned}
\|K\phi\| &= \sup_{\psi \in \mathbb{H}^1(\Omega_\Gamma)} \frac{|k(\phi,\psi)|}{\|\psi\|_{1,\Omega_\Gamma}} \\
&\leq \sup_{\psi \in \mathbb{H}^1(\Omega_\Gamma)} \left\{ (\mu - 1) \left(\int_\Gamma \left| \int_\Sigma \phi(x)\, \mathrm{D}^\Gamma G_n(x,y)\; \mathrm{d}\sigma_x \right|^2 \mathrm{d}\gamma_y \right)^{1/2} \frac{\|\psi\|_{0,\Gamma}}{\|\psi\|_{1,\Omega_\Gamma}} \right\} \\
&\leq C\|\phi\|_{0,\Sigma}.
\end{aligned}$$

The continuity of the trace mapping from $\mathbb{H}^1(\Omega_\Gamma)$ onto $\mathbb{L}^2(\Sigma)$ implies that K is continuous on $\mathbb{H}^1(\Omega_\Gamma)$. From a bounded sequence $(\psi_n)_n$ in $\mathbb{H}^1(\Omega_\Gamma)$ we extract a subsequence $(\tilde{\psi}_n)_n$ that weakly converge in $\mathbb{H}^1(\Omega_\Gamma)$ to a function ψ. Now, as K is continuous on $\mathbb{H}^1(\Omega_\Gamma)$, the sequence $(K\tilde{\psi}_n)_n$ weakly converge in $\mathbb{H}^1(\Omega_\Gamma)$ to $K\psi$. The continuity of the trace mapping γ_0 from $\mathbb{H}^1(\Omega_\Gamma)$ into $\mathbb{H}^{\frac{1}{2}}(\Sigma)$ implies that the sequence $(\tilde{\psi}_n)_n$ weakly converge in $\mathbb{H}^{\frac{1}{2}}(\Sigma)$ to ψ. The compact embedding of $\mathbb{H}^{\frac{1}{2}}(\Sigma)$ in $\mathbb{L}^2(\Sigma)$ enables the extraction of a subsequence $(\hat{\psi}_n)_n$ from $(\tilde{\psi}_n)_n$ that strongly converge to ψ in $\mathbb{L}^2(\Sigma)$. Finally, the inequality

$$\|K\hat{\psi}_n - K\psi\| \leq C\|\hat{\psi}_n - \psi\|_{0,\Sigma}$$

implies that the sequence $(\hat{\psi}_n)_n$ strongly converge in $\mathbb{H}^1(\Omega_\Gamma)$.

The operator A can therefore be considered as a compact perturbation of a continuous elliptic operator on $\mathbb{H}^1(\Omega_\Gamma)$, which ensure existence and uniqueness of the solution to problem (17).

Remark In [10] a similar problem is analysed through a different coupling of a boundary integral method and a finite element method. The authors introduce a mixte variational formulation using $= \frac{\partial\phi}{\partial n}\big|_\Gamma$ as unknown. ◇

3. Discretisation

We compute an approximation of the RSP ϕ in the bounded domain $\Omega_{\Sigma\Gamma}$ using the finite element method. If the value of the RSP is needed in some points on the exterior domain $\complement\overline{\Omega_{\Sigma\Gamma}}$ one use the integral formula (8) to compute them.

3.1. Triangulation of Ω_Γ

We denote by $(\widehat{K}, \mathbb{P}_k, \widehat{\Sigma})$ the degree k Lagrange finite element where $\widehat{K}$ is the reference tetrahedron with vertex the points $(0,0,0), (0,0,1), (0,1,0)$ and $(1,0,0)$. We denote by $\widehat{\Pi}_k$ the degree k Lagrange interpolation operator over $\widehat{K}$. We assume the boundaries Σ and Γ can be decomposed in an union of smooth surfaces each of them with a parameterisation given by

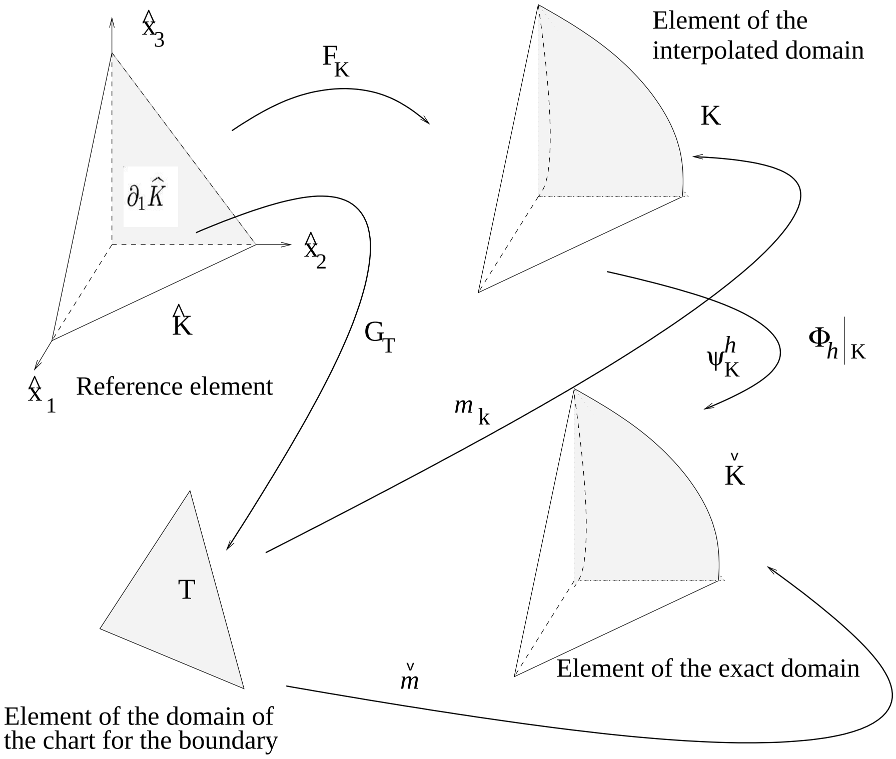

Figure 3. The various mapping involved.

a mapping of the following type:

$$\check{m} : (s,t) \in D \subset \mathbb{R}^2 \longmapsto x = \check{m}(s,t) \in \Sigma \subset \mathbb{R}^3 \, (\text{or} \in \Gamma)$$

where D is a polygonal domain in the plane. We will denote the mapping by $\check{m}^\Sigma$ or $\check{m}^\Gamma$ when an explicit reference to the boundary Σ or Γ will be necessary. For convenience, we will assume in the following that each of the boundaries Σ and Γ can be parameterised with an unique mapping.

Let $(\mathcal{T}_\ell)_\ell$ be a family of regular triangulations of D and let T denote the generic simplex of the triangulation $\mathcal{T}_\ell$. By G_T we denote the affine mapping that map the face $\partial_1 \widehat{K}$ of the tetrahedron $\widehat{K}$ located in the plane $\hat{x}_1 = 0$ onto the triangle T and by m_k the degree k Lagrange interpolation of the mapping $\check{m}$ defined by $m_k \circ G_T = \widehat{\Pi}_k(\check{m} \circ G_T)$, see Fig. 3.

We have the following classical results in surface interpolation theory.

Lemma 1. *For all $T \in \mathcal{T}_\ell$ and for all integer j such that $j \leq k+1$, there exists a positive real α_j (not depending on ℓ) such that:*

$$\|D^j(m - m_k)\|_{\infty,T} = \sup_{x \in T} \|D^j(m - m_k)(x)\| \leq \alpha_j \, \ell^{k+1-j}.$$

Let $J_{\check{m}}$ (resp. J_{m_k}) the absolute value of the Jacobian determinant of $\check{m}$ (resp. m_k). There exists a positive real α (not depending on ℓ) such that:

$$\sup_{x \in T} |J_{\check{m}}(x) - J_{m_k}(x)| \leq \alpha\, \ell^k.$$

We denote by Σ_h (resp. Γ_h) the piecewise degree k Lagrange interpolated boundary obtained from the triangulations $\mathcal{T}_\ell$,

$$\Sigma_h = \bigcup_{T \in \mathcal{T}_\ell} m_k^\Sigma(T) \qquad \Gamma_h = \bigcup_{T \in \mathcal{T}_\ell} m_k^\Gamma(T).$$

As well, we denote by Ω_h the open domain delimited by the boundary Σ_h, by Ω_h^Γ the open domain delimited by the boundary Γ_h and by $\Omega_h^{\Sigma\Gamma}$ the open domain delimited by the boundaries Σ_h and Γ_h.

We consider a regular triangulation $(\mathcal{T}_h)_h$ over the domain Ω_h^Γ of Lagrange type k isoparametric tetrahedral finite elements compatible with the triangulations of the boundaries Σ and Γ in the following sense: $\forall K \in \mathcal{T}_h$ such that $K \cap \Sigma_h \neq \emptyset$ (resp. $K \cap \Gamma_h \neq \emptyset$)

$$\exists T \in \mathcal{T}_\ell \quad \forall \hat{x} \in \partial_1 \widehat{K} \quad F_K(\hat{x}) = m_k \circ G_T(\hat{x})$$

where for all $K \in \mathcal{T}_h$, we have denoted by F_K the one-to-one mapping that maps $\widehat{K}$ onto K (F_K is a degree k polynomial mapping). Under this assumption we have $\ell = O(h)$.

We denote by Φ_h the mapping that maps the approximated domain Ω_h^Γ onto the exact domain Ω_Γ. The procedure to exhibit the mapping Φ_h in the context of approximation by isoparametric finite elements in 3D is given in [11]. The restriction of Φ_h to each element $K \in \mathcal{T}_h$ is denoted by Ψ_K^h. The properties of the mapping Φ_h are given in the following lemma. We refer to [11] for the proof and to [9] for further details.

Lemma 2. - *For all integer $s \leq k+1$, there exists a positive real γ_s such that*

$$\|D^s(\Psi_K^h - I)\|_{\infty,K} = \sup_{x \in K} \|D^s(\Psi_K^h - I)(x)\| \leq \gamma_s\, h^{k+1-s}.$$

- *The mapping Ψ_K^h is a $\mathcal{C}^{k+1}$ diffeomorphism from K onto $\check{K}$.*
- *There exists a positive real γ such that*

$$\sup_{x \in K} |J_{\Psi_K^h}(x) - 1| \leq \gamma\, h^k$$

where $J_{\Psi_K^h}(x)$ denotes the absolute value of the jacobian determinant of Ψ_K^h at point x.

Lemma 3. *The mapping $(\Psi_K^h)^{-1}$ satisfies :*

- *for all integer $s \leq k+1$, there exists a positive real δ_s such that*

$$\|D^s((\Psi_K^h)^{-1} - I)\|_{\infty,\check{K}} = \sup_{x \in \check{K}} \|D^s((\Psi_K^h)^{-1} - I)(x)\| \leq \delta_s\, h^{k+1-s};$$

- *there exists a positive real δ such that*

$$\sup_{x \in \check{K}} |J_{(\Psi_K^h)^{-1}}(x) - 1| \leq \delta\, h^k.$$

3.2. Finite element approximation

Let V_h be the Lagrange isoparametric degree k finite element space over the triangulation $\mathcal{T}_h$ of Ω_h^Γ. For every $K \in \mathcal{T}_h$ we denote by Π_k the interpolation operator over K and by π_h the interpolation operator over the whole triangulation $\mathcal{T}_h$:

$$\forall v \in \mathcal{C}^0(\overline{\Omega_h^\Gamma}) \quad \forall K \in \mathcal{T}_h \quad (\pi_h v)_{|K} = \Pi_k(v_{|K}).$$

We define the approximation g_h of g by the following formula

$$g_h \circ m_k^\Sigma \circ G_T = \widehat{\Pi}_k(g \circ \check{m}^\Sigma \circ G_T) \qquad \forall T \in \mathcal{T}_l. \tag{21}$$

As well, we define the respective approximations $\mathrm{D}_h^\Gamma G_n$ and $\mathrm{D}_h^\Gamma G$ of $\mathrm{D}^\Gamma G_n$ and $\mathrm{D}^\Gamma G$ by the formulae :

$$\mathrm{D}_h^\Gamma G_n(m_k^\Sigma \circ G_T, m_k^\Gamma \circ G_T) = (\widehat{\Pi}_k \times \widehat{\Pi}_k)\mathrm{D}^\Gamma G_n(\check{m}^\Sigma \circ G_T, \check{m}^\Gamma \circ G_T), \tag{22}$$

and

$$\mathrm{D}_h^\Gamma G(m_k^\Sigma \circ G_T, m_k^\Gamma \circ G_T) = (\widehat{\Pi}_k \times \widehat{\Pi}_k)\mathrm{D}^\Gamma G(\check{m}^\Sigma \circ G_T, \check{m}^\Gamma \circ G_T). \tag{23}$$

We can thus consider the following approximate problem:

$$\left\{ \begin{array}{l} \text{find } \ \phi_h \in V_h \text{ such that for all } \ v_h \in V_h, \\ \qquad\qquad b_h(\phi_h, v_h) + k_h(\phi_h, v_h) = f_h(v_h), \end{array} \right. \tag{24}$$

where the bilinear forms b_h and k_h over V_h are defined by

$$b_h(u_h, v_h) \;=\; \int_{\Omega_h^{\Sigma\Gamma}} \nabla u_h . \nabla v_h \ \mathrm{d}x + \mu \int_{\Omega_h} \nabla u_h . \nabla v_h \ \mathrm{d}x + \int_{\Gamma_h} u_h v_h \ \mathrm{d}\gamma \tag{25}$$

$$k_h(u_h, v_h) \;=\; (\mu - 1) \int_{\Gamma_h} v_h(y) \int_{\Sigma_h} u_h(x)\, \mathrm{D}_h^\Gamma G_n(x, y) \ \mathrm{d}\sigma_x \ \mathrm{d}\gamma_y, \tag{26}$$

and the linear form f_h over V_h is defined by

$$f_h(v_h) \;=\; \int_{\Sigma_h} g_h(x)\, v_h(x) \ \mathrm{d}\sigma_x + \int_{\Gamma_h} v_h(y) \int_{\Sigma_h} g_h(x)\, \mathrm{D}_h^\Gamma G(x, y) \ \mathrm{d}\sigma_x \ \mathrm{d}\gamma_y. \tag{27}$$

As well we introduce the bilinear form a_h over V_h defined by $a_h = b_h + k_h$.

If $\{\xi_i\}_{1 \le i \le N}$ denotes a basis of the approximation space V_h and $\{\phi_i\}_{1 \le i \le N}$ denotes the components of the solution ϕ_h to problem (24) in this basis then problem (24) reads,

$$\left\{ \begin{array}{l} \text{find } \ \{\phi_i\}_{1 \le i \le N} \text{ such that for all } \ i \in \{1, \dots, N\} \\ \qquad\qquad \displaystyle\sum_{j=1}^N \phi_j b_h(\xi_j, \xi_i) + \sum_{j=1}^N \phi_j k_h(\xi_j, \xi_i) = f_h(\xi_i). \end{array} \right. \tag{28}$$

In a matrix notation, we have:

$$\left\{ \begin{array}{l} \text{find } \ U \in \mathbb{R}^N \text{ such that} \\ \qquad\qquad (\mathcal{B} + \mathcal{K})U = F \end{array} \right. \tag{29}$$

where $U \in \mathbb{R}^N$ is the vector of the ϕ_i, $1 \leq i \leq N$ $F \in \mathbb{R}^N$ is the vector of the $f_h(\xi_i)$, $1 \leq i \leq N$ and $\mathcal{B} \in \mathcal{M}_N(\mathbb{R})$ and $\mathcal{K} \in \mathcal{M}_N(\mathbb{R})$ stand respectively for the matrices of generic term $b_{ij} = b_h(\xi_j, \xi_i)$ and $k_{ij} = k_h(\xi_j, \xi_i)$. The matrix $\mathcal{B}$ is a sparse and symmetric matrix but $\mathcal{K}$ is a full and not symmetric matrix, see Fig. 5 for the matrix profiles.

We can notice that V_h a subspace of $\mathbb{H}^1(\Omega_h^\Gamma)$ but it is not a subspace of $\mathbb{H}^1(\Omega_\Gamma)$. The mapping Φ_h previously defined will enable us to recover the framework of internal approximation. Let us define

$$\check{V}_h = \left\{\check{v}_h; \exists v_h \in V_h, \check{v}_h = v_h \circ \Phi_h^{-1}\right\}.$$

If $\phi_h \in V_h$ denotes the solution of problem (24), then $\check{\phi}_h = \phi_h \circ \Phi_h^{-1}$ is a solution of the following problem:

$$\left\{\begin{array}{l} \text{find } \check{\phi}_h \in \check{V}_h \text{ such that for all } \check{v}_h \in \check{V}_h, \\ \qquad\qquad \check{b}_h(\check{\phi}_h, \check{v}_h) + \check{k}_h(\check{\phi}_h, \check{v}_h) = \check{f}_h(\check{v}_h), \end{array}\right. \tag{30}$$

where the bilinear forms $\check{b}_h$ and $\check{k}_h$ are defined by

$$\check{b}_h(\check{\phi}_h, \check{v}_h) = b_h(\check{\phi}_h \circ \Phi_h, \check{v}_h \circ \Phi_h), \qquad \check{k}_h(\check{\phi}_h, \check{v}_h) = k_h(\check{\phi}_h \circ \Phi_h, \check{v}_h \circ \Phi_h)$$

and the linear form $\check{f}_h$ by $\check{f}_h(\check{v}_h) = f_h(\check{v}_h \circ \Phi_h)$. We set $\check{a}_h = \check{b}_h + \check{k}_h$ and we denote by $\check{A}_h$, $\check{B}_h$ and $\check{K}_h$ the linear operators associated to $\check{a}_h$, $\check{b}_h$ and $\check{k}_h$ respectively.

We finish the section by giving some classical results (see [12] and [11]) useful for the error estimate carried out in the next section.

Lemma 4. - *For all integer $s \leq k+1$, the norms $v_h \in \mathcal{C}^0(K) \mapsto \|v_h\|_{s,K}$ and $v_h \in \mathcal{C}^0(K) \mapsto \|v_h \circ \Phi_h^{-1}\|_{s,\Omega}$ are uniformly equivalent with respect to h.*

- *For all integer $s \leq k+1$, the semi-norms $v_h \in \mathcal{C}^0(K) \mapsto |v_h|_{s,K}$ and $v_h \in \mathcal{C}^0(K) \mapsto |v_h \circ \Phi_h^{-1}|_{s,\Omega}$ are uniformly equivalent with respect to h.*

Lemma 5. *Assume that v belongs to $\mathcal{C}^0(\overline{\Omega_\Gamma}) \cap \mathbb{H}^{k+1}(\Omega_\Gamma)$ and set $\check{v}_h = (\Pi_h(v \circ \Phi_h)) \circ \Phi_h^{-1}$. Then*

$$\|v - \check{v}_h\|_{1,\Omega_\Gamma} \leq C h^k \|v\|_{k+1,\Omega_\Gamma} \tag{31}$$

Furthermore, the mapping $v \in \mathbb{H}^s(\Omega) \longmapsto \int_\Omega \|\mathrm{D}^s v(x)\| \, \mathrm{d}x$ define a semi-norm over $\mathbb{H}^s(\Omega)$ equivalent to the canonical semi-norm $|\ |_{s,\Omega}$ over $\mathbb{H}^s(\Omega)$, see [12].

4. Abstract error estimate

4.1. The abstract error estimate

Our aim in this section is to obtain an estimate of the error $\|\phi - \check{\phi}_h\|_{1,\Omega_\Gamma}$ where ϕ denotes the solution to problem (17) and $\check{\phi}_h$ denotes the solution to problem (30). The classical abstract error estimate (see [12], [13])

$$\begin{aligned} \|\phi - \check{\phi}_h\|_{1,\Omega_\Gamma} \quad \leq \quad & C\left(\inf_{\check{v}_h \in \check{V}_h} \left\{ \|\phi - \check{v}_h\|_{1,\Omega_\Gamma} + \sup_{\check{w}_h \in \check{V}_h} \frac{|a(\check{v}_h, \check{w}_h) - \check{a}_h(\check{v}_h, \check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} \right\} \right. \\ & \left. + \sup_{\check{w}_h \in \check{V}_h} \frac{|f(\check{w}_h) - \check{f}_h(\check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} \right) \end{aligned} \tag{32}$$

who usually arise from the coerciveness of the bilinear form $a = b + k$ on $\mathbb{H}^1(\Omega_\Gamma)$ is deduced for our problem from the coerciveness of the bilinear form b on $\mathbb{H}^1(\Omega_\Gamma)$ and the compactness of K as detailed below under the assumption:

($\mathcal{H}$) there exists a continuous function α of h which vanishes with h such that

$$|a(\check{v}_h, \check{w}_h) - \check{a}_h(\check{v}_h, \check{w}_h)| \leq \alpha(h)\|\check{v}_h\|_{1,\Omega_\Gamma}\|\check{w}_h\|_{1,\Omega_\Gamma}.$$

Lemma 6. *Under the assumption ($\mathcal{H}$), there exists $\beta \in \mathbb{R}_+^*$ such that for all sufficiently small h*

$$\sup_{\check{w}_h \in \check{V}_h} \frac{\left|(\check{A}_h \check{v}_h, \check{w}_h)\right|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} \geq \beta\|\check{v}_h\|_{1,\Omega_\Gamma} \qquad \forall \check{v}_h \in \check{V}_h. \tag{33}$$

Proof If the inequality doesn't hold, one can find a sequence $(h_n)_n$ that converges to 0 and a sequence $(\check{v}_{h_n})_n$ such that $\check{v}_{h_n}$ belongs to $\check{V}_{h_n}$ and $\|\check{v}_{h_n}\|_{1,\Omega_\Gamma} = 1$ for any integer n with

$$\lim_{n\to\infty} \sup_{\check{w}_{h_n} \in \check{V}_{h_n}} \frac{\left|(\check{A}_{h_n} \check{v}_{h_n}, \check{w}_{h_n})\right|}{\|\check{w}_{h_n}\|_{1,\Omega_\Gamma}} = 0. \tag{34}$$

For any $w \in \mathbb{H}^1(\Omega_\Gamma)$ and $\check{w}_{h_n} \in \check{V}_{h_n}$ we have

$$\begin{aligned}
|(A\check{v}_{h_n}, w)| &\leq |(A\check{v}_{h_n}, w - \check{w}_{h_n})| + \left|((A - \check{A}_{h_n})\check{v}_{h_n}, \check{w}_{h_n})\right| + \left|(\check{A}_{h_n}\check{v}_{h_n}, \check{w}_{h_n})\right| \\
&\leq \|A\|\|\check{v}_{h_n}\|_{1,\Omega_\Gamma}\|w - \check{w}_{h_n}\|_{1,\Omega_\Gamma} + \alpha(h_n)\|\check{v}_{h_n}\|_{1,\Omega_\Gamma}\|\check{w}_{h_n}\|_{1,\Omega_\Gamma} \\
&\quad + \|\check{w}_{h_n}\|_{1,\Omega_\Gamma} \sup_{\check{s} \in \check{V}_{h_n}} \frac{\left|(\check{A}_{h_n}\check{v}_{h_n}, \check{s})\right|}{\|\check{s}\|_{1,\Omega_\Gamma}}
\end{aligned}$$

Since $\|\check{v}_{h_n}\|_{1,\Omega_\Gamma} = 1$, it follows that

$$\begin{aligned}
|(A\check{v}_{h_n}, w)| &\leq \|A\| \inf_{\check{w}_{h_n} \in \check{V}_{h_n}} \|w - \check{w}_{h_n}\|_{1,\Omega_\Gamma} + \alpha(h_n) \inf_{\check{w}_{h_n} \in \check{V}_{h_n}} \|\check{w}_{h_n}\|_{1,\Omega_\Gamma} \\
&\quad + \left(\sup_{\check{s} \in \check{V}_{h_n}} \frac{\left|(\check{A}_{h_n}\check{v}_{h_n}, \check{s})\right|}{\|\check{s}\|_{1,\Omega_\Gamma}} \right) \inf_{\check{w}_{h_n} \in \check{V}_{h_n}} \|\check{w}_{h_n}\|_{1,\Omega_\Gamma}.
\end{aligned} \tag{35}$$

Now from lemma 5 we have

$$\lim_{n\to+\infty} \inf_{\check{w}_{h_n} \in \check{V}_{h_n}} \|w - \check{w}_{h_n}\|_{1,\Omega_\Gamma} = 0.$$

We deduce from relation (35) and assumption (34) that for any $w \in \mathbb{H}^1(\Omega_\Gamma)$,

$$\lim_{n\to+\infty} |(A\check{v}_{h_n}, w)| = 0.$$

It follows that the sequence $(A\check{v}_{h_n})_n$ weakly converge in $\mathbb{H}^1(\Omega_\Gamma)$ to 0 and therefore that the sequence $(\check{v}_{h_n})_n$ weakly converge in $\mathbb{H}^1(\Omega_\Gamma)$ to 0. In particular the sequence $(\check{v}_{h_n})_n$ is bounded in $\mathbb{H}^1(\Omega_\Gamma)$. From the coerciveness of the bilinear form b and under assumption ($\mathcal{H}$) there exists a non negative real γ such that

$$\begin{aligned}
\gamma\|\check{v}_{h_n}\|^2_{1,\Omega_\Gamma} &\leq (B\check{v}_{h_n}, \check{v}_{h_n}) \\
&= ((A - \check{A}_{h_n})\check{v}_{h_n}, \check{v}_{h_n}) - (K\check{v}_{h_n}, \check{v}_{h_n}) + (\check{A}_{h_n}\check{v}_{h_n}, \check{v}_{h_n}) \\
&\leq \alpha(h)\|\check{v}_{h_n}\|^2_{1,\Omega_\Gamma} + \|K\check{v}_{h_n}\|_{1,\Omega_\Gamma}\|\check{v}_{h_n}\|_{1,\Omega_\Gamma} + \|\check{v}_{h_n}\|_{1,\Omega_\Gamma} \sup_{\check{w}_{h_n} \in \check{V}_{h_n}} \frac{\left|(\check{A}_{h_n}\check{v}_{h_n}, \check{w}_{h_n})\right|}{\|\check{w}_{h_n}\|_{1,\Omega_\Gamma}}.
\end{aligned}$$

We thus have

$$\gamma\|\check{v}_{h_n}\|_{1,\Omega_\Gamma} \leq \alpha(h_n)\|\check{v}_{h_n}\|_{1,\Omega_\Gamma} + \|K\check{v}_{h_n}\|_{1,\Omega_\Gamma} + \sup_{\check{w}_{h_n}\in\check{V}_{h_n}} \frac{\left|(\check{A}_{h_n}\check{v}_{h_n},\check{w}_{h_n})\right|}{\|\check{w}_{h_n}\|_{1,\Omega_\Gamma}}. \tag{36}$$

Now as K is compact in $\mathbb{H}^1(\Omega_\Gamma)$, the sequence $(K\check{v}_{h_n})_n$ strongly converge to 0 in $\mathbb{H}^1(\Omega_\Gamma)$. This implies the following contradiction:

$$\lim_{n\to+\infty}\|\check{v}_{h_n}\|_{1,\Omega_\Gamma} = 0 \qquad \text{and} \qquad \|\check{v}_{h_n}\|_{1,\Omega_\Gamma} = 1 \qquad \forall n\in\mathbb{N}.$$

◇

To establish the abstract error estimate (32) we proceed as follows. First, the triangular inequality yields

$$\|\phi - \check{\phi}_h\|_{1,\Omega_\Gamma} \leq \inf_{\check{v}_h\in\check{V}_h}\left\{\|\phi - \check{v}_h\|_{1,\Omega_\Gamma} + \|\check{v}_h - \check{\phi}_h\|_{1,\Omega_\Gamma}\right\} \tag{37}$$

and from lemma 6 we have

$$\|\check{\phi}_h - \check{v}_h\|_{1,\Omega_\Gamma} \leq \frac{1}{\beta}\sup_{\check{w}_h\in\check{V}_h}\frac{\left|(\check{A}_h(\check{\phi}_h - \check{v}_h),\check{w}_h)\right|}{\|\check{w}_h\|_{1,\Omega_\Gamma}}. \tag{38}$$

Now

$$\begin{aligned}
(\check{A}_h(\check{\phi}_h - \check{v}_h),\check{w}_h) &= (\check{A}_h\check{\phi}_h,\check{w}_h) - (\check{A}_h\check{v}_h,\check{w}_h)\\
&= \check{f}_h(\check{w}_h) + ((A-\check{A}_h)\check{v}_h,\check{w}_h) - (A\check{v}_h,\check{w}_h)\\
&= \check{f}_h(\check{w}_h) + ((A-\check{A}_h)\check{v}_h,\check{w}_h) - (A(\check{v}_h - \check{\phi}_h),\check{w}_h) - f(\check{w}_h),
\end{aligned}$$

therefore

$$\left|(\check{A}_h(\check{\phi}_h - \check{v}_h),\check{w}_h)\right| \leq \left|\check{f}_h(\check{w}_h) - f(\check{w}_h)\right| + \left|((A-\check{A}_h)\check{v}_h,\check{w}_h)\right| + \|A\|\,\|\check{v}_h - \check{\phi}_h\|_{1,\Omega_\Gamma}\|\check{w}_h\|_{1,\Omega_\Gamma}$$

and

$$\begin{aligned}
\frac{\left|(\check{A}_h(\check{\phi}_h - \check{v}_h),\check{w}_h)\right|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} &\leq \frac{\left|\check{f}_h(\check{w}_h) - f(\check{w}_h)\right|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} + \frac{\left|((A-\check{A}_h)\check{v}_h,\check{w}_h)\right|}{\|\check{w}_h\|_{1,\Omega_\Gamma}}\\
&\quad + \|A\|\,\|\check{v}_h - \phi\|_{1,\Omega_\Gamma} + \|A\|\,\|\phi - \check{v}_h\|_{1,\Omega_\Gamma}.
\end{aligned} \tag{39}$$

The estimate (32) is then a consequence of relations (37), (38) and (39).

4.2. Estimation of the various components of the error

We must now estimate the terms in the right hand side member of relation (32) and check assumption $(\mathcal{H})$. In the following the same letter C may represent several different non negative constants.

Proposition 2. *For all $\check{w}_h, \check{v}_h \in \check{V}_h$ we have*

$$|a(\check{w}_h,\check{v}_h) - \check{a}_h(\check{w}_h,\check{v}_h)| \leq Ch^k\|\check{w}_h\|_{1,\Omega_\Gamma}\|\check{v}_h\|_{1,\Omega_\Gamma}. \tag{40}$$

Proof Let $\breve{w}_h, \breve{v}_h \in \breve{V}_h$ and set $w_h = \breve{w}_h \circ \Phi_h$, $v_h = \breve{v}_h \circ \Phi_h$. We have

$$
\begin{aligned}
b(\breve{w}_h, \breve{v}_h) - \breve{b}_h(\breve{w}_h, \breve{v}_h) &= b(\breve{w}_h, \breve{v}_h) - b_h(\breve{w}_h \circ \Phi_h, \breve{v}_h \circ \Phi_h) = b(\breve{w}_h, \breve{v}_h) - b_h(w_h, v_h) \\
&= \int_{\Omega_{\Sigma\Gamma}} \nabla \breve{w}_h(\breve{x}) \cdot \nabla \breve{v}_h(\breve{x})\ \mathrm{d}\breve{x} - \int_{\Omega_h^{\Sigma\Gamma}} \nabla w_h(x) \cdot \nabla v_h(x)\ \mathrm{d}x \\
&\quad + \mu \left(\int_{\Omega} \nabla \breve{w}_h(\breve{x}) \cdot \nabla \breve{v}_h(\breve{x})\ \mathrm{d}\breve{x} - \int_{\Omega_h} \nabla w_h(x) \cdot \nabla v_h(x)\ \mathrm{d}x \right) \\
&\quad + \left(\int_{\Gamma} \breve{w}_h(\breve{x})\, \breve{v}_h(\breve{x})\ \mathrm{d}\gamma_{\breve{x}} - \int_{\Gamma_h} w_h(x)\, v_h(x)\ \mathrm{d}\gamma_x \right) \\
&= E_1 + \mu E_2 + E_3.
\end{aligned}
$$

Let us first consider the term E_1 :

$$
\begin{aligned}
E_1 &= \int_{\Omega} \nabla \breve{w}_h(\breve{x}) \cdot \nabla \breve{v}_h(\breve{x})\ \mathrm{d}\breve{x} - \int_{\Omega_h} \nabla w_h(x) \cdot \nabla v_h(x)\ \mathrm{d}x \\
&= \sum_{\substack{K \in \mathcal{T}_h \\ K \subset \overline{\Omega}_h}} \int_K \left\{ \nabla \breve{w}_h(\Psi_h^K(x)) \cdot \nabla \breve{v}_h(\Psi_h^K(x))\, J_{\Psi_h^K}(x) - \nabla w_h(x) \cdot \nabla v_h(x) \right\}\ \mathrm{d}x \\
&= \sum_{\substack{K \in \mathcal{T}_h \\ K \subset \overline{\Omega}_h}} \int_K \sum_{i=1}^{3} \left\{ \partial_i \breve{w}_h(\Psi_h^K(x)) \cdot \partial_i \breve{v}_h(\Psi_h^K(x))\, J_{\Psi_h^K}(x) - \partial_i w_h(x) \cdot \partial_i v_h(x) \right\}\ \mathrm{d}x
\end{aligned}
$$

Since $\breve{v}_h = v_h \circ (\Psi_h^K)^{-1}$ for any $y \in \breve{K}$ we have

$$
\partial_i \breve{v}_h(y) = \mathrm{D}\breve{v}_h(y) \cdot \mathrm{e}_i = \mathrm{D}v_h((\Psi_h^K)^{-1}(y)) \cdot (\mathrm{D} \left((\Psi_h^K)^{-1} \right)(y) \cdot \mathrm{e}_i)
$$

and therefore for any $x \in K$,

$$
\partial_i \breve{v}_h(\Psi_h^K(x)) = \mathrm{D}\breve{v}_h(\Psi_h^K(x)) \cdot \mathrm{e}_i = \mathrm{D}v_h(x) \cdot (\mathrm{D} \left((\Psi_h^K)^{-1} \right)(\Psi_h^K(x)) \cdot \mathrm{e}_i).
$$

It follows that E_1 reads

$$
\begin{aligned}
E_1 &= \sum_{\substack{K \in \mathcal{T}_h \\ K \subset \overline{\Omega}_h}} \int_K \sum_{i=1}^{3} \Big\{ \left(\mathrm{D}v_h(x) \cdot (\mathrm{D} \left((\Psi_h^K)^{-1} \right)(\Psi_h^K(x)) \cdot \mathrm{e}_i) \right) \\
&\quad \times \left(\mathrm{D}w_h(x) \cdot (\mathrm{D} \left((\Psi_h^K)^{-1} \right)(\Psi_h^K(x)) \cdot \mathrm{e}_i) \right)\, J_{\Psi_h^K}(x) - (\mathrm{D}v_h(x) \cdot \mathrm{e}_i)\ (\mathrm{D}w_h(x) \cdot \mathrm{e}_i) \Big\}\ \mathrm{d}x.
\end{aligned}
$$

We decompose E_1 in $E_1 = E_{1,1} + E_{1,2} + E_{1,3}$ where

$$\begin{aligned}
E_{1,1} &= \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \int_K \sum_{i=1}^{3} \left(J_{\Psi_h^K}(x) - 1\right) \left(\mathrm{D}v_h(x)\cdot (\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x))\cdot \mathrm{e}_i)\right)\\
&\qquad \times \left(\mathrm{D}w_h(x)\cdot (\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x))\cdot \mathrm{e}_i)\right)\ \mathrm{d}x,\\
E_{1,2} &= \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \int_K \sum_{i=1}^{3} \left(\mathrm{D}v_h(x)\cdot (\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x)) - I)\cdot \mathrm{e}_i\right) \left(\mathrm{D}w_h(x)\cdot \mathrm{e}_i\right)\ \mathrm{d}x,\\
E_{1,3} &= \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \int_K \sum_{i=1}^{3} \left(\mathrm{D}w_h(x)\cdot (\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x)) - I)\cdot \mathrm{e}_i\right)\\
&\qquad \times \left(\mathrm{D}v_h(x)\cdot (\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x))\cdot \mathrm{e}_i)\right)\ \mathrm{d}x.
\end{aligned}$$

We have to estimate the three terms $E_{1,1}$, $E_{1,2}$ and $E_{1,3}$. The first one can be estimated as follow:

$$\begin{aligned}
|E_{1,1}| &\leq 3 \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \int_K \left|J_{\Psi_h^K}(x) - 1\right| \|\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x))\|^2 \|\mathrm{D}v_h(x)\|\|\mathrm{D}w_h(x)\|\ \mathrm{d}x\\
&\leq 3 \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \|J_{\Psi_h^K} - 1\|_{\infty,K} \|\mathrm{D}\left((\Psi_h^K)^{-1}\right)\|^2_{\infty,\check{K}}\\
&\qquad \times \left(\int_K \|\mathrm{D}v_h(x)\|^2\ \mathrm{d}x\right)^{1/2} \left(\int_K \|\mathrm{D}w_h(x)\|^2\ \mathrm{d}x\right)^{1/2}.
\end{aligned}$$

We deduce from lemmas 2, 3 and 4 the estimate

$$|E_{1,1}| \leq C h^k |\check{v}_h|_{1,\Omega} |\check{w}_h|_{1,\Omega}. \tag{41}$$

The term $E_{1,2}$ can be estimated in the following way:

$$\begin{aligned}
|E_{1,2}| &= \left| \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \int_K \sum_{i=1}^{3} \left(\mathrm{D}v_h(x)\cdot (\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x)) - I)\cdot \mathrm{e}_i\right)\left(\mathrm{D}w_h(x)\cdot \mathrm{e}_i\right)\right|\\
&\leq 3 \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \int_K \|\mathrm{D}v_h(x)\|\|\mathrm{D}w_h(x)\|\|\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x)) - I\|\ \mathrm{d}x\\
&\leq 3 \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \|\mathrm{D}\left((\Psi_h^K)^{-1}\right) - I\|_{\infty,\check{K}} \left(\int_K \|\mathrm{D}v_h(x)\|^2\ \mathrm{d}x\right)^{1/2} \left(\int_K \|\mathrm{D}w_h(x)\|^2\ \mathrm{d}x\right)^{1/2}.
\end{aligned}$$

From lemma 3 we have

$$\|\mathrm{D}\left((\Psi_h^K)^{-1}\right) - I\|_{\infty,\check{K}} \leq C h^k$$

and from lemma 4

$$\left(\int_K \|\mathrm{D}v_h(x)\|^2\ \mathrm{d}x\right)^{1/2} \leq C|v_h|_{1,K} \leq C|\check{v}_h|_{1,\check{K}}$$

and

$$\left(\int_K \|\mathrm{D}w_h(x)\|^2 \,\mathrm{d}x\right)^{1/2} \leq C|w_h|_{1,K} \leq C|\breve{w}_h|_{1,\breve{K}}.$$

Thus we have

$$|E_{1,2}| \leq Ch^k |\breve{v}_h|_{1,\Omega} |\breve{w}_h|_{1,\Omega}. \tag{42}$$

As well we have

$$\begin{aligned}
|E_{1,3}| \quad &\leq \quad 3 \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \int_K \|\mathrm{D}v_h(x)\| \|\mathrm{D}w_h(x)\| \|\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x))\| \\
&\qquad \times \|\mathrm{D}\left((\Psi_h^K)^{-1}\right)(\Psi_h^K(x)) - I\| \,\mathrm{d}x \\
&\leq \quad 3 \sum_{\substack{K\in\mathcal{T}_h\\ K\subset\overline{\Omega}_h}} \|\mathrm{D}\left((\Psi_h^K)^{-1}\right) - I\|_{\infty,\breve{K}} \|\mathrm{D}\left((\Psi_h^K)^{-1}\right)\|_{\infty,\breve{K}} \\
&\qquad \times \left(\int_K \|\mathrm{D}v_h(x)\|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_K \|\mathrm{D}w_h(x)\|^2 \,\mathrm{d}x\right)^{1/2}
\end{aligned}$$

and from lemmas 3 and 4 we deduce

$$|E_{1,3}| \leq Ch^k |v_h|_{1,\Omega} |w_h|_{1,\Omega}. \tag{43}$$

As a consequence of relations (41), (42) and (43) we have

$$|E_1| \leq Ch^k |v_h|_{1,\Omega} |w_h|_{1,\Omega}. \tag{44}$$

We proceed on the same manner to get the following estimate for E_2:

$$|E_2| \leq Ch^k |\breve{v}_h|_{1,\Omega_{\Sigma\Gamma}} |\breve{w}_h|_{1,\Omega_{\Sigma\Gamma}}. \tag{45}$$

The last stage is the estimation of E_3 given by

$$E_3 = \int_\Gamma \breve{w}_h(\breve{x})\, \breve{v}_h(\breve{x}) \,\mathrm{d}\gamma_{\breve{x}} - \int_{\Gamma_h} w_h(x)\, v_h(x) \,\mathrm{d}\gamma_x. \tag{46}$$

We decompose each integral term in a sum of integrals over the triangles of the triangulation of the domains of the charts for the boundary Γ:

$$E_3 = \sum_T \int_T \Big(\breve{w}_h(\breve{m}(s,t))\, \breve{v}_h(\breve{m}(s,t))\, J_{\breve{m}}(s,t) - w_h(m_k(s,t))\, v_h(m_k(s,t))\, J_{m_k}(s,t)\Big) \,\mathrm{d}s \,\mathrm{d}t$$

The relation $\breve{v}_h(\breve{m}(s,t)) = v_h(m_k(s,t))$ leads to

$$\begin{aligned}
|E_3| \quad &\leq \quad \sum_T \int_T \left|\breve{w}_h(\breve{m}(s,t))\, \breve{v}_h(\breve{m}(s,t)) \left(J_{\breve{m}}(s,t) - J_{m_k}(s,t)\right)\right| \,\mathrm{d}s \,\mathrm{d}t \\
&\leq \quad \sum_T \|J_{\breve{m}} - J_{m_k}\|_{\infty,T} \|\breve{v}_h \circ \breve{m}\|_{0,T} \|\breve{w}_h \circ \breve{m}\|_{0,T}.
\end{aligned}$$

From lemma 1, it follows that

$$|E_3| \leq Ch^k \|\breve{v}_h\|_{0,\Gamma} \|\breve{w}_h\|_{0,\Gamma}. \tag{47}$$

From relations (44), (45) and (47) we deduce that

$$\begin{aligned} |b(\check{w}_h,\check{v}_h)-\check{b}_h(\check{w}_h,\check{v}_h)| &\leq Ch^k\left(|\check{v}_h|_{1,\Omega}|\check{w}_h|_{1,\Omega}+|\check{v}_h|_{1,\Omega_{\Sigma\Gamma}}|\check{w}_h|_{1,\Omega_{\Sigma\Gamma}}+\|\check{v}_h\|_{0,\Gamma}\|\check{w}_h\|_{0,\Gamma}\right)\\ &\leq Ch^k\Big(\big(|\check{v}_h|^2_{1,\Omega_\Gamma}+\|\check{v}_h\|^2_{0,\Gamma}\big)^{1/2}\big(|\check{w}_h|^2_{1,\Omega_\Gamma}+\|\check{w}_h\|^2_{0,\Gamma}\big)^{1/2}\Big)\end{aligned}$$

We can therefore conclude the first part of the proof with the estimate:

$$|b(\check{w}_h,\check{v}_h)-\check{b}_h(\check{w}_h,\check{v}_h)|\leq Ch^k\|\check{v}_h\|_{1,\Omega_\Gamma}\|\check{w}_h\|_{1,\Omega_\Gamma}. \tag{48}$$

We now turn out to the estimate of

$$\begin{aligned} k(\check{w}_h,\check{v}_h)-\check{k}_h(\check{w}_h,\check{v}_h) &= \int_\Gamma \check{v}_h(\check{y})\Big(\int_\Sigma \check{w}_h(\check{x})\,\mathrm{D}^\Gamma G_n(\check{x},\check{y})\ \mathrm{d}\sigma_{\check{x}}\Big)\ \mathrm{d}\gamma_{\check{y}}\\ &\quad-\int_{\Gamma_h}\check{v}_h(\Psi_h^K(y))\Big(\int_{\Sigma_h}\check{w}_h(\Psi_h^K(x))\,\mathrm{D}_h^\Gamma G_n(x,y)\ \mathrm{d}\sigma_x\Big)\ \mathrm{d}\gamma_y.\end{aligned}$$

We can decompose each integral term in a sum of integrals over the triangles of the triangulation of the domains of the charts for the boundaries Γ and Σ. On the one hand we have

$$\Sigma=\bigcup_{T\in\mathcal{T}_h^\Sigma}\check{m}^\Sigma(T)\qquad\qquad \Gamma=\bigcup_{T\in\mathcal{T}_h^\Gamma}\check{m}^\Gamma(T)$$

and on the other hand

$$\Sigma_h=\bigcup_{T\in\mathcal{T}_h^\Sigma}m_k^\Sigma(T)\qquad\qquad \Gamma_h=\bigcup_{T\in\mathcal{T}_h^\Gamma}m_k^\Gamma(T).$$

It follows that

$$k(\check{w}_h,\check{v}_h)-\check{k}_h(\check{w}_h,\check{v}_h)=\sum_{T^\Sigma\in\mathcal{T}_h^\Sigma}\ \sum_{T^\Gamma\in\mathcal{T}_h^\Gamma}E_{4,T} \tag{49}$$

where

$$\begin{aligned} E_{4,T}=&\int_{T^\Sigma}\int_{T^\Gamma}\check{v}_h(\check{m}^\Gamma(y))\,\check{w}_h(\check{m}^\sigma(x))\,\mathrm{D}^\Gamma G_n(\check{m}^\Sigma(x),\check{m}^\Gamma(y))J_{\check{m}^\Gamma}(y)J_{\check{m}^\Sigma}(x)\ \mathrm{d}\sigma_x\ \mathrm{d}\gamma_y\\ &-\int_{T^\Sigma}\int_{T^\Gamma}v_h(m_k^\Gamma(y))\,w_h(m_k^\Sigma(x))\,\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))J_{m_k^\Gamma}(y)J_{m_k^\Sigma}(x)\ \mathrm{d}\sigma_x\ \mathrm{d}\gamma_y.\end{aligned}$$

The relations $\check{v}_h(\check{m}^\Gamma(y))=v_h(m_k^\Gamma(y))$ and $\check{w}_h(\check{m}^\Sigma(x))=w_h(m_k^\Sigma(x))$ give

$$\begin{aligned} E_{4,T} =& \int_{T^\Sigma}\int_{T^\Gamma}\check{v}_h(\check{m}_k^\Gamma(y))\,\check{w}_h(\check{m}_k^\Sigma(x))\\ &\times\Big(\mathrm{D}^\Gamma G_n(\check{m}^\Sigma(x),\check{m}^\Gamma(y))J_{\check{m}^\Gamma}(y)J_{\check{m}^\Sigma}(x)-\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))J_{m_k^\Gamma}(y)J_{m_k^\Sigma}(x)\Big)\ \mathrm{d}\sigma_x\ \mathrm{d}\gamma_y.\end{aligned}$$

Thus, we have

$$\begin{aligned} |E_{4,T}| \leq\ & \|\check{v}_h\circ\check{m}_k^\Gamma\|_{0,T^\Gamma}\|\check{w}_h\circ\check{m}_k^\Sigma\|_{0,T^\Sigma}\int_{T^\Sigma}\int_{T^\Gamma}\Big|\mathrm{D}^\Gamma G_n(\check{m}^\Sigma(x),\check{m}^\Gamma(y))J_{\check{m}^\Gamma}(y)J_{\check{m}^\Sigma}(x)\\ &-\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))J_{m_k^\Gamma}(y)J_{m_k^\Sigma}(x)\Big|\ \mathrm{d}\sigma_x\ \mathrm{d}\gamma_y.\end{aligned}$$

Now,

$$\begin{aligned}
&\int_{T^\Sigma}\int_{T^\Gamma}\Big|\mathrm{D}^\Gamma G_n(\check m^\Sigma(x),\check m^\Gamma(y))J_{\check m^\Gamma}(y)J_{\check m^\Sigma}(x)-\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))J_{m_k^\Gamma}(y)J_{m_k^\Sigma}(x)\Big|\,\mathrm{d}\sigma_x\,\mathrm{d}\gamma_y\\
&\quad\le\int_{T^\Sigma}\int_{T^\Gamma}\Big|\big(\mathrm{D}^\Gamma G_n(\check m^\Sigma(x),\check m^\Gamma(y))-\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))\big)\,J_{\check m^\Gamma}(y)J_{\check m^\Sigma}(x)\Big|\,\mathrm{d}\sigma_x\,\mathrm{d}\gamma_y\\
&\quad+\int_{T^\Sigma}\int_{T^\Gamma}\Big|\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))\big(J_{\check m^\Gamma}(y)-J_{m_k^\Gamma}(y)\big)J_{\check m^\Sigma}(x)\Big|\,\mathrm{d}\sigma_x\,\mathrm{d}\gamma_y\\
&\quad+\int_{T^\Sigma}\int_{T^\Gamma}\Big|\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))\big(J_{\check m^\Sigma}(x)-J_{m_k^\Sigma}(x)\big)J_{\check m^\Gamma}(y)\Big|\,\mathrm{d}\sigma_x\,\mathrm{d}\gamma_y.
\end{aligned}$$

The first term in the right hand side can be estimated using classical interpolation results :

$$\begin{aligned}
&\int_{T^\Sigma}\int_{T^\Gamma}\Big|\mathrm{D}^\Gamma G_n(\check m^\Sigma(x),\check m^\Gamma(y))-\mathrm{D}_h^\Gamma G_n(m_k^\Sigma(x),m_k^\Gamma(y))\Big|\,\mathrm{d}\sigma_x\,\mathrm{d}\gamma_y\\
&\qquad=\|(I-\Pi_k^\Sigma\times\Pi_k^\Gamma)\mathrm{D}^\Gamma G_n(\check m^\Sigma,\check m^\Gamma)\|_{0,T^\Sigma\times T^\Gamma}\le Ch^{k+1}.
\end{aligned}$$

Using lemma 1 to estimate the two other terms, we obtain

$$|E_{4,T}|\le Ch^k\|\check v_h\circ\check m_k^\Gamma\|_{0,T^\Gamma}\|\check w_h\circ\check m_k^\Sigma\|_{0,T^\Sigma}.$$

It follows that

$$\big|k(\check w_h,\check v_h)-\check k_h(\check w_h,\check v_h)\big|\le Ch^k\|\check v_h\|_{0,\Gamma}\|\check w_h\|_{0,\Gamma}. \tag{50}$$

From relations (48) and (50) we deduce that

$$\begin{aligned}
|a(\check w_h,\check v_h)-\check a_h(\check w_h,\check v_h)| &\le |b(\check w_h,\check v_h)-\check b_h(\check w_h,\check v_h)|+|k(\check w_h,\check v_h)-\check k_h(\check w_h,\check v_h)|\\
&\le Ch^k\|\check w_h\|_{1,\Omega_\Gamma}\|\check v_h\|_{1,\Omega_\Gamma}.
\end{aligned}\tag{51}$$

The proof is achieved. ◇

Proposition 3. *We have*

$$\sup_{\check w_h\in\check V_h}\frac{\big|f(\check w_h)-\check f_h(\check w_h)_h\big|}{\|\check w_h\|_{1,\Omega_\Gamma}}\le Ch^k\|g\|_{k+1,\Sigma}. \tag{52}$$

Proof It follows from (20) and (27) that

$$\begin{aligned}
\big|f(\check w_h)-\check f_h(\check w_h)\big|&\le\left|\int_\Sigma g(\check x)\,\check w_h(\check x)\,\mathrm{d}\sigma_{\check x}-\int_{\Sigma_h}g_h(x)\,\check w_h(\Phi_h(x))\,\mathrm{d}\sigma_x\right|\\
&+\left|\int_\Gamma\check w_h(\check y)\int_\Sigma g(\check x)\,\mathrm{D}^\Gamma G(\check x,\check y)\,\mathrm{d}\sigma_{\check x}\,\mathrm{d}\gamma_{\check y}-\int_{\Gamma_h}\check w_h(\Phi_h(y))\int_{\Sigma_h}g_h(x)\,\mathrm{D}_h^\Gamma G(x,y)\,\mathrm{d}\sigma_x\,\mathrm{d}\gamma_y\right|.
\end{aligned}\tag{53}$$

To estimate the first term of the right hand side of (53) we decompose the integrals as the sum of integrals over the triangles of the triangulation of the domain of the chart of the boundary Σ. Namely,

$$\begin{aligned}
|E_5| &= \left|\int_\Sigma g(\check x)\,\check w_h(\check x)\,\mathrm{d}\sigma_{\check x}-\int_{\Sigma_h}g_h(x)\,\check w_h(\Phi_h(x))\,\mathrm{d}\sigma_x\right|\\
&= \Bigg|\sum_{T\in\mathcal{T}_h}\int_T\Big(g(\check m(s,t))\,\check w_h(\check m(s,t))\,J_m(s,t)\\
&\qquad -g_h(m_k(s,t))\,\check w_h(\Phi_h(m_k(s,t)))\,J_{m_k}(s,t)\Big)\,\mathrm{d}s\,\mathrm{d}t\Bigg|.
\end{aligned}$$

From the relations $\check{w}_h(\Phi_h(m_k(s,t))) = \check{w}_h(\check{m}(s,t))$ and $g_h(m_k(s,t)) = \Pi_k(g \circ \check{m})(s,t)$ we deduce that

$$\begin{aligned}
|E_5| &= \Big| \sum_{T \in \mathcal{T}_h} \int_T \Big(\big(g(\check{m}(s,t)) - \Pi_k(g \circ \check{m})(s,t)\big)\big) J_{m_k}(s,t)\, \check{w}_h(\check{m}(s,t)) \\
&\quad - g(\check{m}(s,t))\, \check{w}_h(\check{m}(s,t))\, \big(J_{m_k}(s,t) - J_{\check{m}}(s,t)\big)\Big)\, \mathrm{d}s\ \mathrm{d}t \Big| \\
&\leq \sum_{T \in \mathcal{T}_h} \Big(\|J_{m_k}\|_{\infty,T} \|\check{w}_h \circ \check{m}\|_{0,T} \|g \circ \check{m} - \Pi_k(g \circ \check{m})\|_{0,T} \\
&\quad + \|g \circ \check{m}\|_{0,T} \|\check{w}_h \circ \check{m}\|_{0,T} \|J_{\check{m}} - J_{m_k}\|_{\infty,T} \Big).
\end{aligned}$$

From classical interpolation results we have

$$\|g \circ \check{m} - \Pi_k(g \circ \check{m})\|_{0,T} \leq C h^{k+1} \|g \circ \check{m}\|_{k+1,T}$$

and from lemma 1

$$\|J_m - J_{m_k}\|_{\infty,T} \leq \alpha\, h^k.$$

We deduce that

$$|E_5| \quad \leq \quad C h^k \|g\|_{k+1,\Sigma} \|\check{w}_h\|_{0,\Sigma}. \tag{54}$$

Let us now consider the second term in relation (53). Once more, we decompose each integral in a sum of integrals over the triangles of the triangulation of the domains of the charts for the boundaries Γ and Σ. We have

$$\begin{aligned}
E_6 &= \int_\Gamma \check{w}_h(\check{y}) \int_\Sigma g(\check{x})\, \mathrm{D}^\Gamma G(\check{x},\check{y})\ \mathrm{d}\sigma_{\check{x}}\ \mathrm{d}\gamma_{\check{y}} - \int_{\Gamma_h} \check{w}_h(\Phi_h(y)) \int_{\Sigma_h} g_h(x)\, \mathrm{D}_h^\Gamma G(x,y)\ \mathrm{d}\sigma_x\ \mathrm{d}\gamma_y \\
&= \sum_{T^\Sigma \in \mathcal{T}_h^\Sigma} \sum_{T^\Gamma \in \mathcal{T}_h^\Gamma} E_{6,T}
\end{aligned}$$

where

$$\begin{aligned}
E_{6,T} = \int_{T^\Sigma} \int_{T^\Gamma} \Big(&\check{w}_h(\check{m}^\Gamma(s_1,t_1))\, g(\check{m}^\Sigma(s_2,t_2))\, \mathrm{D}^\Gamma G(\check{m}^\Sigma(s_2,t_2), \check{m}^\Gamma(s_1,t_1))\, J_{\check{m}^\Gamma}(s_1,t_1)\, J_{\check{m}^\Sigma}(s_2,t_2) \\
&- \check{w}_h(\Phi_h(m_k^\Gamma(s_1,t_1)))\, g_h(m_k^\Sigma(s_2,t_2))\, \mathrm{D}_h^\Gamma G(m_k^\Sigma(s_2,t_2), m_k^\Gamma(s_1,t_1))\, J_{m_k^\Gamma}(s_1,t_1)\, J_{m_k^\Sigma}(s_2,t_2) \Big)\ \mathrm{d}\sigma\ \mathrm{d}\gamma.
\end{aligned}$$

The relations $\check{w}_h(\Phi_h(\check{m}_k^\Gamma(s_1,t_1))) = \check{w}_h(\check{m}^\Gamma(s_1,t_1))$ and $g_h(m_k^\Sigma(s_2,t_2)) = \Pi_k(g(\check{m}^\Sigma(s_2,t_2)))$ lead to

$$\begin{aligned}
E_{6,T} = \int_{T^\Sigma} \int_{T^\Gamma} \check{w}_h(\check{m}^\Gamma(s_1,t_1)) \times \Big(&g(\check{m}^\Sigma(s_2,t_2))\, \mathrm{D}^\Gamma G(\check{m}^\Sigma(s_2,t_2), \check{m}^\Gamma(s_1,t_1))\, J_{\check{m}^\Gamma}(s_1,t_1)\, J_{\check{m}^\Sigma}(s_2,t_2) \\
&- \Pi_k(g(\check{m}^\Sigma(s_2,t_2)))\, \mathrm{D}_h^\Gamma G(m_k^\Sigma(s_2,t_2), m_k^\Gamma(s_1,t_1))\, J_{m_k^\Gamma}(s_1,t_1)\, J_{m_k^\Sigma}(s_2,t_2) \Big)\ \mathrm{d}\sigma\ \mathrm{d}\gamma.
\end{aligned}$$

In order to estimate $E_{6,T}$ we split its expression as follows:

$$\begin{aligned}
E_{6,T} \;=\; & \int_{T^\Sigma}\int_{T^\Gamma} \check{w}_h(\check{m}^\Gamma(s_1,t_1)) \times \Big(\Big(g_h(m_k^\Sigma(s_2,t_2)) - \Pi_k(g\circ\check{m}^\Sigma)(s_2,t_2)\Big) \\
& \qquad \times \mathrm{D}^\Gamma G(\check{m}^\Sigma(s_2,t_2),\check{m}^\Gamma(s_1,t_1))\, J_{\check{m}^\Gamma}(s_1,t_1) J_{\check{m}^\Sigma}(s_2,t_2) \\
& + \Big(\mathrm{D}^\Gamma G(\check{m}^\Sigma(s_2,t_2),\check{m}^\Gamma(s_1,t_1)) - \mathrm{D}_h^\Gamma G(m_k^\Sigma(s_2,t_2), m_k^\Gamma(s_1,t_1))\Big) \\
& \qquad \times g_h(m_k^\Sigma(s_2,t_2))\, J_{\check{m}^\Gamma}(s_1,t_1)\, J_{\check{m}^\Sigma}(s_2,t_2) \\
& + g_h(m_k^\Sigma(s_2,t_2))\, \mathrm{D}_h^\Gamma G(m_k^\Sigma(s_2,t_2), m_k^\Gamma(s_1,t_1))\, \Big(J_{\check{m}^\Gamma}(s_1,t_1) - J_{m_k^\Gamma}(s_1,t_1)\Big)\, J_{\check{m}^\Sigma}(s_2,t_2) \\
& + g_h(m_k^\Sigma(s_2,t_2))\, \mathrm{D}_h^\Gamma G(m_k^\Sigma(s_2,t_2), m_k^\Gamma(s_1,t_1))\, J_{m_k^\Gamma}(s_1,t_1)\, \Big(J_{\check{m}^\Sigma}(s_2,t_2) - J_{m_k^\Sigma}(s_2,t_2)\Big)\Big)\, \mathrm{d}\sigma\, \mathrm{d}\gamma,
\end{aligned}$$

We then deduce the relation

$$\begin{aligned}
& |E_{6,T}| \le C\|\check{w}_h\circ\check{m}^\Gamma\|_{0,T^\Gamma}\Big(\\
& \|g_h\circ\check{m}_k^\Sigma - \Pi_k(g\circ\check{m}^\Sigma)\|_{0,T^\Sigma}\|\mathrm{D}^\Gamma G\circ(\check{m}^\Sigma,\check{m}^\Gamma)\|_{0,T^\Sigma\times T^\Gamma}\|J_{\check{m}^\Gamma}\|_{\infty,T^\Gamma}\|J_{\check{m}^\Sigma}\|_{\infty,T^\Sigma} \\
& +\|g_h\circ m_k^\Sigma\|_{0,T^\Sigma}\|\mathrm{D}^\Gamma G\circ(\check{m}^\Sigma,\check{m}^\Gamma) - \mathrm{D}_h^\Gamma G\circ(m_k^\Sigma,m_k^\Gamma)\|_{0,T^\Sigma\times T^\Gamma}\|J_{\check{m}^\Gamma}\|_{\infty,T^\Gamma}\|J_{\check{m}^\Sigma}\|_{\infty,T^\Sigma} \\
& +\|g_h\circ m_k^\Sigma\|_{0,T^\Sigma}\|\mathrm{D}_h^\Gamma G\circ(m_k^\Sigma,m_k^\Gamma)\|_{0,T^\Sigma\times T^\Gamma}\|J_{\check{m}^\Gamma} - J_{m_k^\Gamma}\|_{\infty,T^\Gamma}\|J_{\check{m}^\Sigma}\|_{\infty,T^\Sigma} \\
& +\|g_h\circ m_k^\Sigma\|_{0,T^\Sigma}\|\mathrm{D}_h^\Gamma G\circ(m_k^\Sigma,m_k^\Gamma)\|_{0,T^\Sigma\times T^\Gamma}\|J_{m_k^\Gamma}\|_{\infty,T^\Gamma}\|J_{\check{m}^\Sigma} - J_{m_k^\Sigma}\|_{\infty,T^\Sigma}\Big).
\end{aligned}$$

From interpolation theory, we have

$$\|g\circ\check{m}^\Sigma - \Pi_k(g\circ\check{m}^\Sigma)\|_{0,T} \le Ch^{k+1}\|g\circ\check{m}^\Sigma\|_{k+1,T}$$

and

$$\begin{aligned}
\|\mathrm{D}^\Gamma G(\check{m}^\Sigma,\check{m}^\Gamma) - \mathrm{D}_h^\Gamma G(m_k^\Sigma,m_k^\Gamma)\|_{0,T^\Sigma\times T^\Gamma} &= \|(I-\Pi_k^\Sigma\times\Pi_k^\Gamma)\mathrm{D}^\Gamma G_n(\check{m}^\Sigma,\check{m}^\Gamma)\|_{0,T^\Sigma\times T^\Gamma} \\
&\le Ch^{k+1}\|\mathrm{D}^\Gamma G_n(\check{m}^\Sigma,\check{m}^\Gamma)\|_{k+1,T^\Sigma\times T^\Gamma}.
\end{aligned}$$

It follows from lemma 1 that

$$\|J_{\check{m}^\Gamma} - J_{m_k^\Gamma}\|_{\infty,T^\Gamma} \le \alpha\, h^k \qquad \text{and} \qquad \|J_{\check{m}^\Sigma} - J_{m_k^\Sigma}\|_{\infty,T^\Sigma} \le \alpha\, h^k.$$

Therefore we have

$$|E_{6,T}| \le C\|\check{w}_h\circ\check{m}^\Gamma\|_{0,T^\Gamma}\Big(h^{k+1}\|g\circ\check{m}^\Sigma\|_{k+1,T} + h^k\|g_h\circ m_k^\Sigma\|_{0,T^\Sigma}\Big)$$

and

$$|E_6| \le \sum_{T^\Sigma\in\mathcal{T}_h^\Sigma}\sum_{T^\Gamma\in\mathcal{T}_h^\Gamma}|E_{6,T}| \le Ch^k\|\check{w}_h\|_{0,\Gamma}\|g\|_{k+1,\Sigma} \tag{55}$$

Using (54) and (55) we conclude that

$$\sup_{\check{w}_h\in\check{V}_h}\frac{|f(\check{w}_h) - \check{f}_h(\check{w}_h)_h|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} \le Ch^k\|g\|_{k+1,\Sigma}. \tag{56}$$

◇

We are now in position to conclude the error analysis of our method.

Theorem 1. *If ϕ denotes the solution to problem (17) and $\check{\phi}_h$ denotes the solution to problem (30) then*

$$\|\phi - \check{\phi}_h\|_{1,\Omega_\Gamma} \leq Ch^k \left(\|\phi\|_{k+1,\Omega_\Gamma} + \|g\|_{k+1,\Sigma}\right). \tag{57}$$

Proof From (32) we have

$$\|\phi - \check{\phi}_h\|_{1,\Omega_\Gamma} \leq C\left(\inf_{\check{v}_h \in \check{V}_h}\left\{\|\phi - \check{v}_h\|_{1,\Omega_\Gamma} + \sup_{\check{w}_h \in \check{V}_h} \frac{|a(\check{v}_h, \check{w}_h) - \check{a}_h(\check{v}_h, \check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}}\right\} + \sup_{\check{w}_h \in \check{V}_h} \frac{|f(\check{w}_h) - \check{f}_h(\check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}}\right). \tag{58}$$

We must estimate the three terms in the right hand side of the latter inequality. First, from proposition 3 we have

$$\sup_{\check{w}_h \in \check{V}_h} \frac{|f(\check{w}_h) - \check{f}_h(\check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} \leq Ch^k \|g\|_{k+1,\Sigma}. \tag{59}$$

The estimate

$$\sup_{\check{w}_h \in \check{V}_h} \frac{|a(\check{v}_h, \check{w}_h) - \check{a}_h(\check{v}_h, \check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}} \leq Ch^k \|\check{v}_h\|_{1,\Omega_\Gamma}, \tag{60}$$

from proposition 2 implies that

$$\begin{aligned}
&\inf_{\check{v}_h \in \check{V}_h}\left\{\|\phi - \check{v}_h\|_{1,\Omega_\Gamma} + \sup_{\check{w}_h \in \check{V}_h} \frac{|a(\check{v}_h, \check{w}_h) - \check{a}_h(\check{v}_h, \check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}}\right\} \\
&\leq \inf_{\check{v}_h \in \check{V}_h}\left\{\|\phi - \check{v}_h\|_{1,\Omega_\Gamma} + Ch^k \|\check{v}_h\|_{1,\Omega_\Gamma}\right\} \\
&\leq \|\phi - \check{\phi}_h\|_{1,\Omega_\Gamma} + Ch^k \|\check{\phi}_h\|_{1,\Omega_\Gamma}
\end{aligned}$$

where $\check{\phi}_h$ denotes the solution to problem (30). From lemma 5 we have

$$\|\phi - \check{\phi}_h\|_{1,\Omega_\Gamma} \leq Ch^k \|\phi\|_{1,\Omega_\Gamma}$$

and since $\|\check{\phi}_h\|_{1,\Omega_\Gamma} \leq \|\check{\phi}_h - \phi\|_{1,\Omega_\Gamma} + \|\phi\|_{1,\Omega_\Gamma}$ we obtain

$$\inf_{\check{v}_h \in \check{V}_h}\left\{\|\phi - \check{v}_h\|_{1,\Omega_\Gamma} + \sup_{\check{w}_h \in \check{V}_h} \frac{|a(\check{v}_h, \check{w}_h) - \check{a}_h(\check{v}_h, \check{w}_h)|}{\|\check{w}_h\|_{1,\Omega_\Gamma}}\right\} \leq Ch^k \|\phi\|_{1,\Omega_\Gamma}. \tag{61}$$

The estimate (57) is then a consequence of (59) and (61). ◇

Theorem 1 shows that the error is of the same order as the error in the degree k Lagrange finite element method provide the boundaries involved are approximated by piecewise degree k interpolated surfaces according to the way presented in section 3.1.

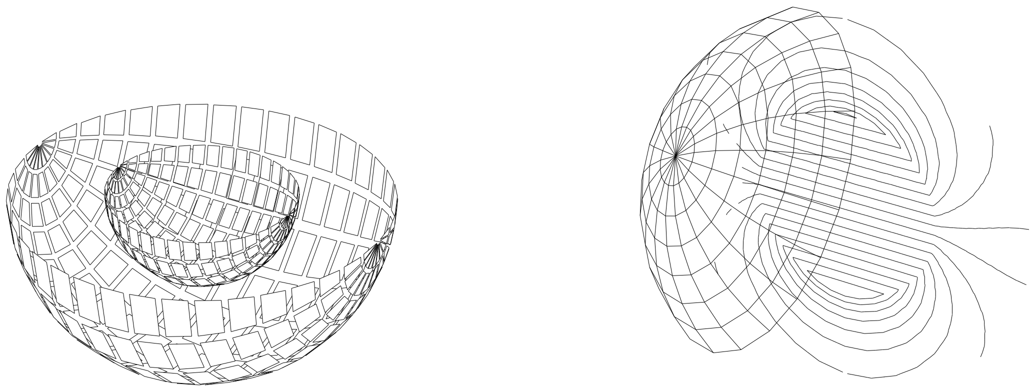

Figure 4. Left: mesh of the domain Ω_Γ (the 2 boundaries Σ and Γ are represented). Right: isolines for the reduced scalar potential in a plane parallel to $\mathbf{H}_s$ passing through the center of the ball.

5. Numerical implementation of the method

The numerical implementation of the above method is achieved using the numerical program MÉLINA developed at the *Institut de Recherche Mathématique de Rennes*, University of Rennes 1 by D. Martin. It is an open collection of Fortran libraries dedicated to the solution of partial differential problem by finite element method. All the computations were done on an INTEL PIII 700Mhz biprocessor personal computer.

5.1. A test example

We have considered the case where the domain Ω is the ball of radius 1 cm and magnetic permeability $\mu = 10^3$. The inductor field $\mathbf{H_s}$ was assumed to be constant in intensity and direction so that an exact expression for ϕ is known. The artificial boundary Γ was the sphere of radius 2 cm. We have meshed the bounded domain Ω_Γ with 3920 elements, see figure 4.

The computation of the reduced scalar potential required 39 seconds. The computation of the coupling terms was fast (0.7 s). The major part of the time was spent in assembling the matrix of the system (19.1 s) and in solving the linear system (16.4 s). The quadratic error over the domain was about 1%. Figure 4 shows the isolines for the reduced scalar potential in a plane parallel to $\mathbf{H}_s$ passing through the center of the ball.

Using the integral representation formula (10) we have computed the total magnetic field $\mathbf{H_m}$ in 408 arbitrary points. It took 31.5 s with the method presented in [14].

In figure 5, the sparsity pattern for the two matrices $\mathcal{B}$ and $\mathcal{K}$ is shown, see (29). The matrix $\mathcal{B}$ is symmetric and sparse but the matrix $\mathcal{K}$ is not.

5.2. Example of an electromagnetic device

We consider an electromagnet which consists of a cylindrical core situated inside a pair of coils, see figure 6. The currents in the two coils are imposed in opposite direction and of constant density (1 A/mm^2). The electromagnet core has the following physical and geometric features: diameter 1 cm, length 3 cm, magnetic permeability $\mu = 10^3$. We have bounded the domain with a sphere of radius 4 cm and have meshed the bounded domain Ω_Γ with 5840 elements.

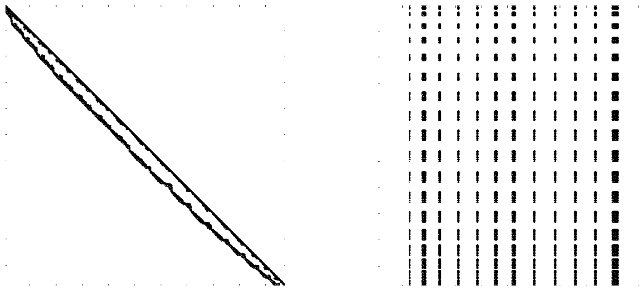

Figure 5. Sparsity pattern for the symmetric matrix $\mathcal{B}$ (on the left) and $\mathcal{K}$ (on the right). As $\mathcal{B}$ is symmetric, only its lower part is shown.

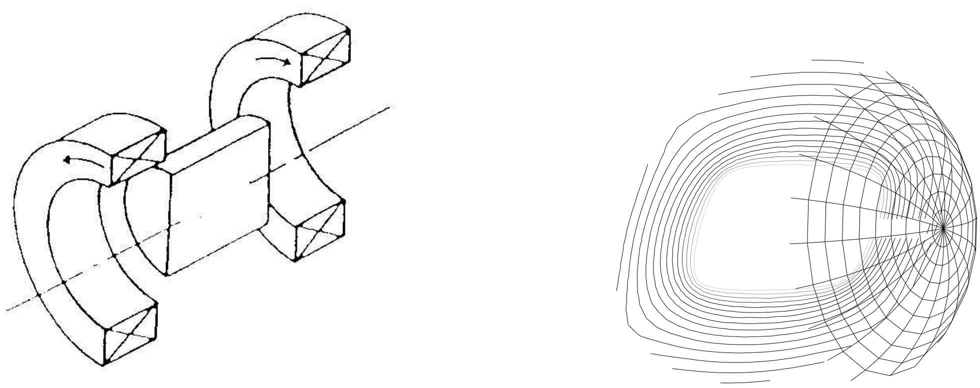

Figure 6. Left: electromagnet which consists of a cylindrical core situated inside a pair of coils. Right: isolines for the reduced scalar potential in the plane shown on the left figure.

The computation of the reduced scalar potential required 136 s while the computation of the magnetic field in 570 arbitrary points took 107 s.

6. Conclusion

We have presented a numerical method to solve magnetostatic problems in three dimensional unbounded domains based on the coupling of the finite element method and integral representation formulae. The basic idea of the method was to bound the exterior domain using an artificial boundary that can be close to the device boundary but always distinct from it. The boundary condition to set on this artificial boundary was obtained using a boundary integral representation formula of the solution, the support for the integral representation being the device boundary. As the two boundaries are distinct, all the involved integrals are regular and standard quadrature schemes can be used. The magnetic potential in the interior domain surrounded by the artificial boundary is computed using a standard finite element approximation. We have carried out an error analysis of the method and we have shown that the convergence order of the method is $\mathcal{O}(h^k)$ when isoparametric Lagrange type k finite elements are used for the discretisation.

This computational approach is well suited for electromagnetic device shape optimisation where the area of interest is localised; with slight modifications it can be used to treat also axisymmetric problems. The coupling boundary can be placed close to the electromagnetic device to reduce the size of the interior domain where the finite element method is employed. The magnetic field at the node of the control surface can be computed using an integral representation formula.

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