Irregular parabolic bundles

Frank Loray (CNRS / Univ Rennes 1)

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a work in progress with Arata Komyo and Masa-Hiko Saito (Kobe Univ)

rank 2 parabolic bundles on curves

Fix $D = t_1 + \cdots + t_n$ reduced divisor on a smooth curve C.

Definition: A rank 2 (quasi-)parabolic bundle on C is a pair $(E, \boldsymbol{l})$ where:

- $E \to C$ rank 2 holomorphic vector bundle

- $\boldsymbol{l} = (l_1, \ldots, l_n)$ a parabolic structure on E: $l_i \subset E|_{t_i}$ linear 1-dim subspace

Stability condition

Fix weights $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in [0,1]^n$

For a line bundle $L \subset E$, define:

$$\mathrm{stab}_L(E, \boldsymbol{l}) = \underbrace{\deg(E) - 2\deg(L)}_{usual} + \sum_{L|_{t_i} \neq l_i} \mu_i - \sum_{L|_{t_i} = l_i} \mu_i \qquad \in \mathbb{R}$$

$$\mathrm{stab}(E, \boldsymbol{l}) = \mathrm{Inf}\,\{\mathrm{stab}_L(E, \boldsymbol{l}) \ ; \ L \subset E \text{ a line bundle}\} \qquad \in \mathbb{R}$$

Definition: semi-stable (resp. stable) $\quad \Leftrightarrow \quad$ stab ≥ 0 (resp. > 0)

Moduli spaces of parabolic bundles

Mehta-Seshadri 80, Bhosle 89, Boden-Hu 95, Biswas-Holla-Kumar 2010, Araujo-Fassarella-Kaur-Massarenti 2021

Theorem: The moduli space of $\boldsymbol{\mu}$-semi-stable degree d parabolic bundles $(E, \boldsymbol{l})$ on (C, D) admits a GIT quotient $\mathrm{Bun}^{ss}_{\boldsymbol{\mu},d}(C, D)$ which is projective and irreducible of dimension $4g - 3 + n$ (or empty).

Moduli spaces of parabolic connections

Nitsure 93, Arinkin-Lysenko 97, Inaba-Iwasaki-Saito 2006, L.-Saito 2015

Theorem: The moduli space of $\boldsymbol{\mu}$-semi-stable $\boldsymbol{\nu}$-parabolic connections $(E, \boldsymbol{l}, \nabla)$ admits a GIT quotient $\mathrm{Con}^{ss}_{\boldsymbol{\mu},\boldsymbol{\nu}}(C, D)$ which is quasi-projective and irreducible of dimension $2(4g - 3 + n)$. It is holomorphic symplectic (on smooth locus). The bundle map

$$\mathrm{Bun} \ : \ \mathrm{Con}^{ss}_{\boldsymbol{\mu},\boldsymbol{\nu}}(C, D) \dashrightarrow \mathrm{Bun}^{ss}_{\boldsymbol{\mu},d}(C, D) \ ; \ (E, \boldsymbol{l}, \nabla) \mapsto (E, \boldsymbol{l})$$

is Lagrangian with affine fibers.

Remarks:

1) locus of stable bundles/connections is open and smooth.

2) for generic weights $\boldsymbol{\mu}$, semi-stable $\Rightarrow$ stable ($\Rightarrow$ smooth)

3) $\mathrm{Bun}^{ss}_{\boldsymbol{\mu},\boldsymbol{d}}(C,D)$ depends on $\boldsymbol{\mu}$, but not its birational type (wall-crossing)

Moving weights: geometry of quartic Del Pezzo surfaces

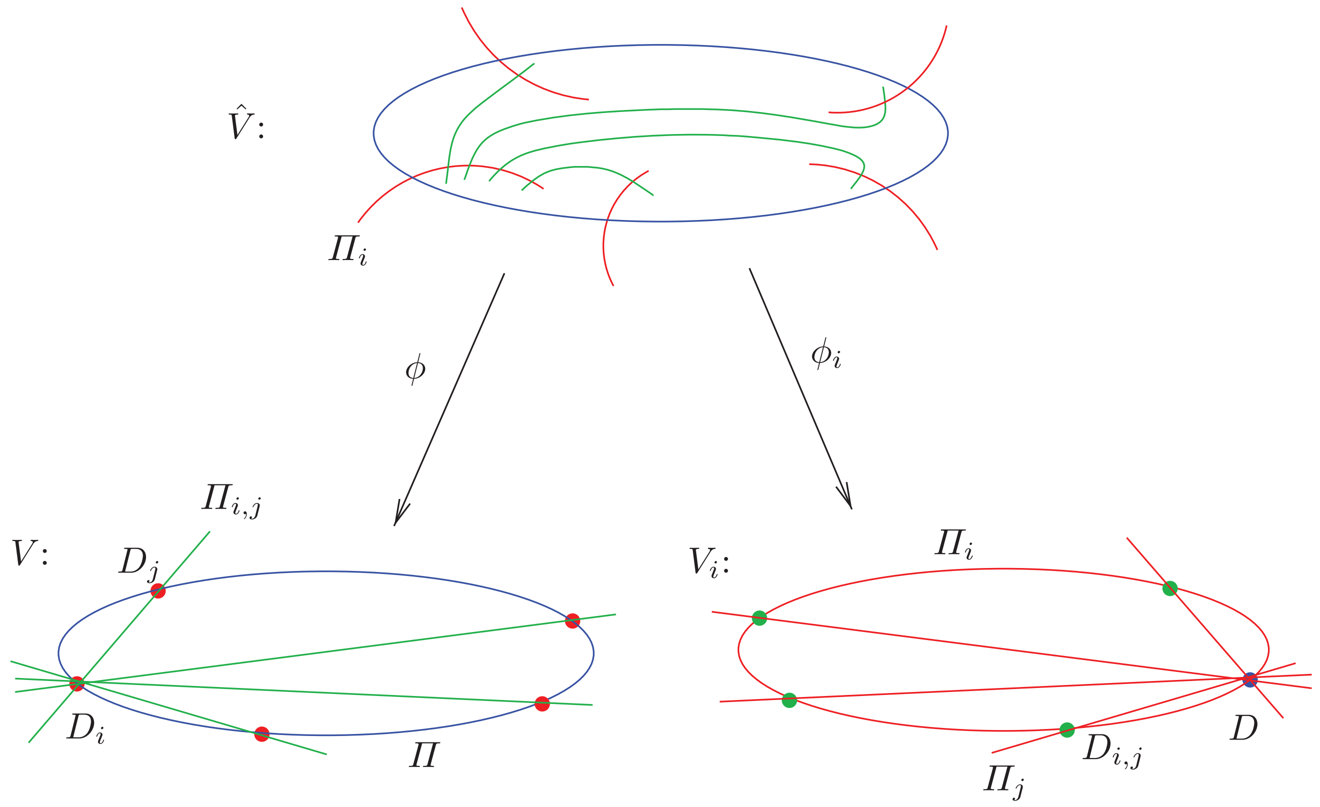


Fig. 2. Projective charts V, $\hat{V}$, and V_i's and Del Pezzo geometry.

L.-Saito, IMRN 2015

Irregular parabolic bundle: ($\leftrightarrow$ irregular connections)
$D = n_1[t_1] + \cdots + n_k[t_k]$ effective on C

Consider D as a non reduced scheme, with structure sheaf $\mathcal{O}_D := \mathcal{O}/\mathcal{O}(D)$.

Parabolic structure on a rank 2 vector bundle $E \to C$:
a line bundle $\boldsymbol{l} \subset E|_D = E \otimes \mathcal{O}_D$. Denote $\boldsymbol{l} = (l_1, \ldots, l_n)$
$\rightsquigarrow (E, \boldsymbol{l})$ irregular quasi-parabolic bundle.

Stability condition: Fix weights $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k) \in [0,1]^k$

For a line bundle $L \subset E$, define:

$$\mathrm{stab}_L(E, \boldsymbol{l}) = \deg(E) - 2\deg(L) + \sum_{i=1}^{k} \Big(n_i - 2\left(l_i \cdot L|_{n_i[t_i]}\right)\Big)\mu_i \qquad \in \mathbb{R}$$

where $l_i \cdot L|_{n_i[t_i]} :=$ intersection multiplicity $\in \{0, 1, \ldots, n_i\}$.

Moduli spaces of irregular parabolic bundles

Yokogawa 93, Bhosle 96

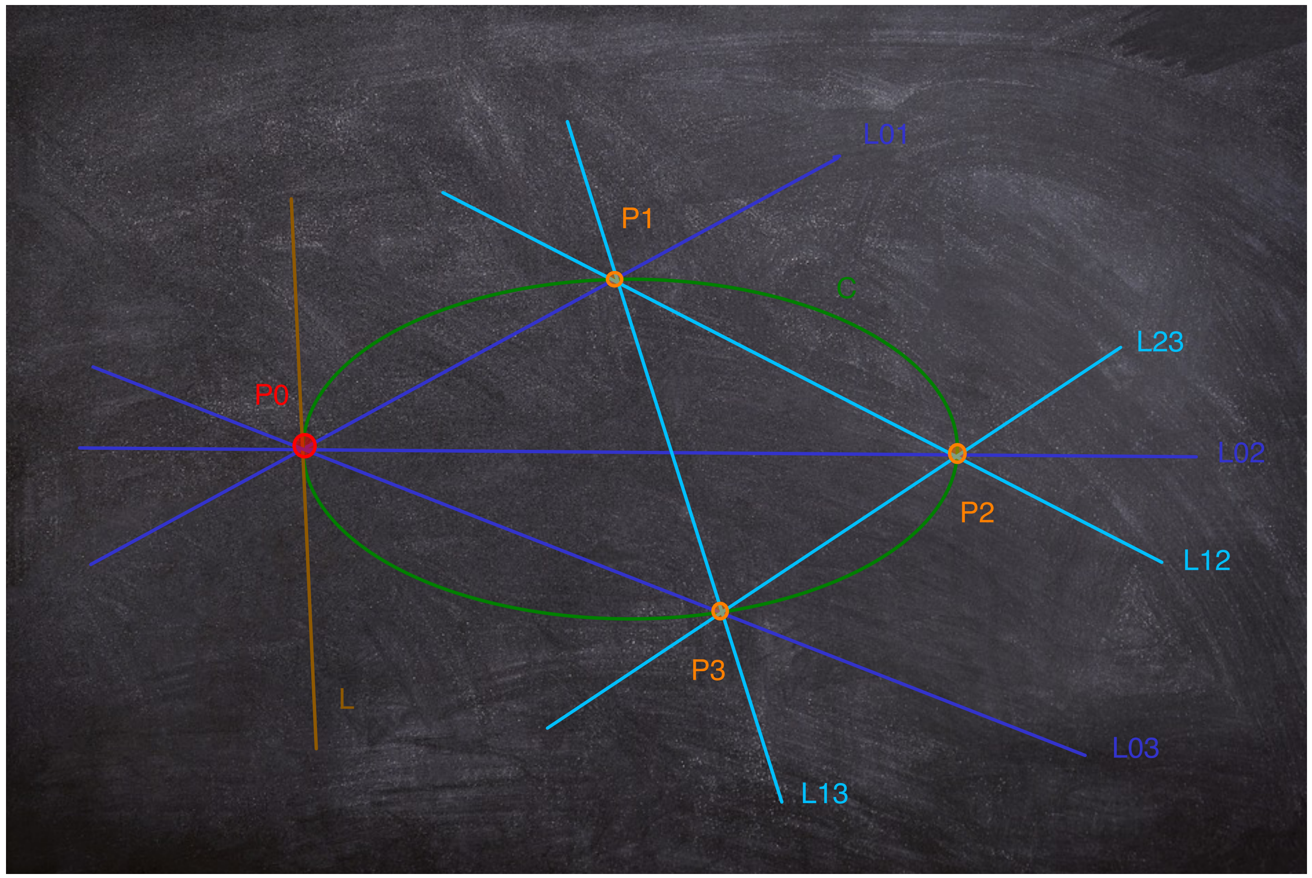
Theorem: The moduli space of $\boldsymbol{\mu}$-semi-stable degree d irregular parabolic bundles $(E, \boldsymbol{l})$ on (C, D) admits a GIT quotient $\mathrm{Bun}^{ss}_{\boldsymbol{\mu},d}(C, D)$ which is projective and irreducible of dimension $4g - 3 + n$ $(n = \sum_k n_k)$.

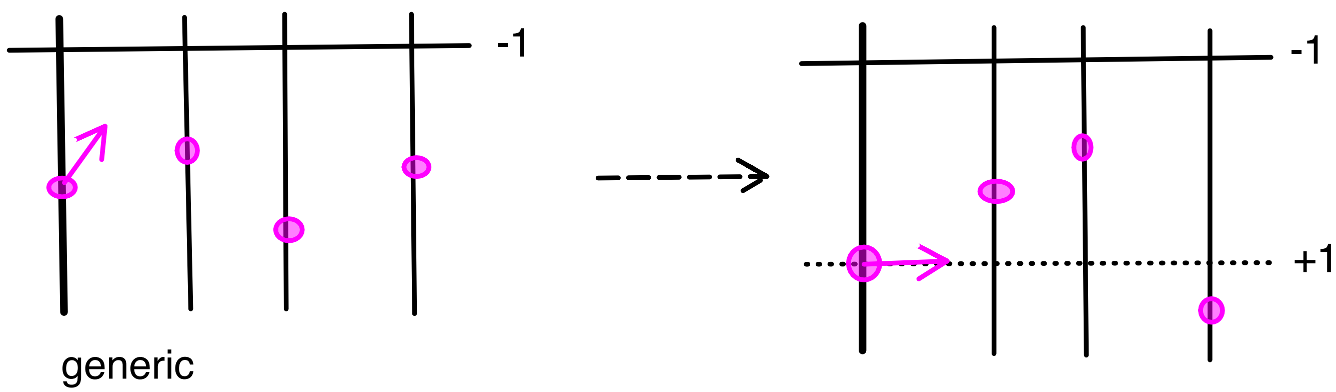
! Here, a more general definition of parabolic bundles is used (filtration of sheaves).

Example: Let $C = \mathbb{P}^1$, $D = n_1[t_1] + \cdots + n_k[t_k]$ and $n = \sum n_i$.
Set $\mu_1 = \ldots = \mu_k =: \mu$ and $\frac{1}{n} < \mu < \frac{1}{n-2}$.

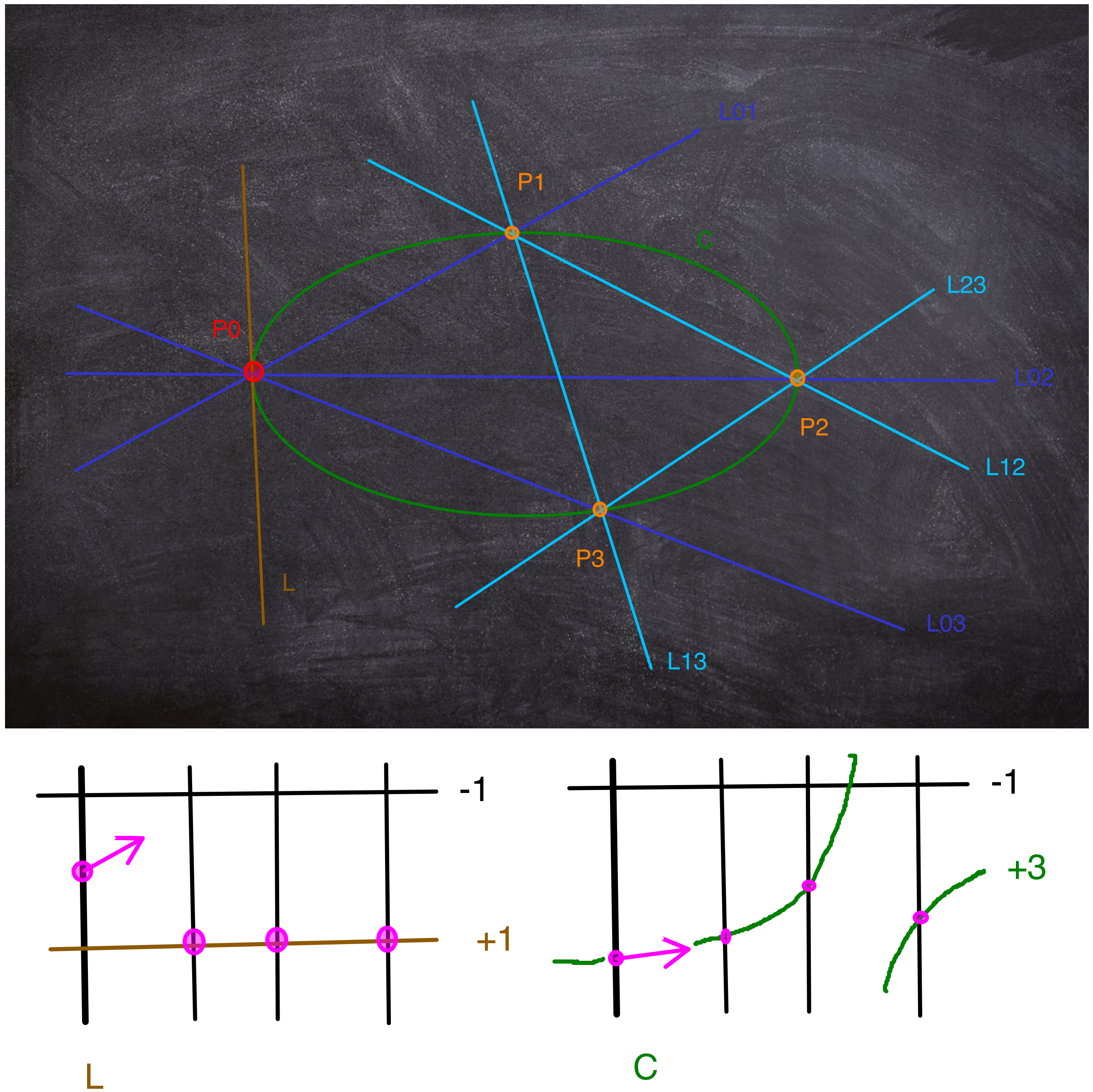
Then $\mathrm{Bun}^{ss}_{\boldsymbol{\mu},-1}(\mathbb{P}^1, D) = \left\{(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \boldsymbol{l}) \ ; \ l_i \cdot \mathcal{O}_{\mathbb{P}^1} = 0\right\} \simeq \mathbb{P}^{n-3}$

Main chart $\mathrm{Bun}^0(\mathbb{P}^1, 2[0] + [t_1] + [t_2] + [t_3]) \simeq \mathbb{P}^2$

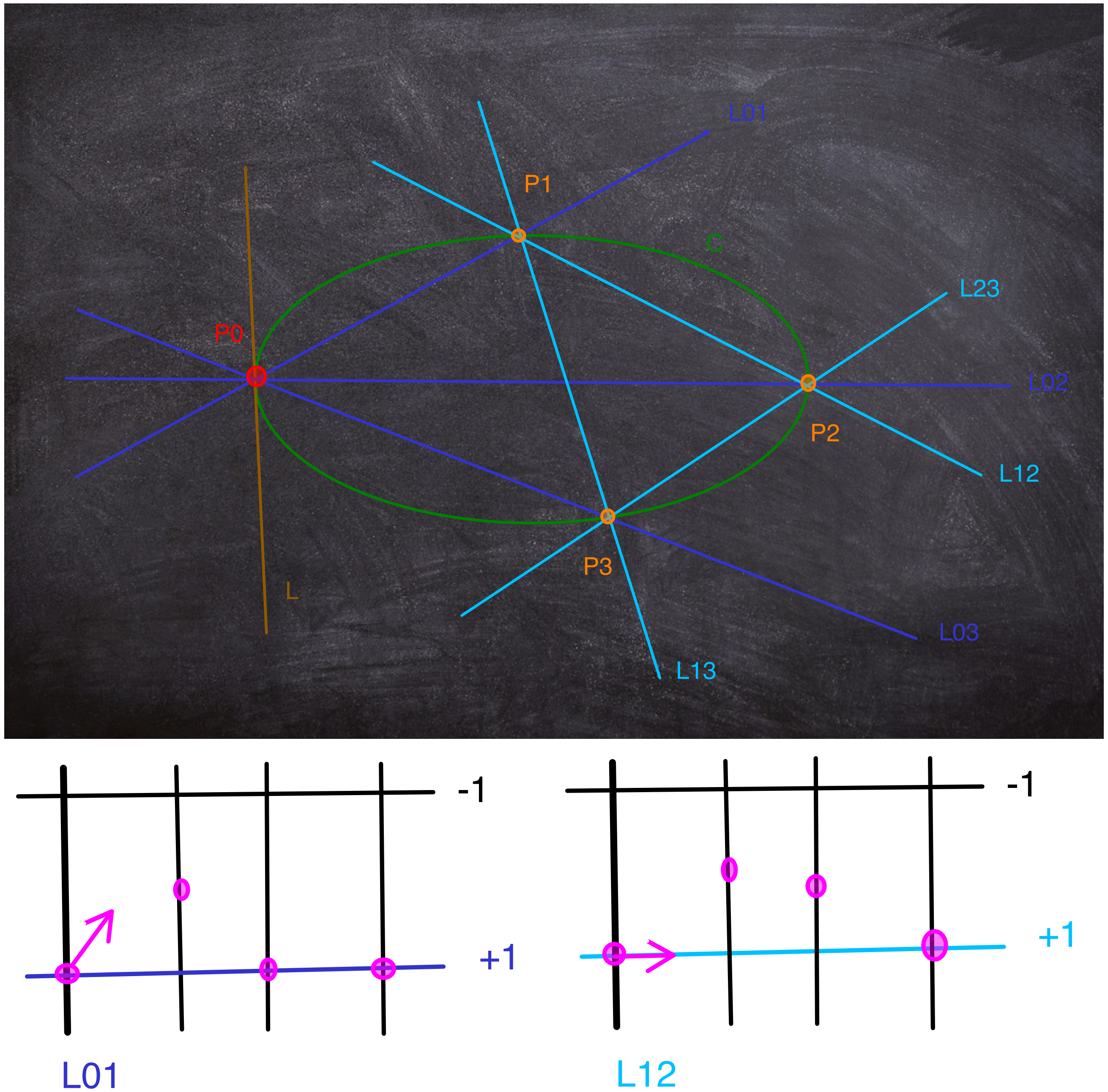




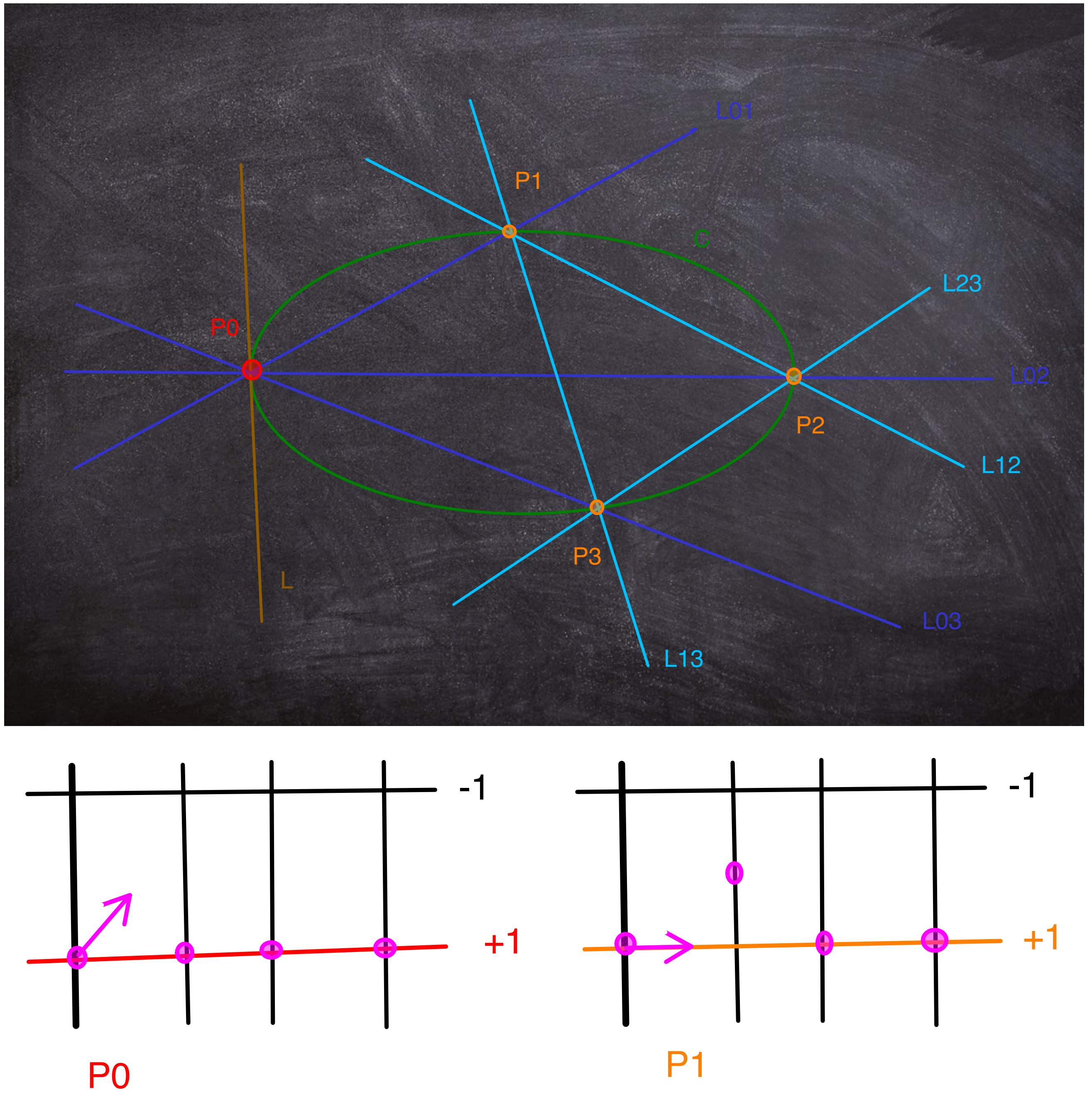
Special bundles in $\mathrm{Bun}^0(\mathbb{P}^1, 2[0]+[t_1]+[t_2]+[t_3])$



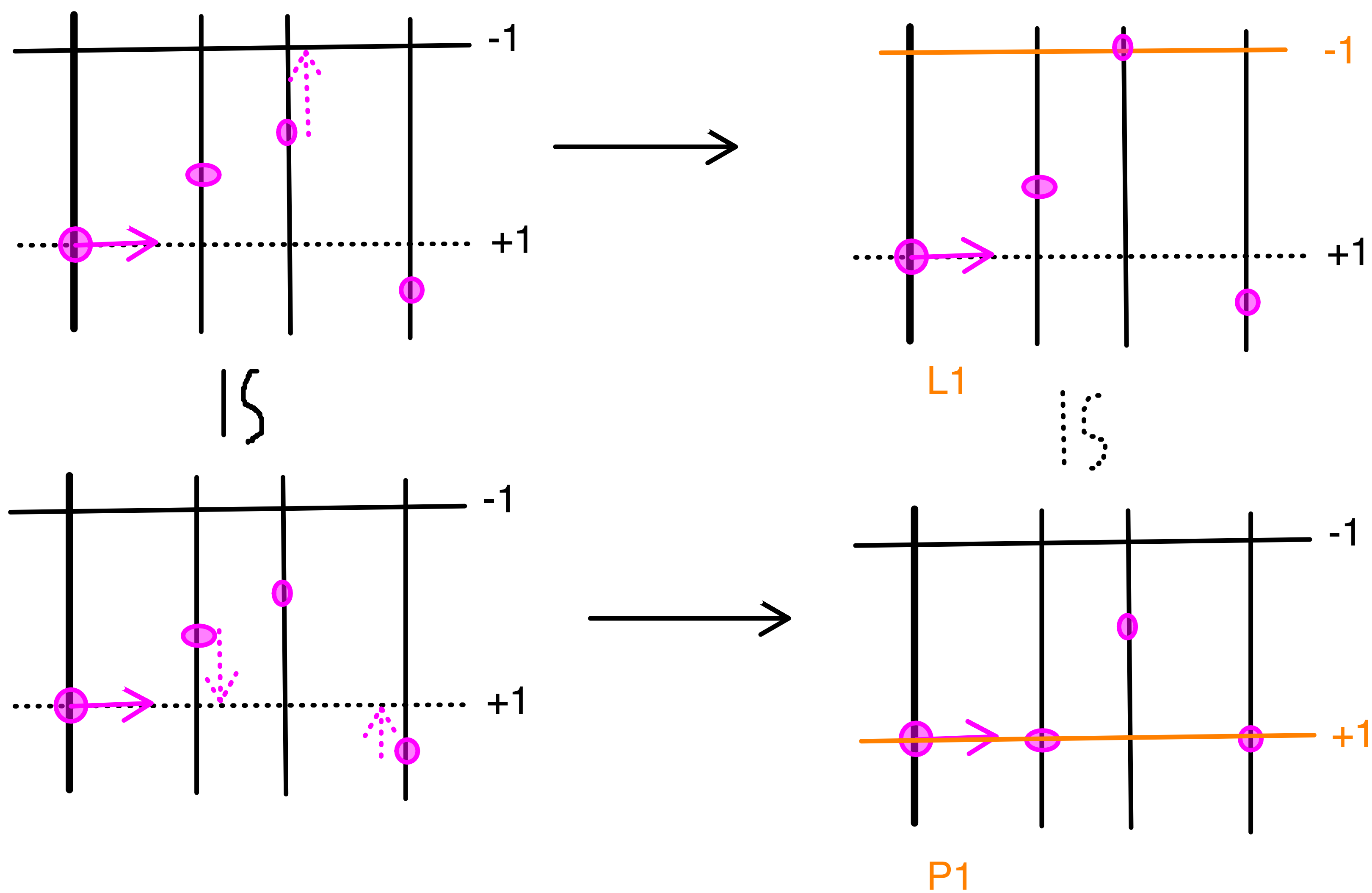
Special bundles in $\mathrm{Bun}^0(\mathbb{P}^1, 2[0] + [t_1] + [t_2] + [t_3])$



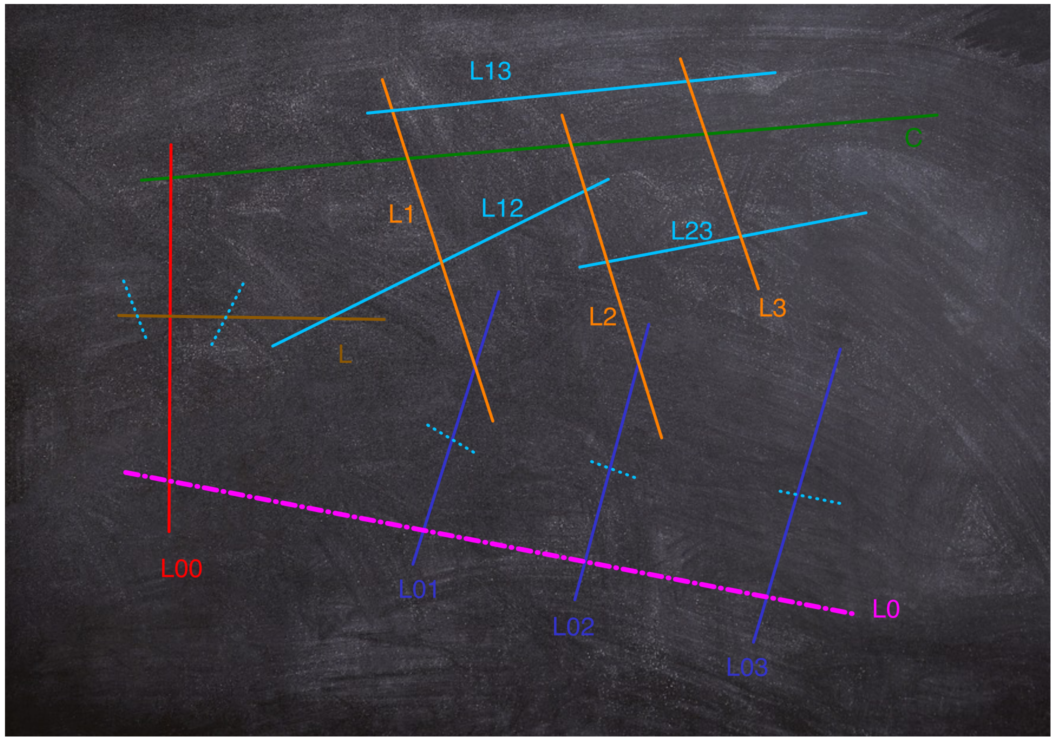
Special bundles in $\mathrm{Bun}^0(\mathbb{P}^1, 2[0] + [t_1] + [t_2] + [t_3])$

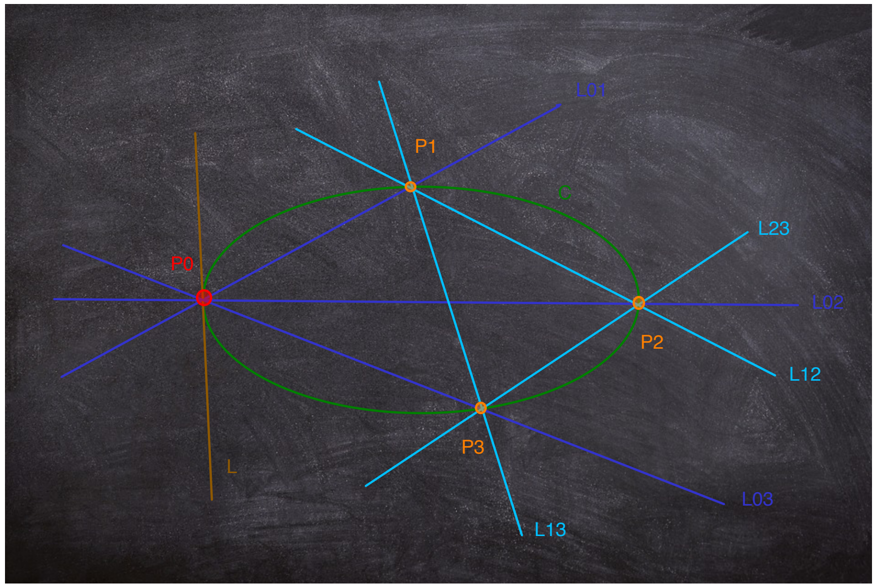


Infinitesimally close bundles in $\mathfrak{Bun}(\mathbb{P}^1, 2[0] + [t_1] + [t_2] + [t_3])$



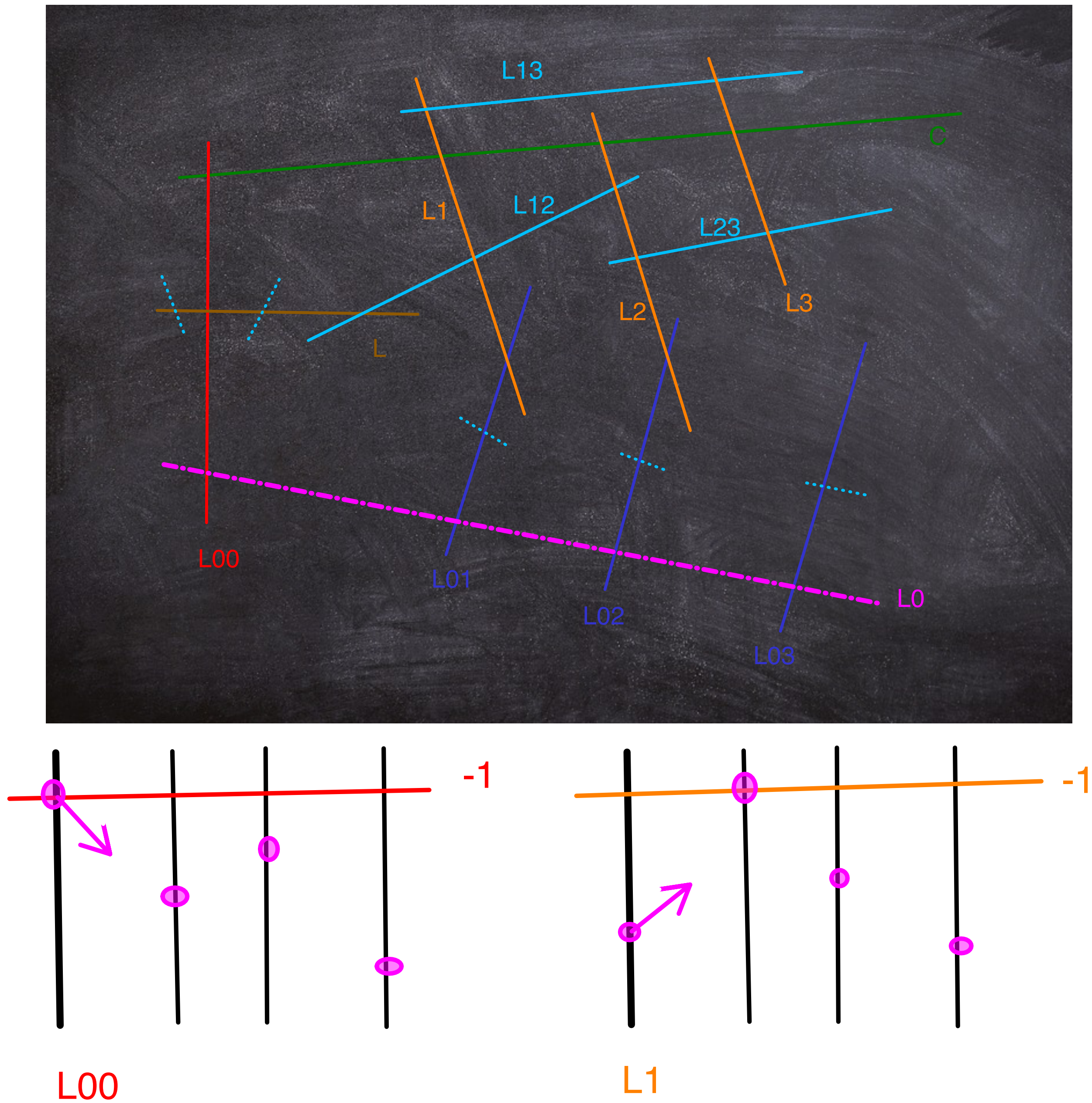
Blowing-up the main chart $\mathrm{Bun}^0(\mathbb{P}^1, 2[0] + [t_1] + [t_2] + [t_3])$



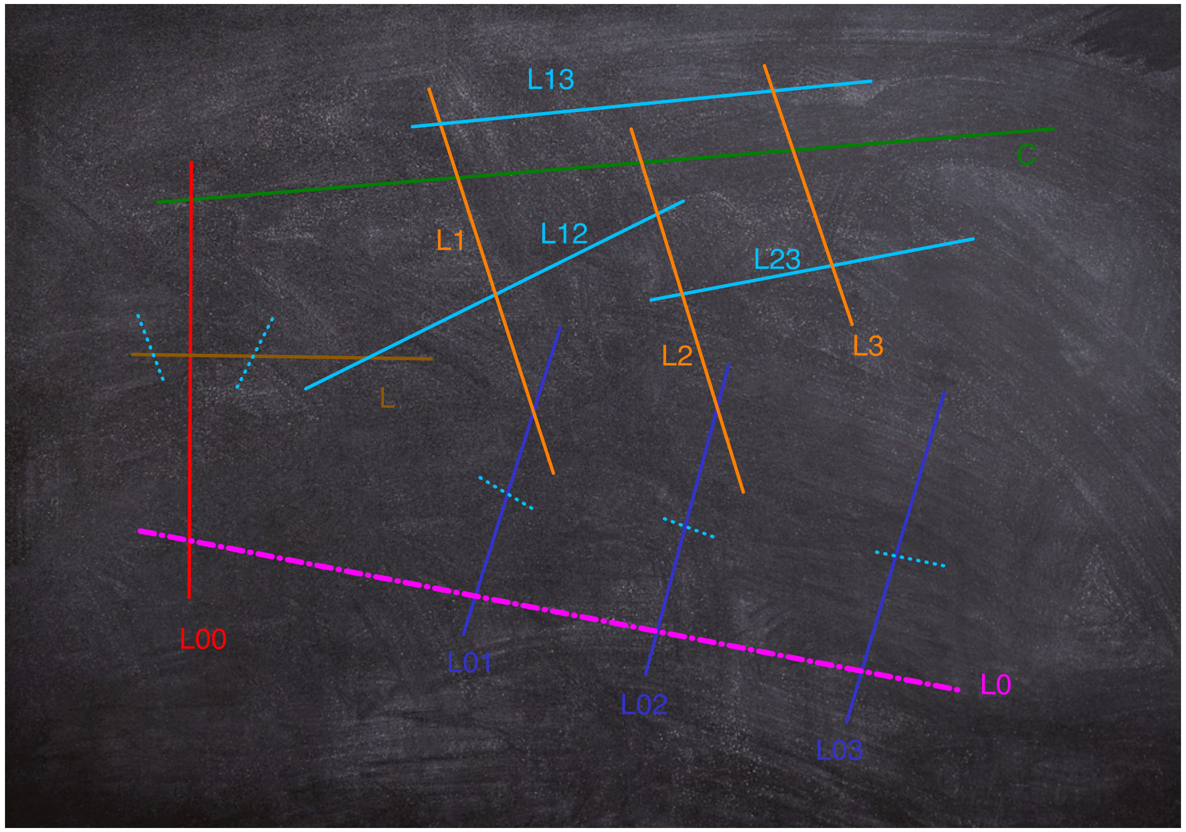


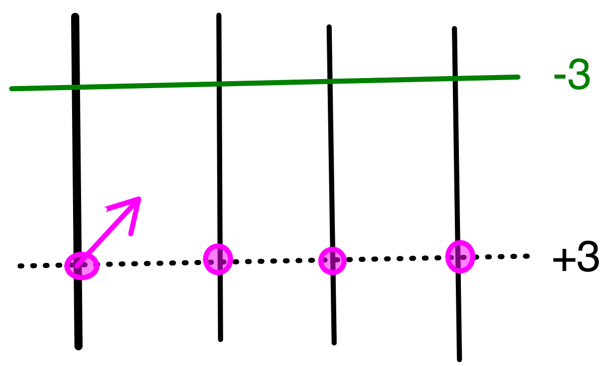
$\rightsquigarrow$ Quartic Del Pezzo surface

New special bundles in $\mathfrak{Bun}(\mathbb{P}^1, 2[0]+[t_1]+[t_2]+[t_3])$

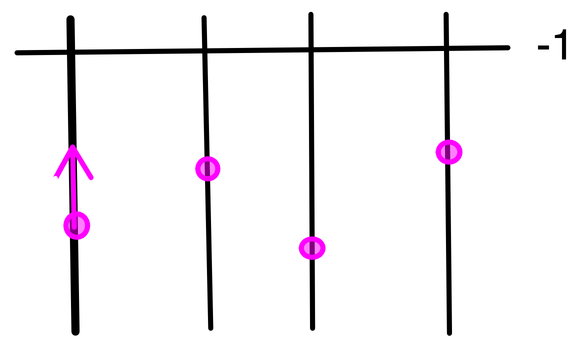


More special bundles in $\mathfrak{Bun}(\mathbb{P}^1, 2[0] + [t_1] + [t_2] + [t_3])$





blow-down of C



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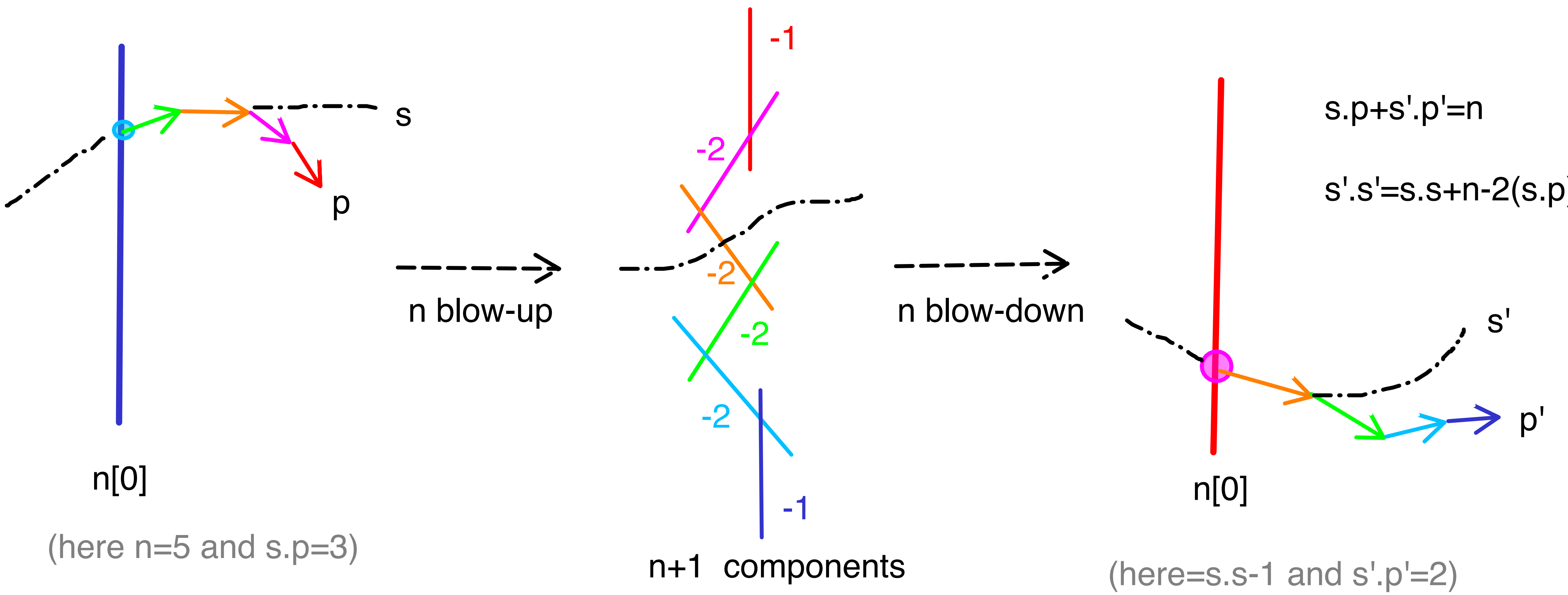
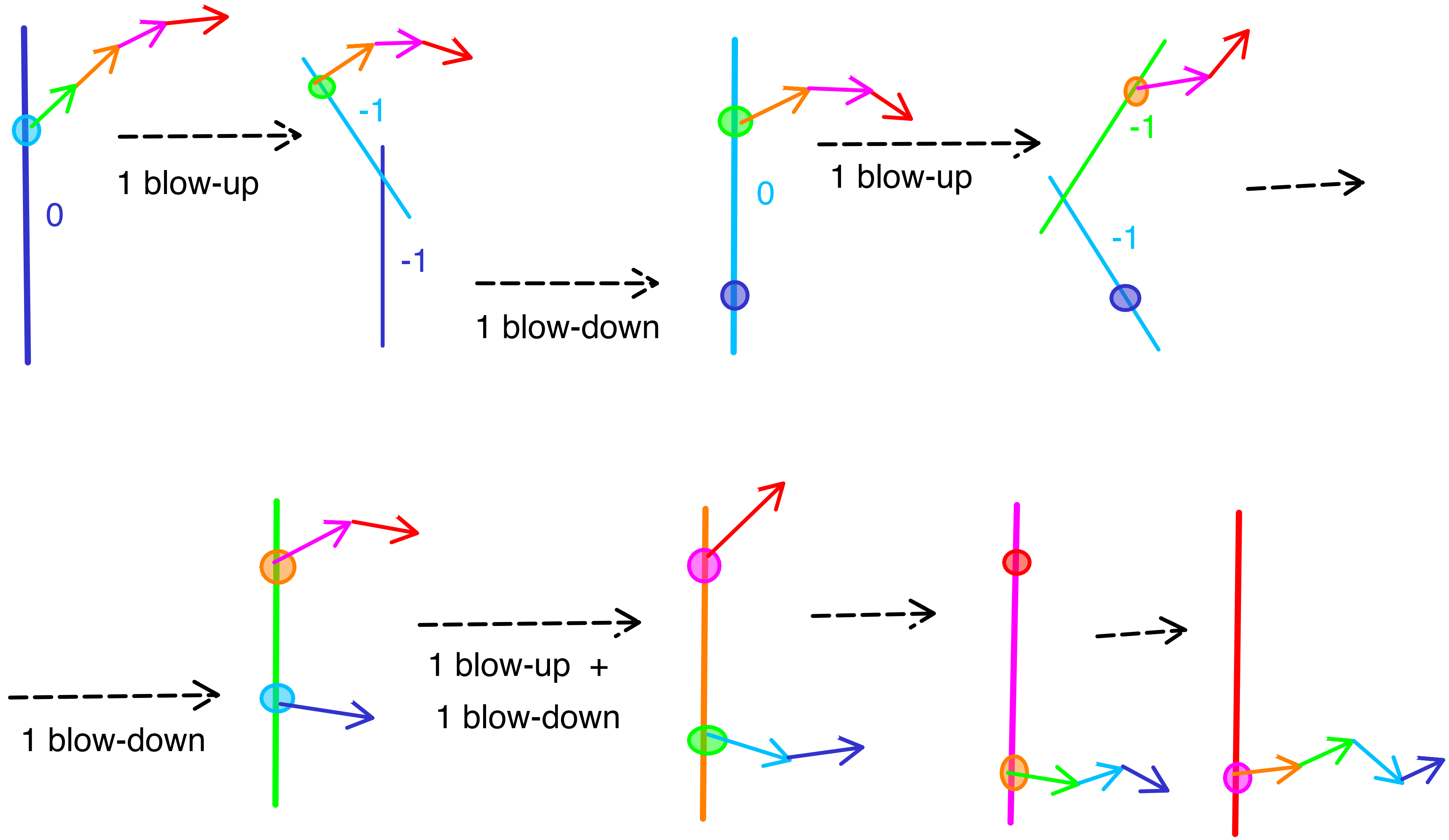
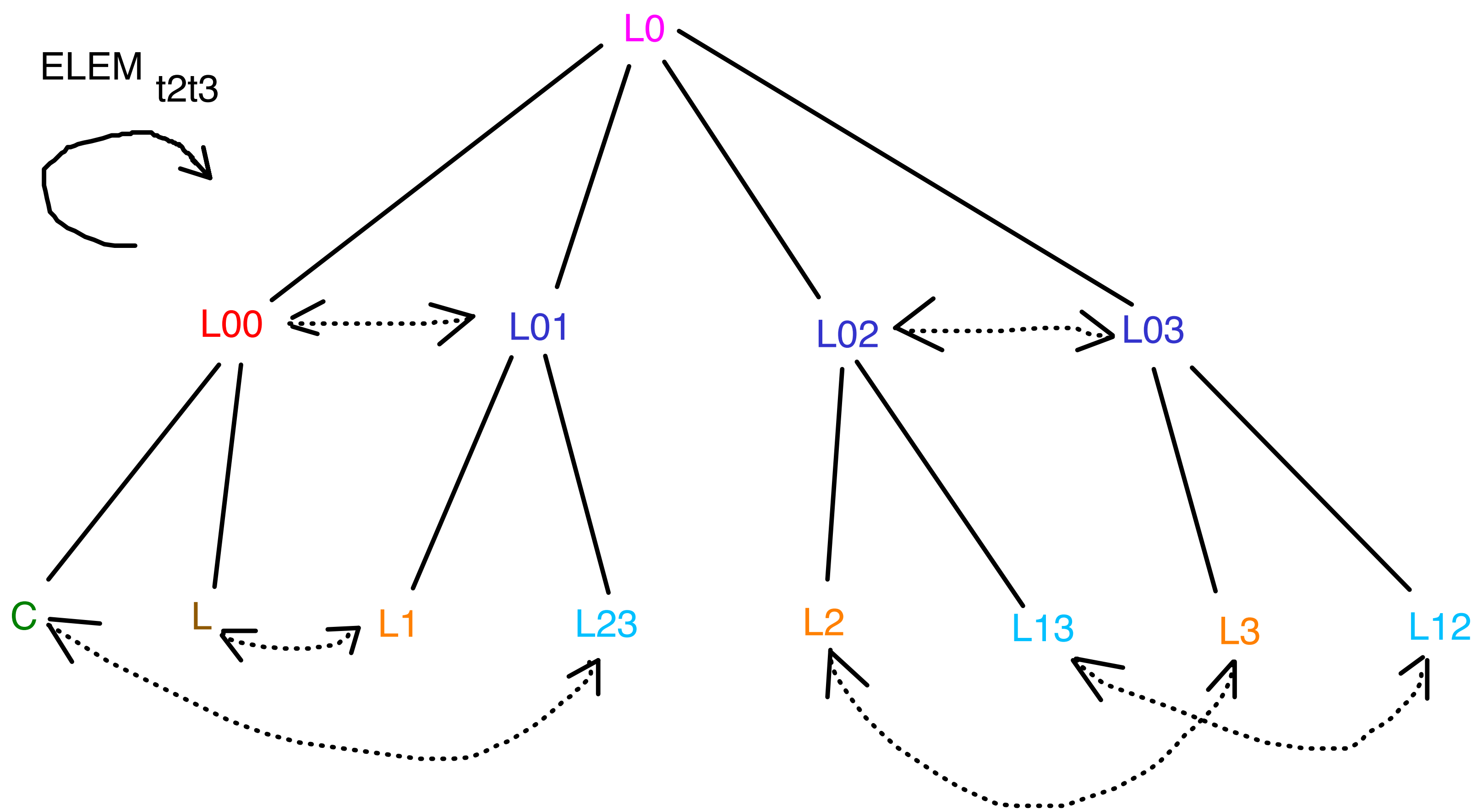
Irregular elementary transformations



Irregular elementary transformations as composition of classical ones

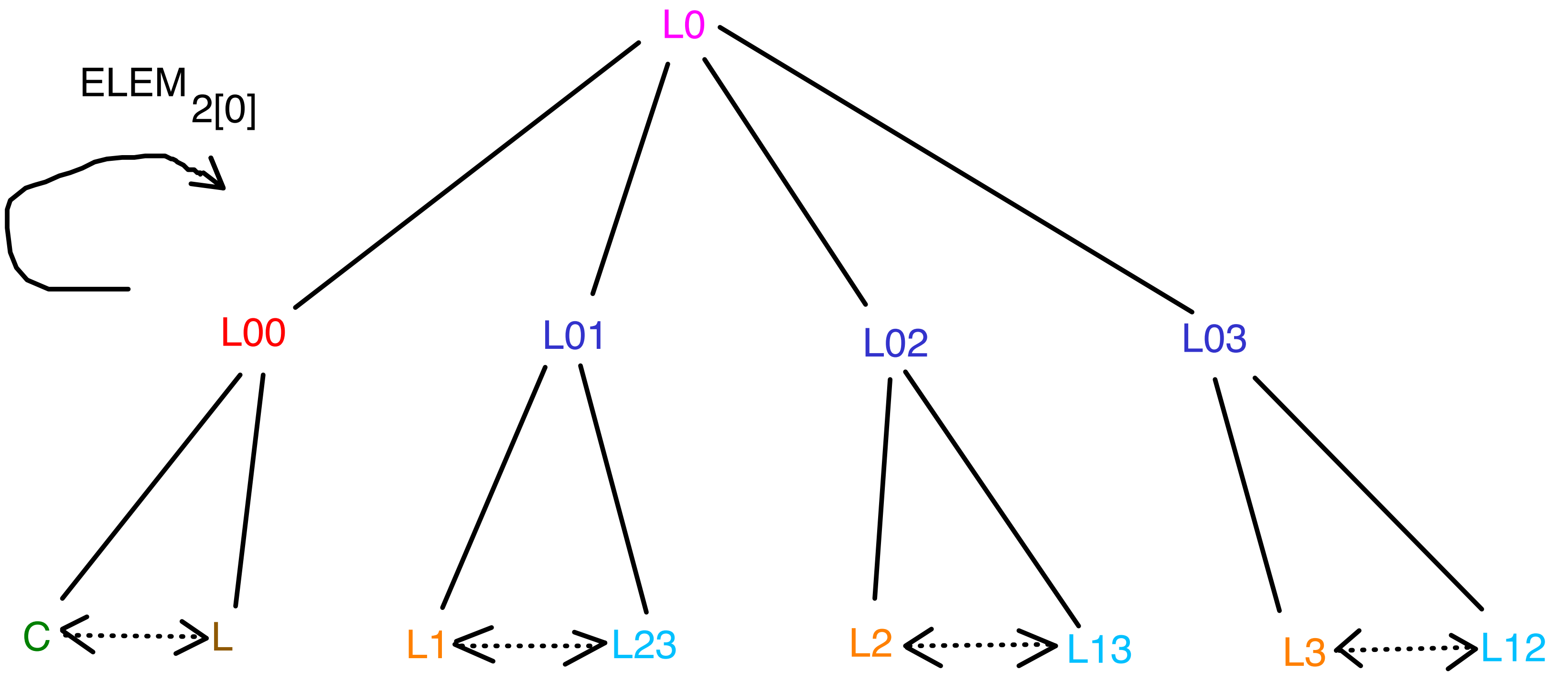


Action of elementary transformations on lines



Automorphism group: $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle \mathrm{elem}_{t_1+t_2}, \mathrm{elem}_{t_2+t_3}, \mathrm{elem}_{2[0]} \rangle$

Action of elementary transformations on lines



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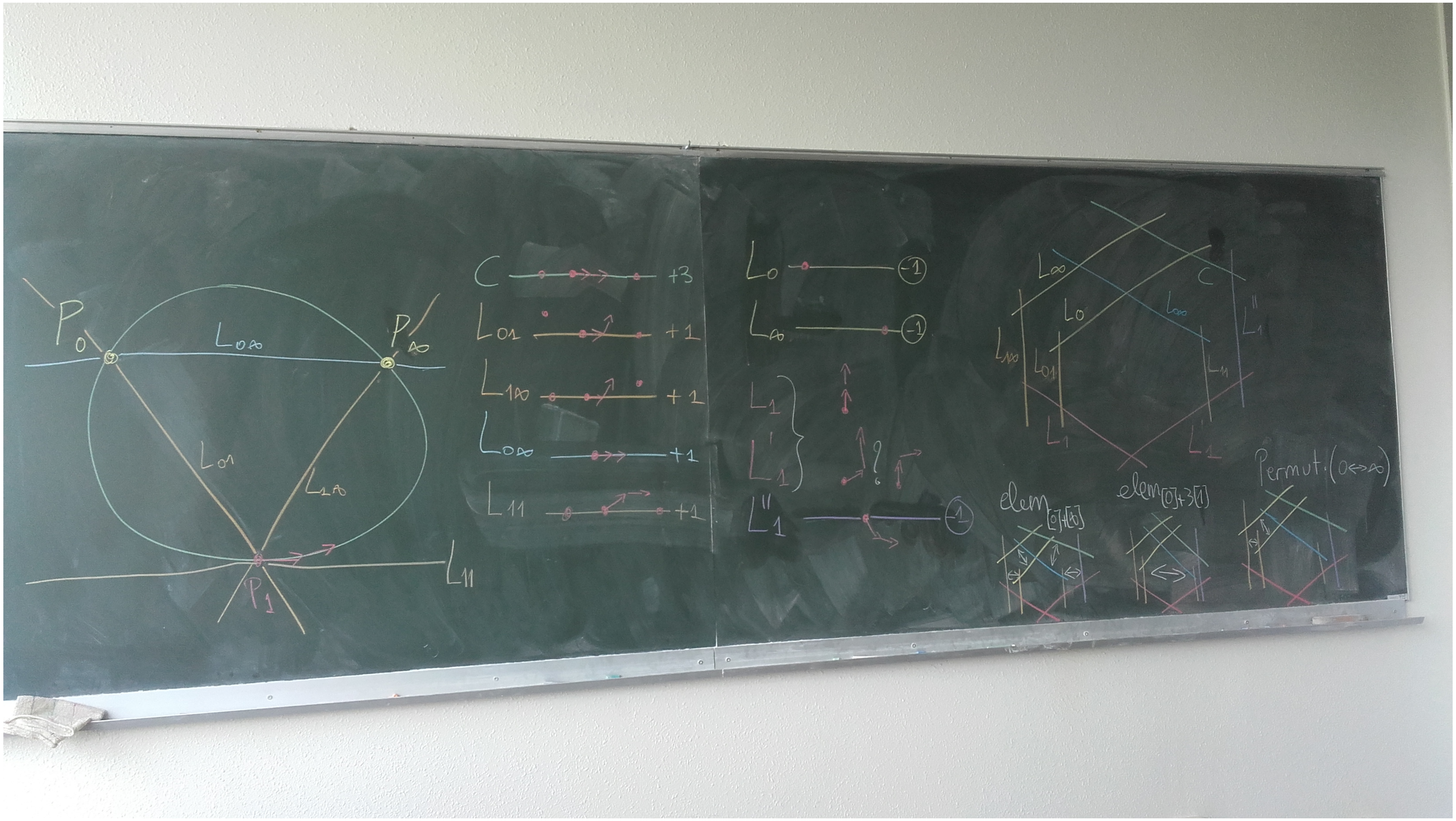

P_0
$L_{0\infty}$
P_∞
L_{01}
$L_{1\infty}$
P_1
L_{11}
C +3
L_{01} +1
$L_{1\infty}$ +1
$L_{0\infty}$ +1
L_{11} +1
L_0 (-1)
L_∞ (-1)
L_1
L'_1
L''_1 (-1)
Permut. (0⟺∞)

Obrigado !