

Foliations with vanishing Chern classes

Jorge Vitório Pereira* and Frédéric Touzet

— *To the memory of Marco Brunella.*

Abstract. In this paper we aim at the description of foliations having tangent sheaf $T\mathcal{F}$ with $c_1(T\mathcal{F}) = c_2(T\mathcal{F}) = 0$ on non-uniruled projective manifolds. We prove that the universal covering of the ambient manifold splits as a product, and that the Zariski closure of a general leaf of \mathcal{F} is an Abelian variety. It turns out that the analytic type of the Zariski closures of leaves may vary from leaf to leaf. We discuss how this variation is related to arithmetic properties of the tangent sheaf of the foliation.

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1 Introduction and statement of results

If X is a compact Kähler manifold (e.g. X is a projective manifold) with real Chern classes satisfying $c_1(TX) = c_2(TX) = 0$ then Yau's solution to Calabi conjecture combined with a result by Apte [2] implies that X admits a flat Hermitian metric. One can thus evoke a classical result by Bierberbach to conclude that there exists a finite étale morphism from a complex torus to X , see [21, Corollary 4.15] and references therein.

In this paper we aim at a generalization of this result where we replace the hypothesis on the tangent bundle of X by the same hypothesis on saturated coherent subsheaves of the tangent bundle of X . We will also assume that X is projective and non-uniruled, i.e. there is no rational curve passing through a general point of X . Even if many of our arguments do work in the more general context of compact Kähler manifolds, at multiple places we will have to restrict to projective manifolds.

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*Corresponding author.

Let X be a complex manifold. A distribution \mathcal{D} on X is determined by a coherent subsheaf $T\mathcal{D}$ of TX (the tangent sheaf of \mathcal{D}) which has torsion free cokernel $TX/T\mathcal{D}$, in other words $T\mathcal{D}$ is saturated in TX . The generic rank of $T\mathcal{D}$ is the dimension of the distribution. A foliation \mathcal{F} on X is a distribution with involutive (i.e. local sections are closed under Lie bracket) tangent sheaf $T\mathcal{F}$.

1.1 Previous results

Before stating our results we recall some partial answers to the problem of classifying distributions with vanishing Chern classes which will be useful in and/or motivate what follows.

1.1.1 Foliation with trivial tangent sheaf

If \mathcal{D} is a distribution of positive dimension with trivial tangent bundle on a compact complex manifold X then $h^0(X, TX) \geq h^0(X, T\mathcal{D}) = \dim \mathcal{D} > 0$ and it follows that X admits holomorphic vector fields. Below we state a theorem of Lieberman [26], see also [1, Theorem 3.2], which completely describes the situation when the ambient manifold is projective and non-uniruled.

Theorem 1.1. *If X is a non-uniruled projective manifold with $h^0(X, TX) > 0$ then up to a finite étale covering, $X = A \times Y$ where A is an Abelian variety and Y satisfies $h^0(Y, TY) = 0$.*

It follows that the distribution \mathcal{D} is smooth, involutive and the underlying foliation is the pull-back under the natural projection to T of a linear foliation.

1.1.2 Foliations with trivial canonical class

In [28], a joint work with F. Loray, we have shown that distributions on non-uniruled projective manifolds satisfying $c_1(T\mathcal{D}) = 0$ are smooth and involutive. Therefore, if we restrict ourselves to the category of non-uniruled projective manifolds then our problem about distributions reduces to a problem about smooth foliations. Under the same hypothesis, we have also proved in [28] the existence of a transverse smooth foliation which together with \mathcal{D} provides a splitting of the tangent bundle of X ; and that the determinant of $T\mathcal{D}$ is a torsion line-bundle. These results, with precise statements, are recalled in §2.1.

As corollary of the statement concerning $\det T\mathcal{D}$ one obtains that a foliation \mathcal{F} of dimension one on a non-uniruled manifolds with $c_1(T\mathcal{F}) = 0$ is defined

by a global holomorphic vector field on a suitable finite étale covering. Hence, we can apply Lieberman's result quoted above to conclude that \mathcal{F} is tangent to an isotrivial fibration by abelian varieties.

1.1.3 Codimension one foliations

Smooth foliations of codimension one with $c_1(T\mathcal{F}) = 0$ on compact Kähler manifolds have been classified in [30]. In particular, when $c_2(T\mathcal{F}) = 0$ we have the following possibilities:

- (1) up to a finite étale covering X is a complex torus and \mathcal{F} is a linear foliation on it; or
- (2) up to a finite étale covering X is the product of a complex torus and a curve and \mathcal{F} is the pull-back of a linear foliation on the torus by the natural projection; or
- (3) X is a \mathbb{P}^1 -bundle over a complex torus, and \mathcal{F} is everywhere transverse to the fibers of this \mathbb{P}^1 -bundle.

In all cases $\det(T\mathcal{F})$ is a torsion line-bundle, and when X is not uniruled then after a finite étale covering X splits as the product of a complex torus and a smooth manifold of dimension 0 or 1.

We will proceed to describe the new results proved in this paper. The remaining of this introduction reflects the structure of the paper with each subsection describing the content of the corresponding section of the paper.

1.2 Splitting of the universal covering

Let X be a projective non-uniruled projective manifold and \mathcal{F} be a foliation on X with $c_1(T\mathcal{F}) = 0$. Using results from ([28]) we prove that $T\mathcal{F}$ is polystable whenever $c_1(T\mathcal{F}) = 0$. Therefore $T\mathcal{F}$ is indeed an Hermite-Einstein bundle by a theorem of Donaldson. Specializing to foliations which satisfy the additional assumption $c_2(T\mathcal{F}) = 0$, we obtain that $T\mathcal{F}$ is a flat hermitian bundle and as such carries a flat connection with unitary monodromy

$$\rho : \pi_1(X) \rightarrow U(r, \mathbb{C}) \subset GL(r, \mathbb{C}).$$

It turns out that this representation is also the monodromy representation of a transversely Euclidean foliation everywhere transverse to \mathcal{F} . Exploiting the transverse geometry of this foliation, similarly to what we have done in a previous joint work with M. Brunella [7], we prove the following result.

Theorem A. *Let \mathcal{F} be a foliation with $c_1(T\mathcal{F}) = c_2(T\mathcal{F}) = 0$ in $H^*(X, \mathbb{R})$ on a projective manifold X . If X is not uniruled then \mathcal{F} is a smooth foliation, there exists a smooth foliation \mathcal{F}^\perp which together with \mathcal{F} induces a splitting $TX = T\mathcal{F} \oplus T\mathcal{F}^\perp$ of the tangent bundle of X , and the universal covering of X splits as a product $\mathbb{C}^{\dim(\mathcal{F})} \times Y$.*

The projectivity of X is used only to prove the polystability of $T\mathcal{F}$, and the proof is based on the pseudo-effectiveness of KX . If we assume that X is a compact Kähler manifold with pseudo-effective KX then the same conclusion probably holds true. Also, if we drop the non-uniruledness assumption but replace it with the existence of decomposition of the tangent bundle then the proof of Theorem A can be adapted to prove the following result.

Theorem B. *Let X be a compact Kähler manifold such that the tangent bundle splits as a direct sum of two subbundles $A \oplus B$. If A is an involutive subbundle of TX and admits a flat hermitian metric then the universal covering of X splits as a product $\mathbb{C}^{\text{rank}(A)} \times Y$ compatible with the splitting of TX .*

This provides further evidence to Beauville's conjecture concerning the universal covering of compact Kähler manifolds with split tangent bundle, see [3, 7, 19] and references therein.

1.3 Shafarevich map and structure theorem

In view of Lieberman's result, one might expect that in the presence of a foliation with vanishing Chern classes there is no need to pass to the universal covering in order to obtain a splitting of the ambient manifold: a finite covering would suffice. It turns out that the situation is more delicate and the existence of such splitting is determined by the holonomy representation $\rho : \pi_1(X) \rightarrow U(r, \mathbb{C})$ of the hermitian flat bundle $T\mathcal{F}$.

If the representation ρ has finite image then, after a finite étale covering, we obtain a foliation with trivial tangent sheaf and we are reduced to Lieberman's Theorem. Otherwise, if the image of ρ is infinite then a result of Zuo on the Shafarevich map of representations [32] allows us to prove the following structure theorem.

Theorem C. *Let \mathcal{F} be a foliation on a non-uniruled projective manifold X . If $c_1(T\mathcal{F}) = c_2(T\mathcal{F}) = 0$ in $H^*(X, \mathbb{R})$ then, after passing to a finite étale covering, there exists an almost holomorphic map on X whose general fiber is an abelian variety, the foliation \mathcal{F} is tangent to the fibers of this map, and the restriction of \mathcal{F} to a general fiber is a linear foliation.*

By an almost holomorphic map we mean a rational map $f : X \dashrightarrow B$ whose restriction to an open subset $X^0 \subset X$ is a proper morphism over an open subset B^0 of the base.

1.4 Infinite monodromy

Perhaps it is worth noticing at this point that the image of ρ can be indeed infinite. This latter property implies the non isotriviality of the abelian fibration and a construction carried out by Faltings in [15] provides examples. In §4 we exhibit foliations of dimension two and codimension three with flat tangent bundle with infinite monodromy. The Abelian fibration given by Theorem C is smooth with fibers of dimension four, and therefore the linear foliations have codimension two in the fibers.

1.5 Codimension two

Studying the variation of Hodge structures determined by the Abelian fibration, we are able to prove that the above mentioned examples are optimal: they have minimal dimension and minimal codimension among the examples with infinite representation ρ .

Theorem D. *Let \mathcal{F} be a foliation of codimension two on a non-uniruled projective manifold X . If $c_1(T\mathcal{F}) = c_2(T\mathcal{F}) = 0$ in $H^*(X, \mathbb{R})$ then, after passing to a finite étale covering, $X = A \times Y$ where A is an Abelian variety and Y is a point, a smooth curve, or a surface. The foliation \mathcal{F} is the pull-back of a linear foliation on A under the projection to the first factor.*

1.6 Arithmetic

Another possible approach to prove our structure Theorem is using reduction to positive characteristic. As the leaves of our foliation are uniformized by Euclidean spaces they are Liouvillian in the sense of pluripotential theory and one might hope to be able to use Bost's Theorem [4] to prove the algebraicity of leaves of the closure of $T\mathcal{F}$ under p -th powers. Although we are able to prove the existence of a non-trivial foliation \mathcal{G} , p -closed for almost every prime p , containing our foliation with vanishing Chern classes, Faltings example shows that \mathcal{G} does not have necessarily vanishing Chern classes and we are unable to control the universal coverings of its leaves. Nevertheless, we can prove the following statement.

Theorem E. *Let \mathcal{F} be a foliation on a non-uniruled complex projective manifold X both defined over a finitely generated \mathbb{Z} -algebra $R \subset \mathbb{C}$. Suppose that \mathcal{F} is maximal, with respect to inclusion, among the foliations with $c_1(T\mathcal{F}) = 0$ and $c_2(T\mathcal{F}) = 0$. Then at least one of the following assertions holds true.*

- (1) *Up to a finite étale covering, X is isomorphic to a product of an Abelian variety A with another projective manifold Y , and \mathcal{F} is the pull-back of a linear foliation on A under the natural projection $A \times Y \rightarrow A$.*
- (2) *For a dense set of maximal primes \mathfrak{p} in $\text{Spec}(R)$ the reduction modulo \mathfrak{p} of $T\mathcal{F}$ is not Frobenius semi-stable. Moreover, there exists a non-empty open subset $U \subset \text{Spec}(R)$ such that for every maximal prime $\mathfrak{p} \in U$ the reduction modulo \mathfrak{p} of \mathcal{F} is either p -closed or the reduction modulo \mathfrak{p} of $T\mathcal{F}$ is not Frobenius semi-stable.*

In particular, the foliations presented in Section 4 have tangent sheaf which are stable but not strongly semi-stable for infinitely many primes. To the best of our knowledge the only previously known examples of this phenomena appeared in [6].

2 Polystability and splitting

2.1 Polystability of the tangent sheaf

Let \mathcal{E} be a coherent sheaf on a n -dimensional smooth projective variety X polarized by an ample line bundle H . The slope of \mathcal{E} (more precisely the H -slope of \mathcal{E}) is defined as the quotient

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\text{rank}(\mathcal{E})}.$$

If the slope of every coherent proper subsheaf \mathcal{E}' of \mathcal{E} satisfies $\mu(\mathcal{E}') < \mu(\mathcal{E})$ (respectively $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$) then \mathcal{E} is called stable (respectively semi-stable).

A vector bundle \mathcal{E} is said to be polystable if it can be expressed as a finite sum

$$\mathcal{E} = \bigoplus_i \mathcal{E}_i$$

where each summand \mathcal{E}_i is a stable subbundle.

The goal of this paragraph is to prove the following lemma.

Lemma 2.1. *If \mathcal{D} is a distribution with $c_1(T\mathcal{D}) = 0$ on a non-uniruled projective manifold X then \mathcal{D} is smooth, integrable, and has polystable tangent sheaf.*

To wit, there exists a finite family $\mathcal{F}_1, \dots, \mathcal{F}_p$ of smooth holomorphic subfoliations of \mathcal{D} whose tangent sheafs are stable with respect to any given polarisation, have zero first Chern class, and satisfy

$$T\mathcal{D} = \bigoplus_{i=1}^p T\mathcal{F}_i.$$

Most of the arguments that will be used in the proof of this lemma already appeared in [28]. We now proceed to briefly recall them.

On the one hand, if X is not uniruled then Boucksom-Demailly-Paun-Peternell characterization of uniruledness [5], which crucially relies on [27], implies that the canonical bundle of X is pseudo-effective. On the other hand, if $c_1(T\mathcal{D}) = 0$ then $T\mathcal{D}$ can be defined as the kernel of a holomorphic q -form with coefficients in the bundle $KX \otimes \det(T\mathcal{D})$. Demailly’s Theorem implies that \mathcal{D} is integrable, i.e. \mathcal{D} is not only a distribution but is also a foliation.

The smoothness of \mathcal{D} is proved in [28]. As the relevant result will be essential in the proof of Lemma 2.1 we reproduce here its statement.

Theorem 2.2. *Let X be a projective manifold with KX pseudo-effective and L be a pseudo-effective line bundle on X . If $v \in H^0(X, \bigwedge^p TX \otimes L^*)$ is a non-zero section then the zero set of v is empty. Moreover, if \mathcal{D} is a codimension q distribution on X with $c_1(T\mathcal{D}) = 0$ then \mathcal{D} is a smooth foliation (i.e. $T\mathcal{D}$ is involutive) and there exists another smooth holomorphic foliation \mathcal{G} of dimension q on X such that $TX = T\mathcal{D} \oplus T\mathcal{G}$.*

Conclusion of the proof of Lemma 2.1. Campana-Peternell in [10] proved that the canonical sheaf of every saturated coherent subsheaf of TX is pseudo-effective. Since $c_1(T\mathcal{D}) = 0$, it follows that $T\mathcal{D}$ is a semi-stable vector bundle with respect to any polarization of X .

Assume now that $T\mathcal{D}$ is not stable. Then there exists a distribution \mathcal{D}_0 tangent to \mathcal{D} , in other words $T\mathcal{D}_0$ is a subbundle of $T\mathcal{D}$, such that $c_1(T\mathcal{D}_0) = 0$. Since \mathcal{D}_0 satisfies the same hypothesis that \mathcal{D} satisfies, we get that it is integrable and we can apply Theorem 2.2 to deduce that \mathcal{D}_0 is a smooth foliation, and to exhibit another smooth foliation \mathcal{D}_0^\perp such that $TX = T\mathcal{D}_0 \oplus T\mathcal{D}_0^\perp$. If we set \mathcal{G} as the foliation obtained as the intersection of \mathcal{D} and \mathcal{D}_0^\perp , i.e.

$$T\mathcal{G} := T\mathcal{D}_0^\perp \cap T\mathcal{D},$$

then

$$T\mathcal{D} = T\mathcal{D}_0 \oplus T\mathcal{G}$$

and, consequently, $c_1(T\mathcal{G}) = 0$. The Lemma follows by induction. □

2.2 Maximal foliations with vanishing Chern classes

Lemma 2.1 ensures the existence of maximal foliation with vanishing Chern classes.

Corollary 2.3. *If \mathcal{F} and \mathcal{G} are foliations on a non-uniruled projective manifold which verify $c_1(T\mathcal{F}) = c_1(T\mathcal{G}) = 0$ then there exists a foliation \mathcal{H} containing \mathcal{F} and \mathcal{G} and with $c_1(T\mathcal{H}) = 0$. Moreover, if $c_2(T\mathcal{F}) = c_2(T\mathcal{G}) = 0$ then we can choose \mathcal{H} with $c_1(T\mathcal{H}) = c_2(T\mathcal{H}) = 0$.*

Proof. Consider the morphism $\varphi : T\mathcal{F} \oplus T\mathcal{G} \rightarrow TX$ sending (v, w) to $v + w$. Since X is not uniruled then the image of φ has non-positive degree with respect to any polarization. The polystability of $T\mathcal{F}$ and $T\mathcal{G}$ implies that the kernel of φ has degree zero and is a polystable summand of both sheaves. On the one hand we have that the image of φ is locally free; and on the other hand both the kernel and the image of φ have vanishing Chern classes. Theorem 2.2 implies the image of φ is an involutive subbundle of TX with Chern polynomial $c(T\mathcal{F}) \cdot c(T\mathcal{G}) \cdot (c(\ker \varphi)^{-1})$, and is the tangent sheaf of the sought foliation. \square

2.3 Hermite-Einstein structure on the tangent sheaf

Let us recall a result by Donaldson [13].

Theorem 2.4. *Let E a polystable holomorphic vector bundle over a projective manifold. Then E carries an Hermite-Einstein metric. In particular, if E satisfies $c_1(E) = c_2(E) = 0$ then this metric is flat.*

We can apply Donaldson's Theorem combined with Lemma 2.1 to deduce the following corollary.

Corollary 2.5. *If \mathcal{F} is a foliation on projective non uniruled manifold X such that $c_1(T\mathcal{F}) = 0$, then there exists an Hermite-Einstein metric on $T\mathcal{F}$. In particular, $T\mathcal{F}$ is flat hermitian whenever $c_2(T\mathcal{F}) = 0$.*

If \mathcal{F} is a foliation satisfying the hypothesis of Theorem A then $T\mathcal{F}$ is a flat unitary vector bundle and as such is defined by an unitary representation

$$\rho : \pi_1(X) \rightarrow U(r, \mathbb{C}).$$

The next result, also from [28], implies that the induced representation $\det(\rho) : \pi_1(X) \rightarrow U(1, \mathbb{C})$ has finite image.

Proposition 2.6. *If \mathcal{D} be a distribution with $c_1(T\mathcal{D}) = 0$ on a non-uniruled projective manifold then its canonical bundle $K\mathcal{D} = \det T\mathcal{D}^*$ is a torsion line bundle.*

As a corollary we obtain the following result.

Corollary 2.7. *If the image $Im \rho$ of ρ is virtually solvable (i.e. contains a solvable group of finite index), then $Im \rho$ is finite.*

Proof. One can find a finite etale covering $e: X^e \rightarrow X$ such that $Im e^*\rho$ is solvable and contained in a connected solvable linear algebraic group. This representation is semi-simple (being unitary), hence splits as a sum of one dimensional representations. The finiteness of $Im \rho$ follows from Proposition 2.6. \square

2.4 Proof of Theorem A

Let \mathcal{F} be a holomorphic foliation on a non uniruled projective manifold X with $c_1(T\mathcal{F}) = c_2(T\mathcal{F}) = 0$. Let us call \mathcal{F}^\perp the (non necessarily unique) complementary foliation of \mathcal{F} whose existence is ensured by Theorem 2.2, and consider the universal covering projection $\pi: \tilde{X} \rightarrow X$.

The foliation \mathcal{F}^\perp is defined by a holomorphic 1-form ω with values in $T\mathcal{F}$. This latter being flat hermitian (Corollary 2.5), we get $\nabla\omega = 0$ using the Kähler identities (here, ∇ denotes the unitary flat connection attached to $E = T\mathcal{F}$). Therefore \mathcal{F}^\perp is transversely Euclidean; moreover, this transversal hermitian structure is complete by compactness of X . In particular, if we take a primitive $F = (F_1, \dots, F_r)$ of $\pi^*\omega$ on \tilde{X} then the restriction of F to any leaf \mathcal{L} of the foliation $\pi^*\mathcal{F}$ must be a surjective covering map onto \mathbb{C}^r . Since \mathbb{C}^r is simply connected this covering map is an isomorphism between \mathcal{L} and \mathbb{C}^r . Moreover, \mathcal{L} is equipped with the Euclidean metric induced by $\sum |dF_i|^2$.

On the other hand, the leaves of $\pi^*\mathcal{F}^\perp$ must coincide with connected components of the levels of F . We claim that for every $x \in \mathbb{C}^r$, $F^{-1}(x)$ is connected. Indeed, assume that $F^{-1}(x)$ is a union of distinct leaves of $\pi^*\mathcal{F}^\perp$, say \mathcal{L}_j for j belonging to some set J . For each $j \in J$ denote by V_j the saturation of \mathcal{L}_j by $\pi^*\mathcal{F}$. As the intersection of any leaf of $\pi^*\mathcal{F}$ with $F^{-1}(x)$ consists in exactly one point (F restricted to the leaves of $\pi^*\mathcal{F}$ is a covering map) it follows that the sets V_j , $j \in J$, are pairwise disjoint open subsets of \tilde{X} such that $\tilde{X} = \cup_{j \in J} V_j$. Connectedness of \tilde{X} implies that J has cardinality one.

Now, fix a leaf $\mathcal{L} \simeq \mathbb{C}^r$ of $\pi^*\mathcal{F}$ and a leaf \mathcal{L}^\perp of $\pi^*\mathcal{F}^\perp$. For every $a \in \tilde{X}$, consider the leaf \mathcal{L}_a of $\pi^*\mathcal{F}$ and the leaf \mathcal{L}_a^\perp such that $\{a\} = \mathcal{L}_a \cap \mathcal{L}_a^\perp$. We get a well defined bijection

$$\tilde{X} \rightarrow \mathcal{L} \times \mathcal{L}^\perp$$

sending a to $(a_{\mathcal{L}}, a_{\mathcal{L}^\perp})$ such that $\{a_{\mathcal{L}}\} = \mathcal{L}_a^\perp \cap \mathcal{L}$ and $\{a_{\mathcal{L}^\perp}\} = \mathcal{L}_a \cap \mathcal{L}^\perp$. This bijection is a biholomorphism by a standard flow-box argument, and we get the expected trivialization $\tilde{X} \simeq \mathbb{C}^r \times Y$. \square

As the reader can easily verify the above argument also proves Theorem B.

3 Structure Theorem

3.1 Shafarevich map

Let X be a smooth projective algebraic variety and $\rho : \pi_1(X) \rightarrow G$ a representation of the fundamental group. One can define the Shafarevich map for ρ as follows: it is a dominant rational morphism with connected fibers

$$\text{sh}_\rho : X \dashrightarrow \text{Sh}_\rho(X)$$

where $\text{Sh}_\rho(X)$ is a normal algebraic variety such that for any irreducible subvariety $V \subset X$ not contained in a union of countably many proper algebraic subvarieties, $\text{sh}_\rho(V) = \text{point}$ iff $\rho(\pi_1(V))$ is finite. It is easy to see that the existence of Sh_ρ is unique up to birational equivalence.

Campana ([9]) and Kollár ([23]) have independently proved that Sh_ρ always exists with the additional property that it is an almost holomorphic map, i.e. it becomes a proper morphism when restricted to a suitable Zariski open set of X .

3.2 Zuo's theorem

Before stating Zuo's Theorem [32], see also [11], let us recall some definitions from the theory of algebraic groups. A connected algebraic group G is called *almost simple* if it is non commutative and all its proper algebraic normal subgroups are finite. If S is a semi-simple connected algebraic group then a classical result asserts that S is isogenous to a product of almost simple algebraic groups.

Theorem 3.1. *Let $\rho : \pi_1(X) \rightarrow G$ be a Zariski dense representation into an almost simple algebraic group. Then there exists a finite étale covering $e : X^e \rightarrow M$ such that the pull-back representation $e^*\rho$ factors through the Shafarevich map $\text{sh}_{e^*\rho} : X^e \rightarrow \text{Sh}_{e^*\rho}(X^e)$ and the Shafarevich variety $\text{Sh}_{e^*\rho}(X^e)$ is a projective algebraic variety of general type.*

3.3 Tangency to the fibers of the Shafarevich map

Recall that $T\mathcal{F}$ is a flat hermitian vector bundle, hence comes from a representation $\rho : \pi_1(X) \rightarrow U(r)$. From now on, let us deal with the Zariski closure G

of $\text{Im } \rho \subset GL(r, \mathbb{C})$. After taking a finite covering of X , we can assume that $\text{Im } \rho$ is torsion free and G is connected.

Write G as $R \times S$ where R is the solvable radical of G and S is a semi-simple group. The latter group decomposes as

$$S = (S_1 \times S_2 \times \cdots \times S_p)/H$$

where the S_i are almost simple and H is a finite subgroup of $S_1 \times S_2 \times \cdots \times S_p$.

Let H_i be the image of H under the projection of $S_1 \times \cdots \times S_p$ onto the i^{th} factor.

Now, projecting ρ to the almost simple group $G_i := S_i/(H_i)$, we obtain a Zariski dense representation

$$\rho_i : \pi_1(X) \rightarrow G_i.$$

Consider the Shafarevich map $\text{sh}_{e^*\rho_i} : X^e \rightarrow \text{Sh}_{e^*\rho_i}(X^e)$ as in Theorem 3.1 (note that the finite etale covering X^e may depend on i).

Proposition 3.2. *The pull-back foliation $e^*\mathcal{F}$ is tangent to the fibers of $\text{sh}_{e^*\rho_i}$. Consequently, the foliation \mathcal{F} is tangent to the fibers of sh_{ρ_i} .*

Proof. Assume that the statement of this proposition is false. We use here the splitting $\tilde{X} = \mathbb{C}^r \times Y$ given by theorem A. There exists an euclidean subspace \mathbb{C}^p , $1 \leq p \leq r$ of the factor \mathbb{C}^r and a local analytic connected subspace Z_Y of Y such that the natural meromorphic map $\mathbb{C}^p \times Z_Y$ induced by the covering projection $\tilde{X} \rightarrow X^e$ and $\text{sh}_{e^*\rho_i}$ is a local biholomorphism near some point; this is absurd because $\text{Sh}_{e^*\rho_i}(X^e)$ is measure hyperbolic (being of general type) whereas $\mathbb{C}^p \times Z_Y$ is not. See [22, Chapter 7] for the related properties of hyperbolicity. □

Proposition 3.3. *The foliation \mathcal{F} is tangent to the fibers of sh_ρ .*

Proof. Denote by ρ_S the representation

$$\pi_1(M) \rightarrow S \simeq G/R$$

induced by ρ . Using Proposition 3.2, one can see that \mathcal{F} is tangent to the fibers of sh_{ρ_S} . Let U be a Zariski open subset of X such that sh_ρ is a smooth proper fibration on U and pick a fiber F_ρ of $\text{sh}_{\rho|U}$. The restriction of ρ to F_ρ takes values into a virtually solvable group. The image of $\rho|_{F_\rho}$ is then finite by Corollary 2.7. Therefore sh_ρ and sh_{ρ_S} coincide. □

3.4 Proof of Theorem C

After replacing X by X^e we can choose an open Zariski subset U of X such that $\Phi_U := \text{sh}_{\rho|_U}$ is a smooth proper fibration onto its image V with the additional property that $h^0(TF)$ is the same for every fiber F (semi-continuity). In particular, one has $h^0(TF) \geq h^0(T\mathcal{F}|_F)$. Since X is non-uniruled the same holds true for every fiber F of Φ_U by a result of Fujiki [16] (stability of uniruledness), and we can apply Theorem 1.1 to deduce that the fibers are foliated by Abelian varieties.

If $i : U \rightarrow X$ denotes the inclusion and TU/V the relative tangent sheaf of the fibration then $i_* \text{sh}_{\rho}^* \text{sh}_{\rho}^* TU/V$ maps to a subsheaf of TX which after saturation becomes the tangent sheaf of a foliation \mathcal{A} . The general leaves of \mathcal{A} are Abelian varieties contained in the fibers of sh_{ρ} , and containing \mathcal{F} as a sublinear foliation. We consider the map to the Hilbert scheme which associates to a point $x \in X$ the point in Hilb corresponding to the leaf of \mathcal{A} through x , for details see [17]. As \mathcal{A} is smooth on U this morphism will give rise to the sought almost holomorphic map. \square

Remark 3.4. *This description shows that the representation ρ arises from a variation of polarized Hodge structures, hence takes values in a number field.*

4 Infinite monodromy

This Section is devoted to presenting examples of foliations with vanishing Chern classes on non-uniruled projective manifolds with infinite monodromy representation. In most of it we follow very closely the presentation of [15, Section 5].

4.1 Quaternion algebras

Let K be a field of characteristic zero. Let A and B be two elements of K and denote by $D = D(A, B)$ the associated quaternion algebra. Concretely, D is the non-commutative associative K -algebra with underlying K -vector space generated by $1, i, j, k$ and subject to relations:

$$i^2 = A, j^2 = B, ij = k, \text{ and } ji = -k.$$

It follows in particular that $jk = i, ki = j$, and $k^2 = -AB$.

The algebra D carries a canonical involution which takes $\alpha = a + bi + cj + dk$ to $\alpha^* = a - bi - cj - dk$, and consequently both $N(\alpha) = \alpha \cdot \alpha^*$ (the norm of α) and $T(\alpha) = \alpha + \alpha^*$ (the trace of α) belong to K .

Notice that $N(a + bi + cj + dk) = a^2 - Ab^2 - Bc^2 + ABd^2$ and therefore N is a quadratic form on D . If we extend the scalars to \overline{K} , the algebraic closure of K , then $D \otimes_K \overline{K}$ is isomorphic to the algebra of 2×2 matrices with coefficients in \overline{K} , i.e. $D \otimes_K \overline{K} \simeq M_2(\overline{K})$. Moreover, such isomorphism can be chosen in such a way that the norm and the trace are respectively identified with the determinant and the trace of matrices.

If there exists a non-zero element in D with zero norm then D is isomorphic to $M_2(K)$, otherwise D is a division algebra with the left inverse of an element α given by $-\alpha^*/N(\alpha)$.

If K is a real number field and we choose an embedding $\sigma : K \rightarrow \mathbb{R}$ then two things can happen: the real quadratic form on $D \otimes_{\sigma(K)} \mathbb{R}$ induced by N is positive definite and then $D \otimes_{\sigma(K)} \mathbb{R}$ is isomorphic to the algebra of quaternions \mathbb{H} ; or the real quadratic form is indefinite and $D \otimes_{\sigma(K)} \mathbb{R}$ is isomorphic to $M_2(\mathbb{R})$.

4.2 Example

From now on K will be a real quadratic number field and we will choose $A, B \in K$ such that for one of the embeddings we get $D \otimes_{\sigma_1(K)} \mathbb{R} \simeq \mathbb{H}$ while for the other we get $D \otimes_{\sigma_2(K)} \mathbb{R} \simeq M_2(\mathbb{R})$. In other words $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{H} \times M_2(\mathbb{R})$. Notice that in this case D is a division K -algebra, the non-invertible elements appear only after extension of scalars.

Let $V_{\mathbb{Q}}$ be the 8-dimensional \mathbb{Q} -vector space underlying D , i.e. $V_{\mathbb{Q}} = D$. Of course, D acts on V by left and right multiplication. If τ is an element of D such that $\tau^* = -\tau$ then it defines a new involution on D : $\alpha \mapsto a^\tau = \tau\alpha^*\tau^{-1}$. If we consider the skew-symmetric on V defined by

$$\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle_\tau = \text{tr}_{K/\mathbb{Q}}(T(\alpha\tau\beta^*))$$

then $\langle \alpha\gamma, \beta \rangle = \langle \alpha, \beta\gamma^\tau \rangle$. Moreover the action by left multiplication of the algebraic group G with rational points

$$G(\mathbb{Q}) = \{\alpha \in D \mid N(\alpha) = 1\},$$

preserves \langle, \rangle .

The group $G(\mathbb{R})$ of real points of G is isomorphic to $SU(2, \mathbb{C}) \times SL(2, \mathbb{R})$ and it acts on $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^2 \oplus (\mathbb{R}^2 \otimes \mathbb{R}^2)$ in such a way that $SU(2, \mathbb{C})$ acts on the first summand by its natural representation and $SL(2, \mathbb{R})$ acts on the the first factor of the tensor product also by its natural representation. The bilinear form \langle, \rangle decomposes as

$$\langle, \rangle = \langle, \rangle_1 \oplus \langle, \rangle_2 \otimes \langle, \rangle_3,$$

where \langle, \rangle_1 in suitable coordinates is a real multiple of the skew-form $(z, w) \mapsto \text{Im}(\bar{z}_1 w_1 + \bar{z}_2 w_2)$ on \mathbb{C}^2 , and \langle, \rangle_2 is a real multiple of the skew-form $(x, y) \mapsto x_1 y_2 - x_2 y_1$ on \mathbb{R}^2 . We will choose σ in such a way that \langle, \rangle_1 is a positive multiple of the skew-form above and that \langle, \rangle_3 is positive definite.

If W_0 is the kernel of the natural multiplication morphism $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathbb{C}^2$ ($W_0 = \{z \otimes 1 + iz \otimes i | z \in \mathbb{C}^2\}$) and $W_+ = \{z \otimes 1 - iz \otimes i | z \in \mathbb{C}^2\}$ then

$$V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = (W_+ \oplus W_0) \oplus (\mathbb{C}^2 \otimes \mathbb{R}^2).$$

Notice that W_0 and W_+ are orthogonal with respect to the complexification of \langle, \rangle_1 and that they are interchanged by complex conjugation. If $z \in W_+$ then $-i^{-1} \langle z, \bar{z} \rangle_1 > 0$. Let now $z = (z_1, z_2) \in \mathbb{C}^2 = \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}$ (the first factor of the tensor product in the decomposition above) be such that

$$-(i)^{-1} \langle z, \bar{z} \rangle_2 > 0$$

and consider the line complex line $L \subset \mathbb{C}^2$ determined by it. Therefore \langle, \rangle defines a weight one polarized Hodge structure on V with

$$V_{\mathbb{C}}^{1,0} = (W_+ \oplus 0) \oplus (L \otimes \mathbb{R}^2).$$

The stabilizer in $G(\mathbb{R})$ of this Hodge structure is a maximal compact subgroup $M \simeq SU(2, \mathbb{C}) \times U(1, \mathbb{C})$, and $G(\mathbb{R})$ acts transitively on the set of polarized Hodge structures of V with polarization given by \langle, \rangle . Thus they are parametrized by $G(\mathbb{R})/M$ which is isomorphic to the Poincaré disc \mathbb{D} .

If we choose a lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$ over which \langle, \rangle takes integral values and which has determinant one then we get a family of 4-dimensional simple Abelian varieties parametrized by the Poincaré disk. If $\Gamma \subset G(\mathbb{Q})$ is a torsion-free arithmetic subgroup which stabilizes $V_{\mathbb{Z}}$ (and, as Γ cannot be contained in M , does not preserve the Hodge structure) then we obtain a family $f : X \rightarrow B$ over the compact Riemann surface $B = \Gamma \backslash \mathbb{D} = \Gamma \backslash G(\mathbb{R})/M$. The subspace $(W_+ \oplus 0) \oplus 0 \subset V_{\mathbb{C}}$ gives rise to a rank two local system \mathbb{W} over B contained in $(R^1 f_* \mathbb{C})^{1,0}$, and with monodromy given by the image of $\Gamma \subset G(\mathbb{Q}) \subset G(\mathbb{R}) \simeq SU(2, \mathbb{C}) \times SL(2, \mathbb{R})$ under the projection to $SU(2, \mathbb{C})$. Therefore $\mathbb{W} \otimes \mathcal{O}_B$ can be seen as a subsheaf of $f_* \Omega_{X/B}^1$. Kollár's decomposition theorem ([24]), implies that $f_* \Omega_{X/B}^1$ splits holomorphically as a direct sum of subbundles

$$f_* \Omega_{X/B}^1 = (\mathbb{W} \otimes \mathcal{O}_B) \oplus \mathcal{L}$$

with \mathcal{L} ample on B (in our case, \mathcal{L} corresponds to the factor $L \otimes \mathbb{R}^2$). It follows the existence of a canonical dual splitting

$$f_* T_{X/B} = (\mathbb{W} \otimes \mathcal{O}_B)^\perp \oplus \mathcal{L}^\perp.$$

The factor $f^*(\mathcal{L}^\perp)$ determines a rank two subsheaf of $T_{X/B} \subset TX$. The corresponding foliation has trivial Chern classes and is tangent to the fibers of f . Over a point $x \in B$ the Abelian fourfold $f^{-1}(x)$ is given by the dual of

$$\frac{(W_+ \oplus 0) \oplus (L_x \otimes \mathbb{R}^2)}{\Lambda},$$

where Λ is the projection of $V_{\mathbb{Z}}$ to the first factor of the decomposition $V_{\mathbb{C}} = V_{\mathbb{C},x}^{1,0} \oplus \overline{V_{\mathbb{C},x}^{1,0}}$. The foliation on $f^{-1}(x)$ is the linear foliation determined by the kernel of $L_x \otimes \mathbb{R}^2 \subset V_{\mathbb{C},x}^{1,0}$.

This example shows that the unitary representation attached to the tangent sheaf of a foliation with vanishing Chern classes can be indeed infinite. It also shows that the hypothesis on the codimension of the foliation in [31, Theorem 1.5] is necessary, contrarily to what was conjectured there.

5 Codimension two

In this section we will prove Theorem D by analyzing the variation of weight one polarized Hodge structures attached to the almost holomorphic map given by Theorem C.

5.1 Settling the notation

Let \mathcal{F} be a codimension two foliation with $c_1(T\mathcal{F}) = c_2(T\mathcal{F}) = 0$ on a non-uniruled projective manifold X . We will denote by n the dimension of X and by r the dimension of \mathcal{F} . Let

$$\rho : \pi_1(X) \rightarrow U(r, \mathbb{C})$$

be the representation attached to $T\mathcal{F}$, and recall that there exists a transverse foliation \mathcal{F}^\perp defined by a closed 1-form with values in the flat vector bundle $T\mathcal{F}$.

Then, by Theorem C, we can assume that \mathcal{F} is tangent to the fibers of an almost holomorphic map $\Phi : X \dashrightarrow Y$ with general fiber being an Abelian variety. Thus there exists open subsets $U \subset X$ and $V \subset Y$ such that $\Phi|_U$ is a proper smooth fibration with connected fibers over $V \subset Y$. With the additional data of an ample divisor on X , this defines a variation of weight one polarized Hodge structure over V . Let F be a fiber of Φ and denote by q_F the polarization form on F . Call ψ the natural representation

$$\psi : \pi_1(V) \rightarrow \text{Aut } H^1(F, \mathbb{C})$$

associated to the local system $R^1\Phi_*\mathbb{C}_U$ and consider $H^1(F, \mathbb{C})$ as a $\pi_1(V)$ -module. Let G be the image of $\pi_1(V)$ by ψ .

Let us define $N^{1,0}$ as the maximal $\pi_1(V)$ -submodule of $H^1(F, \mathbb{C})$ contained in $H^{1,0}(F)$. Because $\pi_1(V)$ acts isometrically with respect to the scalar product

$$N^{1,0} \times N^{1,0} \ni (\omega_1, \omega_2) \mapsto q_F(\omega_1, \overline{\omega_2}),$$

it is indeed a unitary submodule. The orthogonal (non necessarily unitary) submodule to $N = N^{1,0} \oplus \overline{N^{1,0}}$ with respect to q_F is of the form

$$B = B^{1,0} \oplus \overline{B^{1,0}}$$

where $B^{1,0}$ is a complementary subspace of $N^{1,0}$ in $H^{1,0}$.

5.2 Proof of Theorem D

Notice that $H^0(F, i^*(N\mathcal{F}^\perp)^*)$ is a submodule of $N^{1,0}$ (here, i denotes the inclusion of F into V) and that the image of the induced action of $\pi_1(V)$ is precisely $\text{Im } \rho$. Assume for a moment that $N^{1,0} = H^{1,0}(F, \mathbb{C})$, then $H^1(F, \mathbb{C})$ is a unitary module. As G preserves the integer lattice $H^1(F, \mathbb{Z})$, it is a finite group by a theorem of Kronecker.

From now on, we are going to deal with $\text{Im } \rho$ infinite, $\text{codim } \mathcal{F} = 2$ and aim at a contradiction. By the previous observations, we obtain that $N^{1,0} = H^0(F, i^*(N\mathcal{F}^\perp)^*)$ and that $B = B^{1,0} \oplus \overline{B^{1,0}}$ has dimension 2. The group of automorphisms $\text{Gal}(\mathbb{C}/\mathbb{Q})$ acts naturally on the set \mathcal{M} of $\pi_1(V)$ -submodules of $H^1(F, \mathbb{C})$ since the action of $\pi_1(V)$ preserves $H^1(F, \mathbb{Q})$ and therefore is defined over \mathbb{Q} . For $M \in \mathcal{M}$ we will denote M^σ its conjugate by $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$. There exists finitely many irreducible unitary submodules N_1, \dots, N_l such that

$$N^{1,0} = \bigoplus_i N_i$$

corresponding to a splitting of $\rho = \rho_1 \oplus \dots \oplus \rho_l$. Recall that $\text{Im } \rho_i$ is finite whenever $\dim_{\mathbb{C}} N_i = 1$. Hence, there exists some indices i such that $\dim_{\mathbb{C}} N_i \geq 2$. Let us call p_N , respectively p_B , the projection of $H^1(F, \mathbb{C})$ to N , resp. to B . We will distinguish three cases:

- (1) For every $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$, $(N^{1,0})^\sigma \subset N$. This means that the submodule M of N generated by $(N^{1,0})^\sigma$ with σ ranging in $\text{Gal}(\mathbb{C}/\mathbb{Q})$ is defined over \mathbb{Q} , i.e. $M = W_{\mathbb{Q}} \otimes \mathbb{C}$ where $W_{\mathbb{Q}}$ is a subspace of $H^1(F, \mathbb{Q})$. Thus it can be defined over \mathbb{Z} : $M = W_{\mathbb{Z}} \otimes \mathbb{C}$ with $W_{\mathbb{Z}} = W_{\mathbb{Q}} \cap H^1(F, \mathbb{Z})$. Moreover, M is a unitary module (being a submodule of the unitary module N) containing $N^{1,0}$. This implies that $\text{Im } \rho$ is finite, a contradiction.

- (2) There exists an irreducible factor N_{i_0} of dimension at least 2 and $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ such that $p_N((N_{i_0})^\sigma)$ and $p_B((N_{i_0})^\sigma)$ are not $\{0\}$. By irreducibility, these images are irreducible submodules of N and B both isomorphic to $(N_{i_0})^\sigma$. In particular, the second projection is the whole B . Since $p_N((N_{i_0})^\sigma)$ is a unitary $\pi_1(V)$ submodule, the same holds true for B . One can then conclude that G lies in a unitary group and again that G and $\text{Im } \rho$ are finite, absurd.
- (3) There exists an irreducible factor N_{i_0} , $\dim N_{i_0} \geq 2$ and $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ such that $(N_{i_0})^\sigma = B$. Since q_F is defined over \mathbb{Q} , we have that

$$q_F(\omega_1^\sigma, \omega_2^\sigma) = \sigma(q_F(\omega_1, \omega_2)).$$

Here, the contradiction follows from the fact that q_F is trivial on $N_{i_0} \times N_{i_0}$ whereas it is not on $(N_{i_0})^\sigma \times (N_{i_0})^\sigma = B \times B$.

Since at least one of the three possibilities above always holds true, the Theorem follows. □

6 Arithmetic

In this section we analyze the behavior of foliations with vanishing Chern classes under reduction modulo primes. The foliations, varieties, and sheaves defined over a field of characteristic $p > 0$ will be marked with a subscript p , or \mathfrak{p} . For more details about the reduction modulo primes of foliations see [14, 28, 29] and references therein.

6.1 Power map

Let k be an algebraically closed field of characteristic $p > 0$, and let $S_p = \text{Spec}(k)$. In this section X_p will be a smooth irreducible projective S_p -scheme.

A foliation \mathcal{F}_p on X_p is determined by a coherent subsheaf $T\mathcal{F}_p$ of TX_p which is involutive and has torsion free cokernel $TX_p/T\mathcal{F}_p$. Unlike in characteristic zero, where Frobenius integrability theorem implies that a foliation at a formal neighborhood of a general point is nothing but a fibration, a foliation does not need to have a leaf through a general point. This is the case only when $T\mathcal{F}_p$ is not just involutive but also p -closed, i.e., closed under p -th powers.

The p -closedness of $T\mathcal{F}_p$ is equivalent to the vanishing of the morphism of \mathcal{O}_{X_p} -modules

$$\begin{aligned} \times^p : \text{Frob}^*T\mathcal{F}_p &\longrightarrow \frac{TX_p}{T\mathcal{F}_p} \\ a \otimes v &\longmapsto av^p. \end{aligned}$$

Here $Frob: X_p \rightarrow X_p$ denotes the absolute Frobenius morphism. Thus $Frob$ is the identity over the topological space, but the morphism at the level of structural sheaves given by

$$\begin{aligned} Frob^\sharp : \mathcal{O}_{X_p} &\longrightarrow \mathcal{O}_{X_p} \\ f &\longmapsto f^p. \end{aligned}$$

Therefore

$$Frob^*T\mathcal{F}_p = \mathcal{O}_{X_p} \otimes_{Frob^{-1}\mathcal{O}_{X_p}} Frob^{-1}T\mathcal{F}_p \simeq \mathcal{O}_{X_p} \otimes_{Frob^{-1}\mathcal{O}_{X_p}} T\mathcal{F}_p$$

is isomorphic to $T\mathcal{F}_p$ as a sheaf of abelian groups, but not as a sheaf of \mathcal{O}_{X_p} -modules: in it $a^p \otimes v = 1 \otimes av$.

6.2 Canonical connection

If \mathcal{E}_p is an arbitrary coherent sheaf over X then $Frob^*\mathcal{E}_p$ comes equipped with a canonical connection $\nabla: Frob^*\mathcal{E}_p \rightarrow \Omega^1_{X_p} \otimes Frob^*\mathcal{E}_p$, defined as $\nabla(f \otimes v) = df \otimes v$. Clearly the sheaf of flat sections is a sheaf of $\mathcal{O}^p_{X_p}$ -modules and generates $Frob^*\mathcal{E}_p$ as a sheaf of \mathcal{O}_{X_p} -modules.

If \mathcal{G}_p is a subsheaf of $Frob^*\mathcal{E}_p$ then it is natural to inquire if there exists a $\mathcal{H}_p \subset \mathcal{E}_p$ such that $\mathcal{G}_p = Frob^*\mathcal{H}_p$. Such \mathcal{H}_p exists if and only if the \mathcal{O}_{X_p} -morphism

$$\begin{aligned} \mathcal{G}_p &\longrightarrow \Omega^1_{X_p} \otimes \frac{Frob^*\mathcal{E}_p}{\mathcal{G}_p} \\ \sigma &\longmapsto \nabla(\sigma) \quad \text{mod } \Omega^1_{X_p} \otimes \mathcal{G}_p. \end{aligned}$$

induced by the canonical connection ∇ is identically zero, cf. [25, Section 2]. When \mathcal{H}_p does not exist the morphism above induces a non-trivial \mathcal{O}_{X_p} -morphism

$$\begin{aligned} \Phi_{\mathcal{G}_p} : TX_p &\longrightarrow \text{Hom} \left(\mathcal{G}_p, \frac{Frob^*\mathcal{E}_p}{\mathcal{G}_p} \right) \\ v &\longmapsto \left(\sum f_i \otimes \sigma_i \mapsto \sum df_i(v)\sigma_i \quad \text{mod } \mathcal{G}_p \right). \end{aligned}$$

Proposition 6.1. *Let \mathcal{F}_p be a foliation on X_p . If $\mathcal{G}_p \subset Frob^*T\mathcal{F}_p$ is the kernel of \times^p then every germ of infinitesimal automorphism of \mathcal{F}_p is contained in $\ker \Phi_{\mathcal{G}_p}$. In particular, $\ker \Phi_{\mathcal{G}_p}$ contains the smallest p -closed subsheaf of TX_p containing $T\mathcal{F}_p$.*

Proof. Let v be a germ of infinitesimal automorphism of \mathcal{F}_p , i.e., $v \in TX_p(U)$ and $[v, T\mathcal{F}_p(U)] \subset T\mathcal{F}_p(U)$ for some non-empty open subset $U \subset X_p$. If $\sum f_i \otimes v_i \in \text{Frob}_{X_p}^* T\mathcal{F}_p(U)$ is an element in the kernel of \times^p then

$$\sum f_i v_i^p = 0 \pmod{T\mathcal{F}_p(U)}.$$

Since v is an automorphism we have that $[v, \sum f_i v_i^p] = 0 \pmod{T\mathcal{F}_p(U)}$. From $[v, \sum f_i v_i^p] = \sum df_i(v)v_i^p + \sum f_i[v, v_i^p] = \sum df_i(v)v_i^p - \sum f_i \text{ad}_{v_i^p}(v)$ and $\text{ad}_{v_i^p}(v) = (\text{ad}_{v_i})^p(v)$ [20, eq. (60), page 186] we deduce that $\sum f_i \text{ad}_{v_i^p}(v)$ belongs to $T\mathcal{F}_p(U)$ and consequently

$$\sum df_i(v)v_i^p = 0 \pmod{T\mathcal{F}_p(U)}.$$

This last identity proves the first statement. For the second statement it suffices to notice v^{p^n} ($n \geq 1$) are infinitesimal automorphisms of \mathcal{F}_p for any local section $v \in T\mathcal{F}_p(U)$ and that these vector fields generate the p -closure of $T\mathcal{F}_p(U)$. \square

6.3 Lifting the p -envelope of a foliation

Up to the end of this Section, we will denote a maximal prime in a finitely generated \mathbb{Z} -algebra R with residue field of characteristic $p > 0$ by \mathfrak{p} .

Proposition 6.2. *Let \mathcal{F} be a semi-stable foliation on a polarized complex projective manifold (X, H) satisfying $\text{deg}(T\mathcal{F}) := \det(T\mathcal{F}) \cdot H^{n-1} = 0$. If everything in sight is defined over a finitely generated \mathbb{Z} -algebra R then one of the following assertions hold true:*

- (1) *the foliation $\mathcal{F}_{\mathfrak{p}}$ is p -closed for all maximal primes \mathfrak{p} in an non-empty open subset of $\text{Spec}(R)$;*
- (2) *the foliation \mathcal{F} is tangent to a foliation \mathcal{G} with $\dim \mathcal{G} > \dim \mathcal{F}$ and $\det(T\mathcal{G}) \cdot H^{n-1} \geq 0$; or*
- (3) *the foliation \mathcal{F} is tangent to a foliation \mathcal{G} with $\dim X > \dim \mathcal{G} > \dim \mathcal{F}$, $\det(T\mathcal{G}) \cdot H^{n-1} < 0$, and the reduction modulo \mathfrak{p} of $T\mathcal{F}$ is not Frobenius semi-stable for all maximal primes \mathfrak{p} in a dense subset of $\text{Spec}(R)$. Moreover, there exists a non-empty open subset $U \subset \text{Spec}(R)$ such that for every maximal prime $\mathfrak{p} \in U$ the foliation $\mathcal{F}_{\mathfrak{p}}$ is p -closed or $\text{Frob}^* T\mathcal{F}_{\mathfrak{p}}$ is not semi-stable.*

In case (2) we do not exclude the possibility of \mathcal{G} being the foliation on X with only one leaf, i.e. $T\mathcal{G} = TX$, but this happens only if X is uniruled (when KX is not pseudo-effective by [5]) or when X has trivial canonical class.

Proof of Proposition 6.2. Suppose that \mathcal{F}_p the reduction modulo \mathfrak{p} of \mathcal{F} is not p -closed. Then the morphism $\times^p : \text{Frob}^*T\mathcal{F}_p \rightarrow TX_p/T\mathcal{F}_p$ is non-zero. Let \mathcal{I}_p be its image.

If $\text{Frob}^*T\mathcal{F}_p$ is semi-stable then $\deg(\mathcal{I}_p) \geq 0$. Otherwise, according to [29, Corollary 2^p], there exists a constant $C \geq 0$, independent of \mathfrak{p} , such that $\deg(\mathcal{I}_p) \geq -C$.

Let \mathcal{E}_p be the saturation in TX_p of the inverse image of \mathcal{I}_p under the natural quotient morphism $TX_p \rightarrow TX_p/T\mathcal{F}_p$. Notice that \mathcal{E}_p contains $T\mathcal{F}_p$ and, as \mathcal{I}_p , has degree bounded from below by $-C$. Since p -th powers of vector fields tangent to \mathcal{F}_p give rise to infinitesimal automorphisms of \mathcal{F}_p it follows that \mathcal{E}_p is involutive. Also, the cokernel of inclusion of \mathcal{E}_p in TX_p is torsion free and, when \mathfrak{p} varies, have degree uniformly bounded from above by C . Thus the family of involutive sheaves \mathcal{E}_p containing $T\mathcal{F}_p$ belong to a bounded family, [18, Corollaire 2.3], and there exists a foliation \mathcal{G} in characteristic zero strictly containing \mathcal{F} with a tangent sheaf that reduces modulo \mathfrak{p} to \mathcal{E}_p .

If $\text{Frob}^*T\mathcal{F}_p$ is semi-stable for a dense subset of maximal primes $\mathfrak{p} \in \text{Spec}(R)$ then $\deg(T\mathcal{G}) \geq 0$ and there is nothing else to prove. If instead $\text{Frob}^*T\mathcal{F}_p$ is not semi-stable for a dense subset of maximal primes $\mathfrak{p} \in \text{Spec}(R)$ then $\deg(T\mathcal{G}) < 0$ is not excluded. Aiming at a contradiction, let us assume that $T\mathcal{G}$ coincides with TX . Then for infinitely many primes the sheaf \mathcal{E}_p defined above coincides with TX_p and the kernel \mathcal{K}_p of $\times^p : \text{Frob}^*T\mathcal{F}_p \rightarrow TX_p/T\mathcal{F}_p$ has positive degree. Since \mathcal{E}_p is generated, at a general point of X_p , by p -th powers of $T\mathcal{F}_p$ we can apply Proposition 6.1 to deduce that \mathcal{K}_p is a Frobenius pull-back of a subsheaf of $T\mathcal{F}_p$ contradicting the semi-stability of $T\mathcal{F}_p$. \square

6.4 Proof of Theorem E

Let \mathcal{F} be a foliation on a non-uniruled complex projective manifold X defined over a finitely generated Z -algebra $R \subset \mathbb{C}$ maximal among the foliations with tangent bundle with vanishing first and second Chern classes. At a first moment let us assume that R is contained in a number field.

We will start by excluding case (2) of Proposition 6.2. Indeed, if \mathcal{G} is the foliation containing \mathcal{F} given by Proposition 6.2 then since $T\mathcal{G}$ is pseudo-effective by [10] and the polarization in Proposition 6.2 is arbitrary, it follows that $c_1(T\mathcal{G}) = 0$. Theorem 2.2 ensures the existence of a transverse foliation \mathcal{G}^\perp , and \mathcal{G} is tangent to the almost holomorphic map $\pi : X \dashrightarrow S$ given by Theorem C. Restricting to an open subset $U \subset X$ where the fibration is proper and smooth we see that the conormal bundle of \mathcal{G}^\perp has an injective natural morphism

to the relative cotangent bundle of the fibration by Abelian varieties. Let $N^*\mathcal{G}_{\mathcal{F}}^\perp$ denote its image and consider its direct image $(\pi_U)_*N^*\mathcal{G}_{\mathcal{F}}^\perp$. If we take a section of $N^*\mathcal{G}^\perp$ restricted to a fiber F of π , thus a closed holomorphic 1-form, then the parallel transport along the leaves of \mathcal{G}^\perp provides canonical extensions to a neighborhood of F on X which is still closed. Thus $(\pi_U)_*N^*\mathcal{G}_{\mathcal{F}}^\perp$ is flat for the Gauss-connection. It follows that $N^*\mathcal{G}^\perp$ is hermitian flat on U . Since the complement of U has no irreducible components invariant by \mathcal{G}^\perp this hermitian flat structure on $N^*\mathcal{G}^\perp$ extends to the whole X . Hence not only $c_1(T\mathcal{G}) = 0$ but also $c_2(T\mathcal{G}) = 0$, contradicting the maximality of \mathcal{F} . Thus either \mathcal{F} is p -closed for almost every prime of R or $T\mathcal{F}_{\mathfrak{p}}$ is not Frobenius semi-stable for infinitely many primes \mathfrak{p} .

Assume we are in the first case: the foliation is p -closed for almost every prime \mathfrak{p} . There exists a complex manifold $X_{\mathcal{F}}$ endowed with a holomorphic submersion $\pi : X_{\mathcal{F}} \rightarrow X$ and a section $\sigma : X \rightarrow X_{\mathcal{F}}$ such that the fiber of π over $x \in X$ is the holonomy covering of the leaf of \mathcal{F} through x , see for instance [8, Section 4.2].

Since X is compact and the leaves of \mathcal{F} are uniformized by Euclidean spaces according to Theorem A, it follows that $X_{\mathcal{F}}$ is Liouvillian in the sense of pluripotential theory: every plurisubharmonic function bounded from above is constant. Moreover there exists a map $\varphi : X_{\mathcal{F}} \rightarrow X \times X$ such that (a) $\varphi \circ \sigma(x) = (x, x)$; (b) the restriction of φ to $\pi^{-1}(x)$ is the holonomy covering of the leaf through x of a copy of \mathcal{F} contained in $\{x\} \times X$. Therefore our foliation satisfies all the hypothesis of [4, Theorem 2.2] and we conclude that all the leaves of \mathcal{F} are algebraic. Since $c_1(T\mathcal{F}) = c_2(T\mathcal{F}) = 0$ then the leaves of \mathcal{F} also have $c_1 = c_2 = 0$ and, as recalled in the Introduction, it follows that they are all finite coverings of Abelian varieties. The transverse foliation \mathcal{F}^\perp defines isomorphisms between the distinct leaves and establishes the isotriviality of the family of Abelian varieties. Therefore after an unramified covering we arrive at the product of an Abelian variety A with another projective manifold Y and the pull-back of \mathcal{F} is given by the relative tangent sheaf of the projection $A \times Y \rightarrow Y$. This shows that assertion (1) in the statement of Theorem E holds true.

If instead we are in case (3) of Proposition 6.2 then it is assertion (2) in the statement of Theorem E that holds true. The Theorem follows.

The general case, where R is a arbitrary finitely generated \mathbb{Z} -algebra, can be proved along the same lines. If \mathcal{F} is p -closed for every maximal prime \mathfrak{p} in a non-empty open subset of $Spec(R)$ and K , the field of fractions of R , have positive transcendence degree over \mathbb{Q} then we replace the p -closed foliation on the projective manifold X defined over R , by a family of p -closed foliations on

projective manifolds over an affine base (with function field K) defined over a number field. As the conditions on the Chern classes of $T\mathcal{F}$ are algebraic and stable under specialization, we can conclude that every foliation in the family (perhaps after restricting to a non-empty open subset of B) has Liouvillian leaves. As affine manifolds are also Liouvillian we can apply Bost's Theorem to this family in order to conclude the algebraicity of leaves. \square

6.5 A final remark

We do believe that stable foliations on non-uniruled projective manifolds with $c_1(T\mathcal{F}) = 0$ and $c_2(T\mathcal{F}) \neq 0$ have compact leaves. Although the example presented in Section 4 tell us that case (3) of Proposition 6.2 (the p -envelope \mathcal{G} of a foliation with $c_1(T\mathcal{F}) = 0$ has negative $c_1(T\mathcal{G})$) can happen, we point out that [29, Theorem 5] implies that when $\dim(\mathcal{F}) = 2$ and the p -envelope has negative first Chern class then $c_2(T\mathcal{F}) = 0$. A generalization of Shepherd-Barron's result to stable foliations with $c_1(T\mathcal{F}) = 0$ and arbitrary dimension would leave open the possibility of using reduction modulo primes to prove the compactness of leaves.

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Jorge Vitório Pereira

IMPA

Estrada Dona Castorina, 110

Horto, Rio de Janeiro

BRAZIL

E-mail: jvp@impa.br

Frédéric Touzet

IRMAR

Campus de Beaulieu

35042 Rennes Cedex

FRANCE

E-mail: frederic.touzet@univ-rennes1.fr