Moduli spaces of parabolic Higgs bundles

Thiago Fassarella

Universidade Federal Fluminense (uff)
Niterói – Brasil

COFECUB

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introduction

Joint work in progress with Frank Loray

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Settings

- X: a smooth curve (over $\mathbb{C}$) of genus $g \geq 0$
- $t_1, \ldots, t_n$ distinct points on X

$$D = t_1 + \cdots + t_n$$

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- We can construct:
  1. Moduli spaces of connections: parametrizing vector bundles E with a $\mathbb{C}$-linear map $\nabla : E \to E \otimes \Omega^1_X(D)$ satisfying the Leibniz's rule.
  2. Moduli spaces of Higgs bundles: parametrizing vector bundles E with an $\mathcal{O}_X$-homomorphism $\theta : E \to E \otimes \Omega^1_X(D)$

In this talk the rank is two.

Connections

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- Fixed data: Usually we fix a determinant linebundle, a trace connection on it, and the eigenvalues of the residual part over poles. Moreover, the construction needs a choice of weights to impose stability conditions.
- It turns out to be a quasi-projective variety, mostly irreducible.

  (Nitsure 1993, Simpson 1994, Inaba-Iwasaki-M.-H.Saito 2003, Inaba 2013)

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4. $g(X) = 1$ and $n > 1$:
    - F - F.Loray, *Flat parabolic vector bundles on elliptic curves*, (2017)
    - F - F.Loray - A.Muniz, *On the moduli of logarithmic connections on elliptic curves*, (2020)

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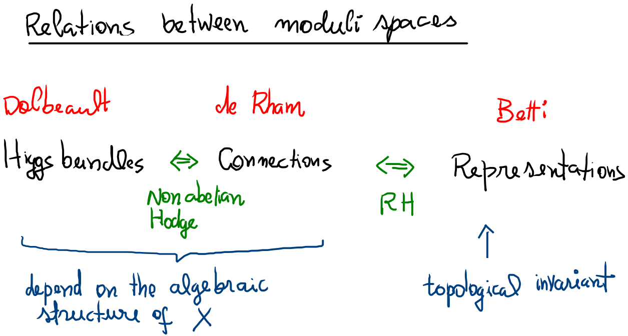

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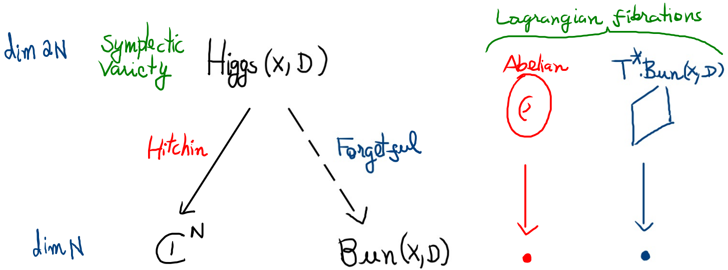
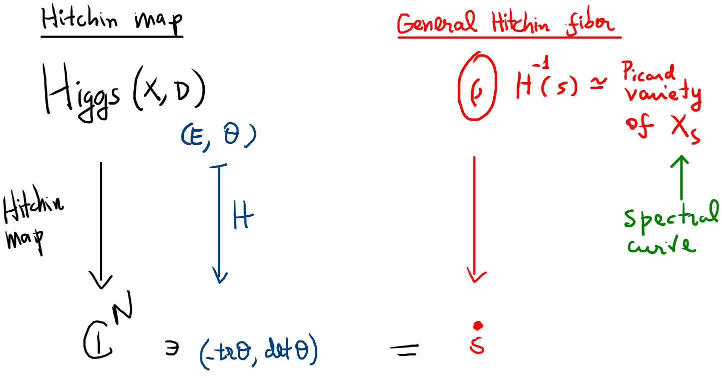


An important paper:

- Beauville-Narasimhan-Ramanan, spectral curves and the generalized theta divisor (1989)

B-N-R correspondence



locally: $X_s = \{ (x, z) \mid z^2 - (\mathrm{tr}\,\theta(x)) \cdot z + \det\theta(x) = 0 \}$

Remaining Hitchin fibers...

One of our motivation is to give a description of the remain special fibers of the Hitchin map, at least in the following cases

- $\mathrm{Higgs}(\mathbb{P}^1, 5)$: $X = \mathbb{P}^1$ and $n = 5$
- $\mathrm{Higgs}(C, 2)$: $X = C$ is an elliptic curve and $n = 2$

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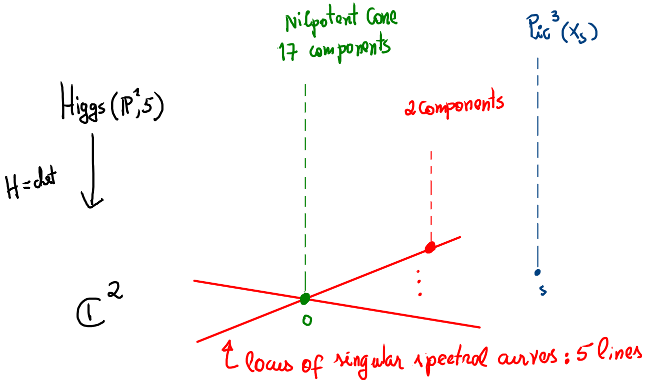

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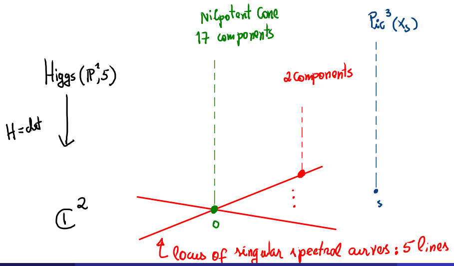
3. the fiber over zero (nilpotent cone) has 17 components: Bun and one component for each one of the 16 lines
4. other singular fibers are birational to an elliptic curve times a degenerate elliptic curve (i.e. two copies of $\mathbb{P}^1$ meeting in two points). This confirms a guess of C. Simpson in
   - C. Simpson, An explicit view of the Hitchin fibration on the Betti side for $\mathbb{P}^1$ minus 5 points, (2017)



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3. Φ preserves the Hitchin fibration and its restriction to a general fiber gives a degree two map
$$\Phi_s : \mathrm{Pic}^3(X_s) \to \mathrm{Prym}^2(C_r) \subset \mathrm{Pic}^2(C_r)$$

The modular map Φ

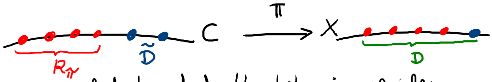
finite map ⇝ modular map

$$C \xrightarrow{\ \pi\ } X$$

$$\mathrm{Higgs}(X, D) \xrightarrow{\ \Phi\ } \mathrm{Higgs}(C, \tilde{D})$$

$$\Phi = \mathrm{elem}_{R_\pi} \circ \pi^*$$

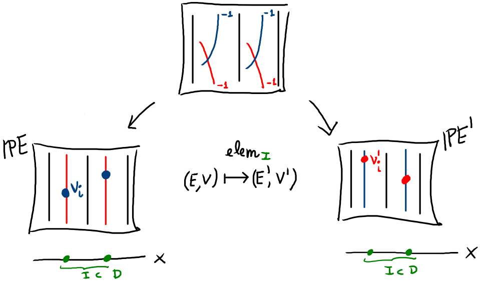
Poles over ramification points became regular after elem



We are led to study the behavior of elem w.r.t the B-N-R correspondence

Elementary transformations

- From the point of view of ruled surfaces



- For vector bundles:

$$0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} \bigoplus_{i \in I} E/V_i \to 0$$

$$\mathrm{elem}_{B_\pi} \circlearrowright \mathrm{Higgs}(\mathbb{P}^1, 5) \xrightarrow{2:1} \mathrm{Higgs}(C, 2)$$

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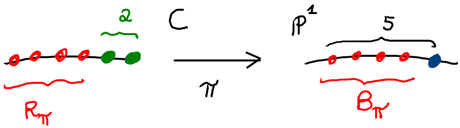

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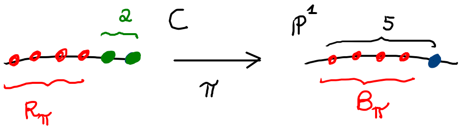

Thank you!