1. Introduction

Thm (Paun 98). Let (X, ω) be a compact Kähler manifolds with $-K_X$ nef. Then $\pi_1(X)$ has polynomial growth.

Def. For a group G with a finite system of generators $\Gamma = \{\gamma_1 \cdots \gamma_r\}$, set $S(n)$ = number of elements of G that can be written as a product of length $\leq n$ of γ_i, γ_i^{-1}. We say G has polynomial growth, if there exists a polynomial $P \in \mathbb{R}[X]$. s.t. $S(n) \leq P(n)$.

The main tool in the proof:

Margulis lemma.

The lemma gives a sufficient condition for certain subgroups of $\pi_1(X)$ to be of polynomial growth.

Paun proves his result by showing one can find such a subgroup, who maps on to $\pi_1(X)$.

Question. Can we generalise this result to orbifold case ?

Some known generalisation.

Thm (Kobayashi 61.)

(X, ω) compact Kähler manifold. $Ric_\omega > 0$.

Then $\pi_1(X)$ is trivial.

Thm (Campana - Claudon, 12).

$(\mathcal{X}, \omega)$ compact kähler orbifold. $Ric_\omega > 0$.

Then $\pi_1^{orb}(\mathcal{X})$ is finite.

Tools known in orbifold theory.

① Galois theory of covering ~~orbifolds~~ by (Thurston).

② results in Riemannian orbifolds obtained by Borzellino in 93.

"Simpson's Principle"

Any results in manifolds obtained by applying local theory should have a generalisation in orbifolds.

2. orbifolds and its metric geometry.

We begin by recall the definition of orbifolds.

Def. Let X be a topological space. Fix $n \geq 0$.

(1). An n-dimensional real orbifold chart on X is a triple consisting of an open subset $\hat{U} \subset \mathbb{R}^n$, a finite sub-group $G \subset \mathrm{Aut}(\hat{U})$ and a homeomorphism $\phi: \hat{U}/G \to U$, where U is an open subset of X.

(2). Suppose $U \subset V \subset X$ be two open subsets. A chart embedding $\lambda: (\hat{U}, G, \phi) \to (\hat{V}, H, \psi)$ is a smooth embedding $\lambda: \hat{U} \to \hat{V}$ s.t.

$$\begin{tikzcd} \hat{U} \arrow[r, "\lambda"] \arrow[d] & \hat{V} \arrow[d] \\ U \arrow[r, hook] & V \end{tikzcd}$$

commutes.

(3). An orbifold atlas on X is a family $\mathcal{U} = \{(\tilde{U}, G, \phi)\}$ of orbifold charts such that $\{U = \phi(\tilde{U})\}$ covers X. and for any $x \in X$. covered by U and V there exists a third orbifold chart $(\hat{W}, K, \mu)$ with $x \in W$ and two chart embedding $(\hat{W}, K, \mu) \to (\hat{U}, G, \phi)$ and $(\hat{W}, K, \mu) \longrightarrow (\tilde{V}, H, \psi)$.

Def. We define a real orbifold $\mathcal{X}$ of dimension n. to be a pair $(X, \mathcal{U})$, where X is a topological space and $\mathcal{U}$ an orbifold chart. (real).

We could do the same with $\mathbb{C}^n$.

Slogan. An orbifold is a space modeled by $\mathbb{R}^n/G$ or $\mathbb{C}^n/G$. G finite.

For ~~any~~ two chart $(\hat{U}, G, \phi)$, $(\tilde{V}, H, \psi)$, if $\lambda : (\hat{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a chart embedding. then $T\lambda : (T\tilde{U}, G) \to (T\tilde{V}, H)$.

One could glue all these $\{(T\hat{U}_i, G_i)\}$ to get a new orbifold $T\mathfrak{X}$, the tangent bundle of $\mathfrak{X}$.

The description of tensor / form

An orbi-tensor / orbi-form is a section of the respective bundle. An equivalent description is that:

A (p,q)-tensor T on $\mathfrak{X}$ is a collection of G_i-invariant (p,q)-tensors $\tilde{T}_i$ on $(\hat{U}_i, G_i)$, where $(\hat{U}_i, G_i, \varphi_i : \hat{U}_i \to U_i)$ are orbifold charts, and $\cup U_i = X$. If $\lambda : \hat{U}_i \to \tilde{U}_j$ is an embedding, then $\lambda^* \tilde{T}_j = \tilde{T}_i$.

Hence, we could say a Riemannian metric g and a Kähler form ω on $\mathfrak{X}$.

Let $(\mathfrak{X}, g)$ be ~~an~~ a Riemannian orbifold. For any continuous map $c : [0,1] \to X$, it is possible to assign a length $l_g(c) \in [0, +\infty]$.

We thus turn X to a metric space.

with $d(x,y) = \inf\limits_{c : x \sim y} L_g(c)$.

<u>Thm</u>. (Borzellino). <u>(X, d) is a length space.</u>

This paves way for the generalisation of classical Hopf-Rinow and Gromov-Bishop thm.

- <u>Covering orbifold</u>.

A covering orbifold $\mathcal{Y} \to \mathcal{X}$ is a map that is locally modelled by $\mathrm{id} : \mathbb{C}^n/H \to \mathbb{C}^n/G$. with $H < G$.

<u>Thm</u> (Thurston). For any orbifold $\mathcal{X}$, there exists a universal covering $\widehat{\mathcal{X}}$.

We define the fundamental group of $\mathcal{X}$

$$\pi_1^{orb}(\mathcal{X}) = \mathrm{Aut}(\widehat{\mathcal{X}}/\mathcal{X}).$$

<u>Thm</u> (Thurston) We have Galois correspondence between subgroups of $\pi_1^{orb}(\mathcal{X})$ and isomorphic classes of orbifold covering space.

Let $p: \widetilde{\mathcal{X}} \to \mathcal{X}$ be the universal covering.

If g is a Riemannian metric on $\mathcal{X}$. then we can consider $p^*(g)$ on $\widetilde{\mathcal{X}}$ and the metric space $(|\widetilde{\mathcal{X}}|, d_{\tilde{g}})$.

<u>Prop.</u> $\pi_1^{orb}(\mathcal{X})$ acts on $(\widetilde{\mathcal{X}}, d_{\tilde{g}})$ by isometry.

3. Sketch of the proof

Now we try to give the proof of the polynomial growth of $\pi_1^{orb}(\mathcal{X})$. with $-K_{\mathcal{X}}$ nef.

<u>New tool</u>: <u>Generalised Margulis lemma</u>.

This is a set of results obtained by

Breuillard - Green - Tao 2012.

For our convenience, we state the following.

<u>Lemma</u>. Let $n \geq 1$ be an integer. $\exists\, \alpha = \alpha(n) > 0$. such that the following holds true.:

$(\mathcal{X}, g)$ complete Riemannian orbifolds. $\mathrm{Ric}_g \geq -(n-1)$. $\Gamma \subset \mathrm{Isom}(|\mathcal{X}|)$ acting properly discontinuously. Then

$$\Gamma_\alpha(x) := \{\gamma \in \Gamma : d(\gamma \cdot x, x) < \alpha\}$$

is of polynomial growth.

We note to obtain this results, we make use of the orbifold version of Bishop-Gromov comparison thm.

We now recall

Def. Let $(\mathcal{X}, \omega)$ be a compact Kähler orbifold.
We say $-K_{\mathcal{X}} = \det \mathcal{T}_{\mathcal{X}}$ nef. If there exists for each $\varepsilon > 0$, a ~~metric~~ h_ε on $-K_{\mathcal{X}}$. s.t. the chern curvature Θ_{h_ε} satisfies

$$\Theta_{h_\varepsilon} \geq -\varepsilon\omega.$$

The ideal of the proof is by ~~tryin~~ finding a suitable metric ω', s.t. the group

$$\Gamma_\alpha(x) = \langle\langle \{\gamma \in \pi_1 : d(\gamma \cdot x, x) < \alpha\} \rangle\rangle$$

is the whole π_1

here $\langle\langle \cdot \rangle\rangle$ means the normal group generated.

* $-K_{\mathcal{X}}$ nef. implies existence of $\{\omega_\varepsilon\}$ kähler form. $\omega_\varepsilon \sim \omega$ and $\mathrm{Ricci}_{\omega_\varepsilon} \geq -\varepsilon\omega_\varepsilon$.

let h_ε be the metric such that

$$u_\varepsilon = \Theta_{h_\varepsilon} \geq -\varepsilon\omega.$$

One may search ω_ε such that

$$\mathrm{Ricci}_{\omega_\varepsilon} = -\varepsilon\omega_\varepsilon + \varepsilon\omega + u_\varepsilon. \quad (*)$$

note $u_\varepsilon = \mathrm{Ricci}_\omega + \sqrt{-1}\,\partial\bar\partial f.$

$\omega_\varepsilon = \omega + \sqrt{-1}\,\partial\bar\partial\phi_\varepsilon.$

then $(*)$ is equivalent to

$$\frac{(\omega + \sqrt{-1}\,\partial\bar\partial\phi_\varepsilon)^n}{\omega^n} = \exp(\varepsilon\phi_\varepsilon - f_\varepsilon).$$

<u>Thm.</u> (orbifold Aubin-Yau).

on a compact Kähler orbifold $(\mathcal{X}, \omega)$.

the equation $M(\varphi) = \lambda\varphi + f$. $\lambda \geq 0$

has $\begin{cases} \text{unique solution} & \lambda > 0 \quad (\text{Aubin}) \\ \text{unique solution up to constant} & \lambda = 0. \quad (\text{Yau}) \end{cases}$

Thus we have the sequence ω_ε s.t.

$$\mathrm{Ricci}_{\omega_\varepsilon} \geq -\varepsilon\omega_\varepsilon.$$

Let $\hat{\mathfrak{X}} \to \mathfrak{X}$ be the universal covering.

$\{\gamma_i\}$ a finite system of generators..

<u>lemma.</u> (Demailly-Peternell-Schneider 93)

$U \subset \hat{X} = |\hat{\mathfrak{X}}|$ compact. $\forall \delta > 0$ $\exists U_{\varepsilon,\delta} \subset U$

closed, s.t. $\mathrm{vol}_{\omega}(U \setminus U_{\varepsilon,\delta}) < \delta$ and

$\mathrm{diam}_{\omega_\varepsilon}(U_{\varepsilon,\delta}) < C_1 \delta^{\frac{1}{2}}$.

C_1 is a constant independant of ε and δ.

Take U a compact subset of $\hat{X}$ which contains the Dirichlet domain $F \subset \hat{X}$ of $\pi_1^{orb}(\mathfrak{X})$.

We may take U large enough s.t. $U \cap \gamma_j U \neq \phi$ $\forall \gamma_j$.

Take $\delta << 1$ s.t. $\begin{cases} \delta < \frac{1}{4} \mathrm{vol}_\omega(U \cap \gamma_j U) \\ \delta < \frac{1}{2} \mathrm{vol}_\omega(F). \end{cases}$

lemma above $\exists U_{\varepsilon,\delta} \subset U$ s.t.

$\mathrm{diam}_{\omega_\varepsilon}(U_{\varepsilon,\delta}) < C_1 \delta^{\frac{1}{2}} := C$.

By compate volume, we know $U_{\varepsilon,\delta} \cap \gamma_j U_{\varepsilon,\delta} \neq \phi$ $\forall \gamma_j$.

Hence for $\hat{x} \in U_{\varepsilon,\delta}$. $d_{\omega_\varepsilon}(\hat{x}, \gamma_j \hat{x}) < C$.

set $\tilde{\omega}_\varepsilon = \frac{\varepsilon}{n-1}\omega_\varepsilon$. we have

$$\begin{cases} \mathrm{Ric}(\tilde{\omega}_\varepsilon) \geq -(n-1)\tilde{\omega}_\varepsilon \\ d_{\tilde{\omega}_\varepsilon}(\tilde{x}, \gamma_j\tilde{x}) < \frac{\varepsilon}{n-1} C. \end{cases}$$

For $\varepsilon << 1$. $\frac{\varepsilon}{n-1} C < \alpha(n)$.

Hence $\gamma_j \in \Gamma_\alpha(\tilde{x})$. $\Rightarrow \pi_1^{orb}(\mathcal{X}) = \Gamma_\alpha(\tilde{\mathcal{X}})$.