Cancellation errors in an integral for calculating magnetic field from reduced scalar potential

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Abstract

In magnetostatic field computation with regions containing current sources, it is classical to write the corresponding magnetostatic problem in terms of the reduced scalar magnetic potential ϕ. Usually numerical differentiation is used to get the magnetic field $\mathbf{H}$ from the potential values, which implies loss in accuracy. An alternative is to compute $\mathbf{H}$ from ϕ from an integral formula. In fact the formula does not give a straightforward method due to a cancellation in the integral. In this paper we investigate the mathematical reason why the formula is not suited for numerical purposes. We do a careful numerical analysis with illustrations on a test example and propose a way to circumvent this difficulty based on a sort of decomposition method.

1 Introduction

The situation under consideration is the computation of the magnetic field generated by an electromagnetic device composed of a weakly ferromagnetic core Ω and an inductor Ω_s characterized by a time independent current density $\mathbf{j}$ in a three dimensional geometry. The domain Ω is a simply connected open set in $\mathbb{R}^3$. We denote by Σ and Σ_s the boundary of Ω and Ω_s respectively and by Ω^c and Ω_s^c their complement in $\mathbb{R}^3$. We assume for simplicity that the metallic core Ω has a constant permeability. This assumption is not a limitation of the study and we give in appendix the way to deal with the general case of a domain Ω with piecewise constant permeabilities.

It is classical to express the total magnetic field $\mathbf{H}$ as $\mathbf{H} = \mathbf{H_s} + \mathbf{H_m}$ where $\mathbf{H_s}$, the field due to the source currents, satisfies

$$\left\{ \begin{array}{rcll} \operatorname{div} \mathbf{H_s} &=& 0 & \text{in } \mathbb{R}^3, \\ \mathbf{rot}\, \mathbf{H_s} &=& \mathbf{j} & \text{in } \Omega_s, \\ \mathbf{rot}\, \mathbf{H_s} &=& \mathbf{0} & \text{in } \Omega_s^c, \\ [\mathbf{H_s} \wedge \mathbf{n}] &=& \mathbf{0} & \text{across } \Sigma_s, \end{array} \right. \tag{1}$$

and $\mathbf{H_m}$, the reaction of the ferromagnetic piece, satisfies

$$\left\{ \begin{array}{rcll} \mathbf{rot}\, \mathbf{H_m} &=& \mathbf{0} & \text{in } \mathbb{R}^3, \\ \operatorname{div} \mathbf{H_m} &=& 0 & \text{in } \Omega \text{ and } \Omega^c, \\ (\mu - 1)\, \mathbf{H_m} \cdot \mathbf{n} &=& (1-\mu)\, \mathbf{H_s} \cdot \mathbf{n} & \text{across } \Sigma, \end{array} \right. \tag{2}$$

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where μ is the relative magnetic permeability of the ferromagnetic core, $\mathbf{n}$ the outward unit normal to Σ or Σ_s, and $[\,]$ the jump across these surfaces.
$\mathbf{H_s}$ can be efficiently computed, see [1], [2], through the evaluation of the Biot and Savart integral

$$\mathbf{H_s}(x) = \frac{1}{4\pi} \int_{\Omega_s} \left(\mathbf{j}(y) \wedge \frac{\mathbf{r}}{r^3}\right) \, \mathrm{d}y \tag{3}$$

with $\mathbf{r} = \mathbf{x} - \mathbf{y}$. Thus the computation of the total magnetic field is reduced to the computation of $\mathbf{H_m}$. The advantage of this approach is that although the conductors lie within the computational area, they do not need to be meshed when using a finite element discretization to solve problem (2). Moreover all what is needed for the computation of $\mathbf{H_m}$ is the value of the field $\mathbf{H_s}$ over the surface Σ.

As the field $\mathbf{H_m}$ is curl free, we can introduce the so-called *reduced scalar magnetic potential* (RSP) ϕ such that $\mathbf{H_m} = -\nabla\phi$. The RSP satisfies the following problem deduced from (2),

$$\left\{ \begin{array}{ll} \Delta\phi = 0 & \text{in } \Omega \text{ and } \Omega^c, \\ \mu \dfrac{\partial\phi}{\partial n}\Big|_{\Omega} - \dfrac{\partial\phi}{\partial n}\Big|_{\Omega^c} = (\mu - 1)\, g & \text{across } \Sigma, \end{array} \right. \tag{4}$$

where $g = \mathbf{H_s} \cdot \mathbf{n}$ is considered as a given function since $\mathbf{H_s}$ is computed using (3).

The main advantage of introducing the RSP is to transform a problem for a 3 components vector field throughout the space into a problem for a scalar function, thus reducing the number of degrees of freedom in the discretization. A numerical method widely used to solve magnetostatic problems is the finite element method (FEM). As problem (4) is set in an unbounded domain an artificial boundary can be introduced at a finite distance from the electromagnetic device. The behavior of the solution at infinity is then handled through an approximate boundary condition set on this artificial boundary that is derived from an asymptotic expansion of the solution at infinity, see [3]. Another way to proceed is to set an exact boundary condition obtained using a boundary integral representation of the solution and to write a variational formulation in a bounded domain, see [4]. For the magnetostatic problem (4) the method is discussed in [5] where numerical results are presented. The goal in the paper is to focus on the numerical computation of the vector field $\mathbf{H_m}$ from the RSP ϕ, supposed to be numerically computed.

The evident way to compute the field $\mathbf{H_m}$ from the RSP is to numerically differentiate the FEM potential solution on each element of the mesh. This leads to loss in accuracy. For instance, when problem (4) is discretized using Lagrange finite element of degree 1, a constant field value is obtained on each element. Another way to proceed is to compute $\mathbf{H_m}$ through the integral representation formula

$$\begin{aligned} \mathbf{H_m}(y) \quad = \quad & (1-\mu) \int_{\Sigma} g(x)\ \nabla_y G(x,y)\ \mathrm{d}\sigma_x \\ & -(1-\mu) \int_{\Sigma} \phi(x)\ \nabla_y \frac{\partial G}{\partial n_x}(x,y)\ \mathrm{d}\sigma_x, \end{aligned} \tag{5}$$

where G denotes the Green kernel for the Laplacian. This formula would allow the computation of $\mathbf{H_m}$ in any point from the values of the RSP on Σ. We have an explicit representation formula for $\mathbf{H_m}$ since the integration applies only on the surface Σ. This formula is of great interest when the magnetic field is needed in very few points on a localized area. For an overview of derivative extraction methods we refer to [6] and [7].

Here we present a way to use formula (5) to compute the magnetic field from the RSP. Indeed while attractive, formula (5) cannot be used in its present form. We observe from numerical experiments that for large values of μ ($\mu \sim 10^3$), which is the case in most applications, the two integrals in the right-hand side of (5) nearly cancel. Another similar phenomenon occurs when using the reduced scalar potential to compute the field within a permeable region, see [8]. To compute the total magnetic field $\mathbf{H}$ it is necessary to add the field $\mathbf{H_s}$ to the field $\mathbf{H_m}$. If the total field is small (for instance due to a shielding effect of the permeable region), the 2 components of the field would tend to be of same magnitude, but of different sign leading to an oscillating, error prone, solution for the total field. However there are many situations where the total magnetic field is not needed in the permeable region but in the exterior domain. For instance, our work originates from a shape optimization study where the magnetic field is needed in a very small region on the air-gap of an electromagnet. In such an example, formula (5) is the ideal way to compute the magnetic field.

We investigate the mathematical reason for this cancellation and we propose a way to compute these integrals based on an asymptotic expansion for ϕ function of μ. This is an example of an elegant and attractive mathematical formula useless for numerical purposes without a careful mathematical analysis. Let us remark that in [9], the limit $1/\mu \to 0$ of ϕ function of μ is studied to validate the boundary condition on a highly conducting wall. The approach concerns the first term of the development. Our goal is different and we use the expansion to compute the magnetic field out of an integral formula; our method remains valid for a large range of μ.

The content of the paper is the following. Section II is devoted to set up the integral formula (5). In section III a careful analysis is done to know the dependence of ϕ on μ. In section IV we present the way relation (5) has to be used to compute the magnetic field from the RSP accurately and we conclude in section V with numerical experiments.

For $\Omega \subset \mathbb{R}^3$ let $\mathbb{L}^2(\Omega)$ be the set of square integrable functions over Ω. For $m \in \mathbb{N}^*$ $\mathbb{H}^m(\Omega)$ denotes the set of functions with derivatives up to the order m in $\mathbb{L}^2(\Omega)$. To handle functions defined over the unbounded domain Ω^c we will use the Sobolev spaces $\mathbb{W}_0^1(\mathbb{R}^3)$ and $\mathbb{W}_0^1(\Omega^c)$ defined by

$$\mathbb{W}_0^1(\Omega^c) = \Big\{\psi \ ; \ \frac{\psi}{\sqrt{1+|x|^2}} \in \mathbb{L}^2(\Omega^c), \quad \nabla\psi \in \mathbb{L}^2(\Omega^c)^3\Big\}.$$

2 The integral representation formula

The representation formula (5) is obtained by applying potential theory results to the reduced scalar potential ϕ. Let G denotes the Green kernel associated with the three-dimensional Laplacian,

$$G(x,y) = \frac{1}{4\pi|x-y|} \qquad \text{for } \ x, y \in \mathbb{R}^3, x \neq y,$$

and $G_n(x,y) = \nabla_x G(x,y).\mathbf{n}, x \in \Sigma, y \in \mathbb{R}^3$, denotes its normal derivative on Σ.

One can prove that problem (4) has a unique solution in the Sobolev space $\mathbb{W}_0^1(\mathbb{R}^3)$ and that this solution is continuous in $\mathbb{R}^3$. Furthermore, since the magnetic potential ϕ is harmonic in the exterior domain Ω^c we have the Green representation formula for $y \in \Omega^c$, see [10],

$$\phi(y) = \int_\Sigma \phi\Big|_{\Omega^c}(x)\ G_n(x,y)\ \mathrm{d}\sigma_x - \int_\Sigma \frac{\partial\phi}{\partial n}\Big|_{\Omega^c}(x)\ G(x,y)\ \mathrm{d}\sigma_x. \tag{6}$$

Let's take $x \in \Omega$ and $y \in \Omega^c$. We deduce from Green second identity the relations,

$$\begin{aligned} 0 &= \int_\Omega \left(\Delta_x\phi(x)\ G(x,y) - \phi(x)\Delta_x G(x,y)\right)\ \mathrm{d}x \\ &= \int_\Sigma \left(\frac{\partial\phi}{\partial n}\Big|_{\Omega}(x)\ G(x,y) - \phi\Big|_{\Omega}(x)\ G_n(x,y)\right)\ \mathrm{d}\sigma_x. \end{aligned} \tag{7}$$

Then we multiply (7) by μ and add it to (6) to get for $y \in \Omega^c$

$$\begin{aligned} \phi(y) &= \int_\Sigma \left(\phi\Big|_{\Omega^c}(x) - \mu\ \phi\Big|_{\Omega}(x)\right)\ G_n(x,y)\ \mathrm{d}\sigma_x \\ &- \int_\Sigma \left(\frac{\partial\phi}{\partial n}\Big|_{\Omega^c}(x) - \mu\ \frac{\partial\phi}{\partial n}\Big|_{\Omega}(x)\right)\ G(x,y)\ \mathrm{d}\sigma_x. \end{aligned} \tag{8}$$

Using the boundary condition on Σ in (4), we obtain the following representation formula for $y \in \Omega^c$,

$$\phi(y) = (\mu - 1)\int_\Sigma g(x)\ G(x,y)\ \mathrm{d}\sigma_x - (\mu-1)\int_\Sigma \phi(x)\ G_n(x,y)\ \mathrm{d}\sigma_x. \tag{9}$$

As a byproduct we can express for $y \in \Omega^c$ the reaction field $\mathbf{H_m}(y) = -\nabla\phi(y)$ as

$$\mathbf{H_m}(y) = (1-\mu)\int_\Sigma g(x)\ \nabla_y G(x,y)\ \mathrm{d}\sigma_x - (1-\mu)\int_\Sigma \phi(x)\ \nabla_y G_n(x,y)\ \mathrm{d}\sigma_x. \tag{10}$$

Suppose we have computed ϕ on Σ with a certain numerical method (finite element method or boundary integral method), then $\mathbf{H_m}(y)$ can be computed via the formula (10) since g and G are known functions. In fact for large values of μ ($\mu \sim 10^3$) the two integrals in (10) nearly cancel. This can be observed from numerical computations. Table 1 shows the values of the 2 integral terms as well as the value of $\mathbf{H_m}$ computed using (10) in the case of a metallic ball of relative magnetic permeability $\mu = 10^3$ for arbitrarily chosen points.

However one can observe, see section 5, that formula (10) gives the correct values of $\mathbf{H_m}$ for small values of μ ($\mu \sim 1+10^{-3}$). Such values are valid for paramagnetic materials. It follows that the numerical efficiency of formula (10) strongly depends on the value of the relative magnetic permeability μ. In the next section we investigate through an asymptotic expansion how ϕ, and therefore $\mathbf{H_m}$, depends on μ.

3 How ϕ depends on μ

Let us study how the solution ϕ to (4) depends on μ considered now as a parameter. We write down an asymptotic expansion of the RSP ϕ in power of $\frac{1}{\mu}$. For convenience ϕ^i

Table 1: Values of the third component of the 2 integral terms in (10), value of $\mathbf{H_m} \cdot \mathbf{z}$ computed using (10) and exact value of $\mathbf{H_m} \cdot \mathbf{z}$ for $\mu = 10^3$.

	1st term	2nd term	$\mathbf{H_m}$ by(10)	$\mathbf{H_m}$ exact
P_1	$0.9266\ 10^4$	$0.9327\ 10^4$	$-0.4792\ 10^6$	$-0.2284\ 10^6$
P_2	$0.8897\ 10^4$	$0.8954\ 10^4$	$-0.4522\ 10^6$	$-0.2193\ 10^6$
P_3	$0.7437\ 10^4$	$0.7529\ 10^4$	$-0.7301\ 10^6$	$-0.1847\ 10^6$
P_4	$0.8534\ 10^4$	$0.8629\ 10^4$	$-0.7577\ 10^6$	$-0.2116\ 10^6$
P_5	$0.6962\ 10^4$	$0.7087\ 10^4$	$-0.9959\ 10^6$	$-0.1740\ 10^6$

denotes the restriction of the RSP to the interior domain Ω and ϕ^e the restriction of the RSP to the exterior domain Ω^c. Problem (4) then reads: find $\phi^i \in \mathbb{H}^1(\Omega)$ and $\phi^e \in \mathbb{W}_0^1(\Omega^c)$ such that

$$\begin{cases} \Delta\phi^i = 0 & \text{in} \quad \Omega, \\ \Delta\phi^e = 0 & \text{in} \quad \Omega^c, \\ \phi^e = \phi^i & \text{on} \quad \Sigma, \\ \dfrac{1}{\mu}\dfrac{\partial\phi^e}{\partial n} - \dfrac{\partial\phi^i}{\partial n} = (\dfrac{1}{\mu} - 1)g & \text{on} \quad \Sigma. \end{cases} \tag{11}$$

Let us look for ϕ^i and ϕ^e of the form

$$\phi^i = \phi_0^i + \frac{1}{\mu}\phi_1^i + \frac{1}{\mu^2}\phi_2^i + \dots \tag{12}$$

$$\phi^e = \phi_0^e + \frac{1}{\mu}\phi_1^e + \frac{1}{\mu^2}\phi_2^e + \dots \tag{13}$$

One can easily check that $\phi_k^i \in \mathbb{H}^1(\Omega)$ and $\phi_k^e \in \mathbb{W}_0^1(\Omega^c)$, $k \in \{0,1\}$, should be solutions of the following interior Neumann and exterior Dirichlet coupled problems:

$$(\mathcal{P}_0^i) \quad \begin{cases} \Delta\phi_0^i = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial\phi_0^i}{\partial n} = g & \text{on} \quad \Sigma, \end{cases}$$

$$(\mathcal{P}_0^e) \quad \begin{cases} \Delta\phi_0^e = 0 & \text{in} \quad \Omega^c, \\ \phi_0^e = \phi_0^i & \text{on} \quad \Sigma, \end{cases}$$

$$(\mathcal{P}_1^i) \quad \begin{cases} \Delta\phi_1^i = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial\phi_1^i}{\partial n} = \dfrac{\partial\phi_0^e}{\partial n} - g & \text{on} \quad \Sigma, \end{cases}$$

$$(\mathcal{P}_1^e) \quad \begin{cases} \Delta\phi_1^e = 0 & \text{in} \quad \Omega^c, \\ \phi_1^e = \phi_1^i & \text{on} \quad \Sigma. \end{cases}$$

For $k \geq 2$, ϕ_k^i and ϕ_k^e should satisfy to

$$(\mathcal{P}_k^i) \quad \begin{cases} \Delta \phi_k^i = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial \phi_k^i}{\partial n} = \dfrac{\partial \phi_{k-1}^e}{\partial n} & \text{on} \quad \Sigma, \end{cases}$$

$$(\mathcal{P}_k^e) \quad \begin{cases} \Delta \phi_k^e = 0 & \text{in} \quad \Omega^c, \\ \phi_k^e = \phi_k^i & \text{on} \quad \Sigma. \end{cases}$$

The key point in the above decomposition method is to prove there exists unique sequences of functions $(\phi_k^i)_{k\in\mathbb{N}}$, $(\phi_k^e)_{k\in\mathbb{N}}$ solution to the coupled problems $(\mathcal{P}_k^i)$, $(\mathcal{P}_k^e)$, $k \geq 0$. First to have existence for $(\mathcal{P}_0^i)$ we need the compatibility condition $\int_\Sigma g \; \mathrm{d}\sigma = 0$ which is satisfied since

$$\int_\Sigma g \; \mathrm{d}\sigma = \int_\Sigma \mathbf{H_s} \cdot \mathbf{n} \; \mathrm{d}\sigma = \int_\Omega \operatorname{div} \mathbf{H_s} \; \mathrm{d}x = 0. \tag{14}$$

Therefore there exists a unique solution to $(\mathcal{P}_0^i)$ in the functional space

$$\left\{ \psi \in \mathbb{H}^1(\Omega), \int_\Omega \psi \; \mathrm{d}x = 0 \right\}$$

and we numerically compute a solution $\widetilde{\phi}_0^i$ to the Neumann problem $(\mathcal{P}_0^i)$ such that $\widetilde{\phi}_0^i = \phi_0^i - C_0$ where C_0 is an unknown constant.

The solution to the exterior Dirichlet problem $(\mathcal{P}_0^e)$ is uniquely determined.

To have existence for $(\mathcal{P}_1^i)$, the compatibility condition reads $\displaystyle\int_\Sigma \frac{\partial \phi_0^e}{\partial n} \; \mathrm{d}\sigma = 0$. This condition is satisfied if problem $(\mathcal{P}_0^e)$ has been solved with a good choice for C_0 (namely if the boundary values of ϕ_0^i on Σ have been retrieved from the computed solution $\widetilde{\phi}_0^i$). Let us consider $v \in \mathbb{W}_0^1(\Omega^c)$ the unique solution of

$$(\mathcal{P}_v) \quad \begin{cases} \Delta v = 0 & \text{in} \quad \Omega^c, \\ v = 1 & \text{on} \quad \Sigma. \end{cases}$$

Clearly $\displaystyle\int_\Sigma \frac{\partial v}{\partial n} \; \mathrm{d}\sigma > 0$ and we can set $C_0 = -\dfrac{\displaystyle\int_\Sigma \frac{\partial \widetilde{\phi}_0^e}{\partial n} \; \mathrm{d}\sigma}{\displaystyle\int_\Sigma \frac{\partial v}{\partial n} \; \mathrm{d}\sigma}$ with $\widetilde{\phi}_0^e$ given by

$$(\widetilde{\mathcal{P}}_0^e) \quad \begin{cases} \Delta \widetilde{\phi}_0^e = 0 & \text{in} \quad \Omega^c, \\ \widetilde{\phi}_0^e = \widetilde{\phi}_0^i & \text{on} \quad \Sigma. \end{cases}$$

Once C_0 is computed we deduce ϕ_0^i from $\widetilde{\phi}_0^i$ and with $\phi_0^e = \widetilde{\phi}_0^e + C_0 v$ the compatibility condition $\displaystyle\int_\Sigma \frac{\partial \phi_0^e}{\partial n} \; \mathrm{d}\sigma = 0$ for $(\mathcal{P}_1^i)$ is satisfied. It is interesting to relate $(\mathcal{P}_v)$ with the cutting surface problem in [9]. In both cases we have a jump condition equal to 1, but in a different context : once on the cutting surface for a non simply connected domain and here on the physical surface Σ.

Similarly we continue to construct in a unique way the functions ϕ_k^i and ϕ_k^e, $k \geq 1$. It is simple to check that the series (12) and (13) converge to ϕ in the energy norm.

4 Computation of the magnetic field

We are now in position to explain why for large values of the relative permeability μ formula (10) is unfitted for numerical computation and to present a way to circumvent the issue.

Writing formula (10) for $\mathbf{H_m}$ with the developments (12) and (13), we get for $y \in \Omega^c$

$$\begin{aligned}
\mathbf{H_m}(y) &= (1-\mu)\int_\Sigma \Big(g(x)\ \nabla_y G(x,y) \\
&\qquad -\phi(x)\ \nabla_y G_n(x,y)\Big)\ \mathrm{d}\sigma_x \qquad (15)\\
&= (1-\mu)\int_\Sigma \Big(g(x)\ \nabla_y G(x,y) \\
&\qquad -\phi_0^i(x)\ \nabla_y G_n(x,y)\Big)\ \mathrm{d}\sigma_x \\
&+(\mu-1)\int_\Sigma \left(\sum_{k=1}^{\infty}\frac{1}{\mu^k}\ \phi_k^i(x)\right)\ \nabla_y G_n(x,y)\ \mathrm{d}\sigma_x. \qquad (16)
\end{aligned}$$

Now, since ϕ_0^i is the solution of problem $(\mathcal{P}_0^i)$, the second Green identity yields for $y \in \Omega^c$

$$\int_\Sigma \left(\phi_0^i(x)\ G_n(x,y) - \frac{\partial \phi_0^i}{\partial n}(x)\ G(x,y)\right)\ \mathrm{d}\sigma_x = 0 \qquad (17)$$

and by differentiation with respect to y,

$$\int_\Sigma \left(\phi_0^i(x)\ \nabla_y G_n(x,y) - g(x)\ \nabla_y G(x,y)\right)\ \mathrm{d}\sigma_x = 0. \qquad (18)$$

Thus (16) reads for $y \in \Omega^c$

$$\mathbf{H_m}(y) = \int_\Sigma \left(\sum_{k=1}^{\infty}\frac{\mu-1}{\mu^k}\ \phi_k^i(x)\right)\ \nabla_y G_n(x,y)\ \mathrm{d}\sigma_x. \qquad (19)$$

Now it is simple to know where the cancellation occurs; it occurs in formula (18). When we compute the function ϕ for large values of μ, we have essentially the first term ϕ_0 which gives a cancellation in (16). From (19) we deduce that the values of $\mathbf{H_m}$ directly depend on ϕ_1^i which is not given by a straightforward numerical resolution of (4). Indeed for large values of μ the RSP ϕ has a major contribution from ϕ_0^i and a lower contribution from ϕ_1^i with a ratio between the two contributions equal to $1/\mu$. In numerical computations the contribution from ϕ_1^i falls under the numerical precision and the relevant information in ϕ to compute $\mathbf{H_m}$ is missing.

For small values of the relative permeability μ formula (10) gives accurate results as shown in table 4. The reason is that ϕ_0^i is now not the largest term in the expansions (12), each term $\dfrac{\phi_k^i}{\mu^k}$ has roughly the same magnitude as ϕ_0^i.

One can wonder whether formula (10) would give an accurate solution if the RSP was computed more accurately. The answer is negative as it can be observed in section 5 where we have used the exact value of the RSP to compute the boundary integral. The reason is connected to the way the boundary integral is computed. A mesh with polygons of the surface Σ is used to decompose the integral over Σ in a sum of integral over the polygons. These integrals are then computed using numerical quadrature. Even if we have the exact value for the RSP, the difference between the exact surface and the approach polyhedral surface can be interpreted as a computational error on ϕ leading to an analogous effect to the one described above.

For μ in the range $[10^2, 10^4]$, relation (19) shows that $\mathbf{H_m}$ can be approximated with accuracy by

$$\begin{aligned}\mathbf{H_m}(y) \quad \approx \quad & \frac{\mu-1}{\mu} \int_\Sigma \phi_1^i(x)\ \nabla_y G_n(x,y)\ \mathrm{d}\sigma_x \\ & + \frac{\mu-1}{\mu^2} \int_\Sigma \phi_2^i(x)\ \nabla_y G_n(x,y)\ \mathrm{d}\sigma_x.\end{aligned} \tag{20}$$

Thus, relation (20) gives a way to compute $\mathbf{H_m}$ once ϕ_1^i and ϕ_2^i are known. To compute ϕ_1^i and ϕ_2^i we can follow the steps presented in section III. The method reduces to solve two Laplace equations, one in Ω and one in Ω^c with several right-hand sides.

Problems $(\mathcal{P}_k^i)$ in the ferromagnetic core can be solved using the finite element method. The matrix of the discretized problem is the same for all the $(\mathcal{P}_k^i)$ and therefore time is saved in the matrix assembling process. Furthermore if a direct method is used to solve the linear system then only one matrix factorization is required. If an iterative method is used, depending on the preconditioning technique used this advantage can be lost. Problems $(\mathcal{P}_k^e)$ are set in an unbounded domain. The method we used to compute ϕ_k^e is discussed in [5]. An artificial boundary is introduced at a close distance from the ferromagnetic core and the behavior of ϕ_k^e at infinity is handle through an exact boundary condition which is set on this artificial boundary. It has the advantage of greatly reducing the size of the domain to be mesh and therefore the size of the linear system. The boundary can be set very closely to Ω since the only relevant information for the following computations are the values of ϕ_k^e on the boundary Σ. There is no need to compute the RSP ϕ in this approach since the magnetic field $\mathbf{H}$ is given from the functions ϕ_1^i and ϕ_2^i.

5 Numerical experiments

In order to illustrate our discussion we consider the case where the domain Ω is the ball of radius 1 cm and of relative magnetic permeability μ. The inductor field $\mathbf{H_s}$ is assumed to be constant in intensity and direction so that an exact expression for the RSP ϕ is known.

To solve the exterior Dirichlet problems we use a coupling between a finite element method and an integral representation method as described in [5]. The RSP is computed on Σ with a quadratic relative error of 0.8% and a maximum relative error of 0.7%. The boundary Σ is meshed with 402 elements.

The numerical implementation is achieved using the program MÉLINA [11] developed at the *Institut de Recherche Mathématique de Rennes*, University of Rennes 1. It is an open collection of Fortran libraries dedicated to the solution of partial differential problem by finite element methods. All the computations are done on an INTEL PIII 700Mhz biprocessor personal computer.

To compare the accuracy of formulae (10) and (20) we choose arbitrarily 5 points in the exterior domain Ω^c,

$$\begin{aligned}& P_1 = (0.44, -1.42, -0.14), \quad P_2 = (-0.07, -1.48, 0.22), \\ & P_3 = (0.55, -1.33, -0.40), \quad P_4 = (0.83, -1.21, -0.27), \\ & P_5 = (-0.14, 1.42, 0.44).\end{aligned}$$

Table 2 shows that accurate values of $\mathbf{H_m}$ are obtained using formula (10) for small values of the parameter μ ($\mu = 10^{-3}$ in the example). In that case the 2 integral terms in (10) are distinct so that there is no difficulty in making their difference.

Table 2: Values of the third component of the 2 integral terms in (10), value of $\mathbf{H_m} \cdot \mathbf{z}$ computed using (10) and exact value of $\mathbf{H_m} \cdot \mathbf{z}$ for $\mu = 10^{-3}$.

	1st term	2nd term	$\mathbf{H_m}$ by(10)	$\mathbf{H_m}$ exact
P_1	$0.3069\ 10^1$	$0.9327\ 10^4$	$-0.7420\ 10^2$	$-0.7635\ 10^2$
P_2	$0.2947\ 10^1$	$0.8954\ 10^4$	$-0.7123\ 10^2$	$-0.7329\ 10^2$
P_3	$0.2462\ 10^1$	$0.7529\ 10^4$	$-0.5989\ 10^2$	$-0.6173\ 10^2$
P_4	$0.2826\ 10^1$	$0.8629\ 10^4$	$-0.6865\ 10^2$	$-0.7073\ 10^2$
P_5	$0.2305\ 10^1$	$0.7087\ 10^4$	$-0.5638\ 10^2$	$-0.5815\ 10^2$

Table 3 confirms that formula (10) is irrelevant for numerical computation even if the exact value of the RSP is used.

Table 3: Values of the third component of the 2 integral terms in (10), value of $\mathbf{H_m} \cdot \mathbf{z}$ computed using (10) with the exact value of ϕ and exact value of $\mathbf{H_m} \cdot \mathbf{z}$ for $\mu = 10^3$.

	1st term	2nd term	$\mathbf{H_m}$ by(10)	$\mathbf{H_m}$ exact
P_1	$0.9351\ 10^4$	$0.9327\ 10^4$	$0.1960\ 10^6$	$-0.2284\ 10^6$
P_2	$0.8979\ 10^4$	$0.8954\ 10^4$	$0.1993\ 10^6$	$-0.2193\ 10^6$
P_3	$0.7499\ 10^4$	$0.7529\ 10^4$	$-0.2343\ 10^6$	$-0.1847\ 10^6$
P_4	$0.8606\ 10^4$	$0.8629\ 10^4$	$-0.1837\ 10^6$	$-0.2116\ 10^6$
P_5	$0.7020\ 10^4$	$0.7087\ 10^4$	$-0.5343\ 10^6$	$-0.1740\ 10^6$

Finally, Table 4 shows the values of $\mathbf{H_m}$ computed using the method proposed in section IV. We compare the value obtained with one term in the expansion (20), to the value obtained with two terms in the expansion and to the exact value of $\mathbf{H_m}$ for $\mu = 10^3$. One can see that taking into account two terms in the expansion does not improve the accuracy of the computed approximation of $\mathbf{H_m}$. Indeed when $\mu = 10^3$, the second term is roughly 10^{-3} smaller in magnitude than the first one. The second term is involved in the third decimal of the approximation of $\mathbf{H_m}$ which is under the precision of the computation of the boundary integral.

Table 4: Values of $\mathbf{H_m} \cdot \mathbf{z}$ computed using the asymptotic expansion taking one and two terms and exact value of $\mathbf{H_m} \cdot \mathbf{z}$ for $\mu = 10^3$.

	1 term	2 terms	$\mathbf{H_m}$ exact
P_1	$-0.2237\ 10^6$	$-0.2233\ 10^6$	$-0.2284\ 10^6$
P_2	$-0.2148\ 10^6$	$-0.2144\ 10^6$	$-0.2193\ 10^6$
P_3	$-0.1796\ 10^6$	$-0.1792\ 10^6$	$-0.1847\ 10^6$
P_4	$-0.2061\ 10^6$	$-0.2057\ 10^6$	$-0.2116\ 10^6$
P_5	$-0.1681\ 10^6$	$-0.1678\ 10^6$	$-0.1740\ 10^6$

6 Conclusion

In this paper we have investigated in detail an integral formula to compute the magnetic field from the reduced potential in magnetostatics. First we have shown that while attractive, the formula cannot be used as it is due to a cancellation effect of the two involved integrals. Then we have investigated the reason for this cancellation. Finally we have proposed a way to accurately compute the magnetic field in the exterior domain using an asymptotic expansion of the reduced potential.

Appendix

We consider the more general case where the core Ω contains N regions of different permeabilities. However with the same order of magnitude μ. We denote by $\Omega_i, i = 1, \cdots, N$ the domains of constant permeability compounding Ω and by μ_i their relative permeability. We have $\mu_i = C_i\mu$ where $C_i \in [0.1, 10]$ are constant data and $\Omega = \bigcup_{i=1}^{N} \Omega_i$. For convenience we denote by Ω_{N+1} the exterior domain Ω^c and by μ_{N+1} its relative magnetic permeability ($\mu_{N+1} = 1$). We denote by Σ_{ij} $i, j \in \{1, \cdots, N+1\}, i \neq j$ the oriented boundary between Ω_i and Ω_j with the normal vector from Ω_i to Ω_j (possibly we have $\Sigma_{ij} = \emptyset$) and by Σ_i the whole boundary of Ω_i, $\Sigma_i = \bigcup_{j=1}^{N+1} \Sigma_{ij}$, $i \in \{1, \cdots, N+1\}$. The RSP satisfies the following equations that generalize (4),

$$\left\{\begin{array}{l} \Delta\phi = 0 \quad \text{in } \Omega_1, \cdots, \Omega_{N+1}, \\ \mu_j \dfrac{\partial \phi}{\partial n}\Big|_{\Omega_j} - \mu_i \dfrac{\partial \phi}{\partial n}\Big|_{\Omega_i} = (\mu_j - \mu_i)\, g \\ \qquad\qquad \text{through } \Sigma_{ij}, 1 \leq i < j \leq N+1, \end{array}\right. \tag{21}$$

where $g = \mathbf{H_s}\cdot\mathbf{n}$. Our concern is the computation of the induced magnetic field $\mathbf{H_m} = -\nabla\phi$ in the exterior domain Ω_{N+1} by the following integral formula which is the generalization of (10),

$$\begin{aligned} \mathbf{H_m}(y) &= \sum_{i=1}^{N+1} \sum_{\substack{j=1 \\ j>i}}^{N+1} \frac{\mu_j - \mu_i}{\mu_{N+1}} \int_{\Sigma_{ij}} \big(g(x)\nabla_y G(x,y) \\ &\qquad -\phi(x)\nabla_y G_n(x,y)\big)\, \mathrm{d}\sigma_x \\ &= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} \mu(C_j - C_i) \int_{\Sigma_{ij}} \big(g(x)\nabla_y G(x,y) \\ &\qquad -\phi(x)\nabla_y G_n(x,y)\big)\, \mathrm{d}\sigma_x \\ &+ \sum_{i=1}^{N} (1 - \mu C_i) \int_{\Sigma_{ij}} \big(g(x)\nabla_y G(x,y) \\ &\qquad -\phi(x)\nabla_y G_n(x,y)\big)\, \mathrm{d}\sigma_x. \end{aligned} \tag{22}$$

As in formula (10) this formula is unfit for the purpose of numerical computation due to a cancellation in the integral terms. In the following we investigate the cancellation phenomenon in (22).

For convenience we again denote by ϕ^i the restriction of the RSP to the domain Ω and by ϕ^e the restriction of the RSP to the exterior domain $\Omega^c = \Omega_{N+1}$. We denote by I the set of integers i such that $\Sigma_{iN+1} \neq \emptyset$. Problem (21) then becomes: find ϕ^i and ϕ^e such that

$$
\begin{cases}
\Delta\phi^i = 0 \quad \text{in} \quad \Omega_j, j\in\{1,\cdots,N\},\\
\Delta\phi^e = 0 \quad \text{in} \quad \Omega_{N+1},\\
\phi^e = \phi^i \quad \text{on} \ \Sigma_{N+1},\\
\dfrac{1}{\mu}\dfrac{\partial\phi^e}{\partial n} - C_i\dfrac{\partial\phi^i}{\partial n} = (\dfrac{1}{\mu} - C_i)\ g \quad \text{on} \ \ \Sigma_{iN+1}, \forall i\in I,\\
C_j\dfrac{\partial\phi^i}{\partial n}\Big|_{\Omega_j} - C_i\dfrac{\partial\phi^i}{\partial n}\Big|_{\Omega_i} = (C_j - C_i)\ g\\
\qquad\qquad \text{through} \ \ \Sigma_{ij}, \forall i,j\in\{1,\cdots,N\}, i<j.
\end{cases}
\tag{23}
$$

Again we look for ϕ^i and ϕ^e of the form

$$
\phi^i = \sum_{k=0}^{+\infty}\frac{1}{\mu^k}\phi^i_k \qquad \text{and} \qquad \phi^e = \sum_{k=0}^{+\infty}\frac{1}{\mu^k}\phi^e_k. \tag{24}
$$

One can easily check that $\phi^i_k \in \mathbb{H}^1(\Omega)$ and $\phi^e_k \in \mathbb{W}^1_0(\Omega^c)$, $k\in\{0,1\}$, should be solutions of the following interior Neumann and exterior Dirichlet coupled problems:

$$
(\mathcal{P}^i_0) \quad
\begin{cases}
\Delta\phi^i_0 = 0 \quad \text{in} \quad \Omega_j, j\in\{1,\cdots,N\},\\
C_j\dfrac{\partial\phi^i_0}{\partial n}\Big|_{\Omega_j} - C_i\dfrac{\partial\phi^i_0}{\partial n}\Big|_{\Omega_i} = (C_j - C_i)\ g\\
\qquad\qquad \text{through} \ \ \Sigma_{ij}, \forall i,j\in\{1,\cdots,N\}, i<j,\\
\dfrac{\partial\phi^i_0}{\partial n} = g \quad \text{on} \ \ \Sigma_{N+1},
\end{cases}
$$

$$
(\mathcal{P}^e_0) \quad
\begin{cases}
\Delta\phi^e_0 = 0 \quad \text{in} \quad \Omega_{N+1},\\
\phi^e_0 = \phi^i_0 \quad \text{on} \ \ \Sigma_{N+1},
\end{cases}
$$

$$
(\mathcal{P}^i_1) \quad
\begin{cases}
\Delta\phi^i_1 = 0 \quad \text{in} \quad \Omega_j, j\in\{1,\cdots,N\},\\
C_j\dfrac{\partial\phi^i_1}{\partial n}\Big|_{\Omega_j} - C_i\dfrac{\partial\phi^i_1}{\partial n}\Big|_{\Omega_i} = 0\\
\qquad \text{through} \ \ \Sigma_{ij}, i,j\in\{1,\cdots,N\}, i<j,\\
C_i\dfrac{\partial\phi^i_1}{\partial n} = \dfrac{\partial\phi^e_0}{\partial n} - g \quad \text{on} \ \ \Sigma_{iN+1}, \forall i\in I,
\end{cases}
$$

$$
(\mathcal{P}^e_1) \quad
\begin{cases}
\Delta\phi^e_1 = 0 \quad \text{in} \quad \Omega_{N+1},\\
\phi^e_1 = \phi^i_1 \quad \text{on} \ \ \Sigma_{N+1}.
\end{cases}
$$

And for $k\geq 2$,

$$
(\mathcal{P}^i_k) \quad
\begin{cases}
\Delta\phi^i_k = 0 \quad \text{in} \quad \Omega_j, j\in\{1,\cdots,N\},\\
C_j\dfrac{\partial\phi^i_k}{\partial n}\Big|_{\Omega_j} - C_i\dfrac{\partial\phi^i_k}{\partial n}\Big|_{\Omega_i} = 0\\
\qquad \text{throught} \ \ \Sigma_{ij}, i,j\in\{1,\cdots,N\}, i<j,\\
C_i\dfrac{\partial\phi^i_k}{\partial n} = \dfrac{\partial\phi^e_{k-1}}{\partial n} \quad \text{on} \ \ \Sigma_{iN+1}, \forall i\in I,
\end{cases}
$$

$$(\mathcal{P}_k^e) \quad \begin{cases} \Delta \phi_k^e = 0 & \text{in } \Omega_{N+1}, \\ \phi_k^e = \phi_k^i & \text{on } \Sigma_{N+1}. \end{cases}$$

We write (22) with the asymptotic expansion (24) and arrange the expression in ascending order in power of μ,

$$\begin{aligned}
\mathbf{H}_{\mathbf{m}}(y) = \mu \Bigg\{ & \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} (C_j - C_i) \int_{\Sigma_{ij}} \Big(g(x) \nabla_y G(x,y) \\
& - \phi_0^i(x) \nabla_y G_n(x,y) \Big) \, \mathrm{d}\sigma_x \\
+ & \sum_{i \in I} C_i \int_{\Sigma_{iN+1}} \Big(g(x) \nabla_y G(x,y) - \phi_0^i(x) \nabla_y G_n(x,y) \Big) \, \mathrm{d}\sigma_x \Bigg\} \\
- & \Bigg\{ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} (C_j - C_i) \int_{\Sigma_{ij}} \phi_1^i(x) \nabla_y G_n(x,y) \, \mathrm{d}\sigma_x \\
& + \sum_{i \in I} C_i \int_{\Sigma_{iN+1}} \phi_1^i(x) \nabla_y G_n(x,y) \, \mathrm{d}\sigma_x \\
& + \sum_{i \in I} \int_{\Sigma_{iN+1}} \Big(g(x) \nabla_y G(x,y) - \phi_0^i(x) \nabla_y G_n(x,y) \Big) \, \mathrm{d}\sigma_x \Bigg\} \\
- & \sum_{k=1}^{+\infty} \frac{1}{\mu^k} \Bigg\{ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} (C_j - C_i) \int_{\Sigma_{ij}} \phi_{k+1}^i(x) \nabla_y G_n(x,y) \, \mathrm{d}\sigma_x \\
& + \sum_{i \in I} C_i \int_{\Sigma_{iN+1}} \phi_{k+1}^i(x) \nabla_y G_n(x,y) \, \mathrm{d}\sigma_x \\
& - \sum_{i \in I} \int_{\Sigma_{iN+1}} \phi_k^i(x) \nabla_y G_n(x,y) \, \mathrm{d}\sigma_x \Bigg\}.
\end{aligned} \tag{25}$$

By using Green second identity for ϕ_0^i solution of $(\mathcal{P}_0^i)$ we get

$$\begin{aligned}
0 = & \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} (C_j - C_i) \int_{\Sigma_{ij}} \Big(g(x) \nabla_y G(x,y) \\
& \qquad - \phi_0^i(x) \nabla_y G_n(x,y) \Big) \, \mathrm{d}\sigma_x \\
+ & \sum_{i \in I} C_i \int_{\Sigma_{iN+1}} \Big(g(x) \nabla_y G(x,y) - \phi_0^i(x) \nabla_y G_n(x,y) \Big) \, \mathrm{d}\sigma_x.
\end{aligned} \tag{26}$$

This means that the leading term (the power one of μ) in the expression of $\mathbf{H}_{\mathbf{m}}$ given in (25) cancels. When we compute the function ϕ for large values of μ, we have essentially the first term ϕ_0 which gives a cancellation in (25). From (25) we deduce that the values of $\mathbf{H}_{\mathbf{m}}$ directly depend on ϕ_1^i which is not given by a straightforward numerical resolution of (21). To circumvent the cancellation, one should use the following formula deduced from (25) and (26) that generalize (19),

$$\begin{aligned}\mathbf{H_m}(y) = -\Bigg\{ &\sum_{i=1}^{N}\sum_{\substack{j=1\\ j>i}}^{N}(C_j - C_i)\int_{\Sigma_{ij}} \phi_1^i(x)\nabla_y G_n(x,y)\ \mathrm{d}\sigma_x \\ &+\sum_{i=1}^{N} C_i \int_{\Sigma_{iN+1}} \phi_1^i(x)\nabla_y G_n(x,y)\ \mathrm{d}\sigma_x \\ &+\sum_{i=1}^{N}\int_{\Sigma_{iN+1}} \Big(g(x)\nabla_y G(x,y) - \phi_0^i(x)\nabla_y G_n(x,y)\Big)\ \mathrm{d}\sigma_x\Bigg\} \\ -\sum_{k=1}^{+\infty}\frac{1}{\mu^k}\Bigg\{ &\sum_{i=1}^{N}\sum_{\substack{j=1\\ j>i}}^{N}(C_j - C_i)\delta_{ij}\int_{\Sigma_{ij}} \phi_{k+1}^i(x)\nabla_y G_n(x,y)\ \mathrm{d}\sigma_x \\ &+\sum_{i=1}^{N} C_i \int_{\Sigma_{iN+1}} \phi_{k+1}^i(x)\nabla_y G_n(x,y)\ \mathrm{d}\sigma_x \\ &-\sum_{i=1}^{N}\int_{\Sigma_{iN+1}} \phi_k^i(x)\nabla_y G_n(x,y)\ \mathrm{d}\sigma_x\Bigg\}.\end{aligned} \tag{27}$$

Here again to use (27) we have to solve 2 Laplace equations, one in Ω and one in Ω^c with several right hand sides.

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