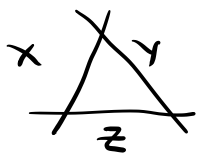
Degenerations of line bundles and divisors along families of curves

(joint w/ Amini, Rodriguez, Santana, Santos and Vital)

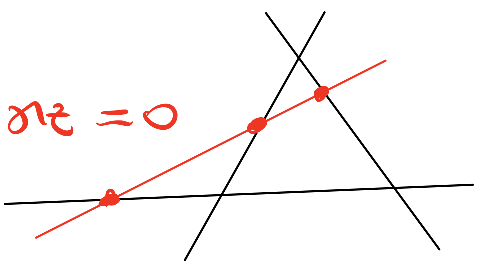
Example: $xyz + tF = 0$ $\quad$ F general cubic



Consider the system cut out by lines.

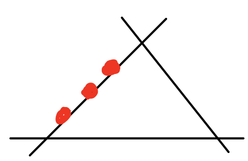
How do the divisors degenerate as $t \to 0$?

$\alpha X + \beta Y + \gamma Z = 0$



How about $X = 0$?

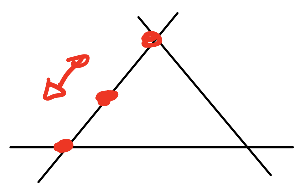
$$\begin{matrix} X = 0 \\ XYZ + tF = 0 \end{matrix} \iff \begin{matrix} X = 0 \\ F = 0 \end{matrix}$$



How does this fit with the others?

We may fix the special curve $XYZ=0$ and move the line:

$$\begin{matrix} X-\beta(t)tY-\gamma(t)tZ=0 \\ XYZ=0 \end{matrix} \quad \xrightarrow[t\to 0]{} \quad \begin{matrix} X=0 \\ (\beta(0)Y+\gamma(0)Z)YZ=0 \end{matrix}$$



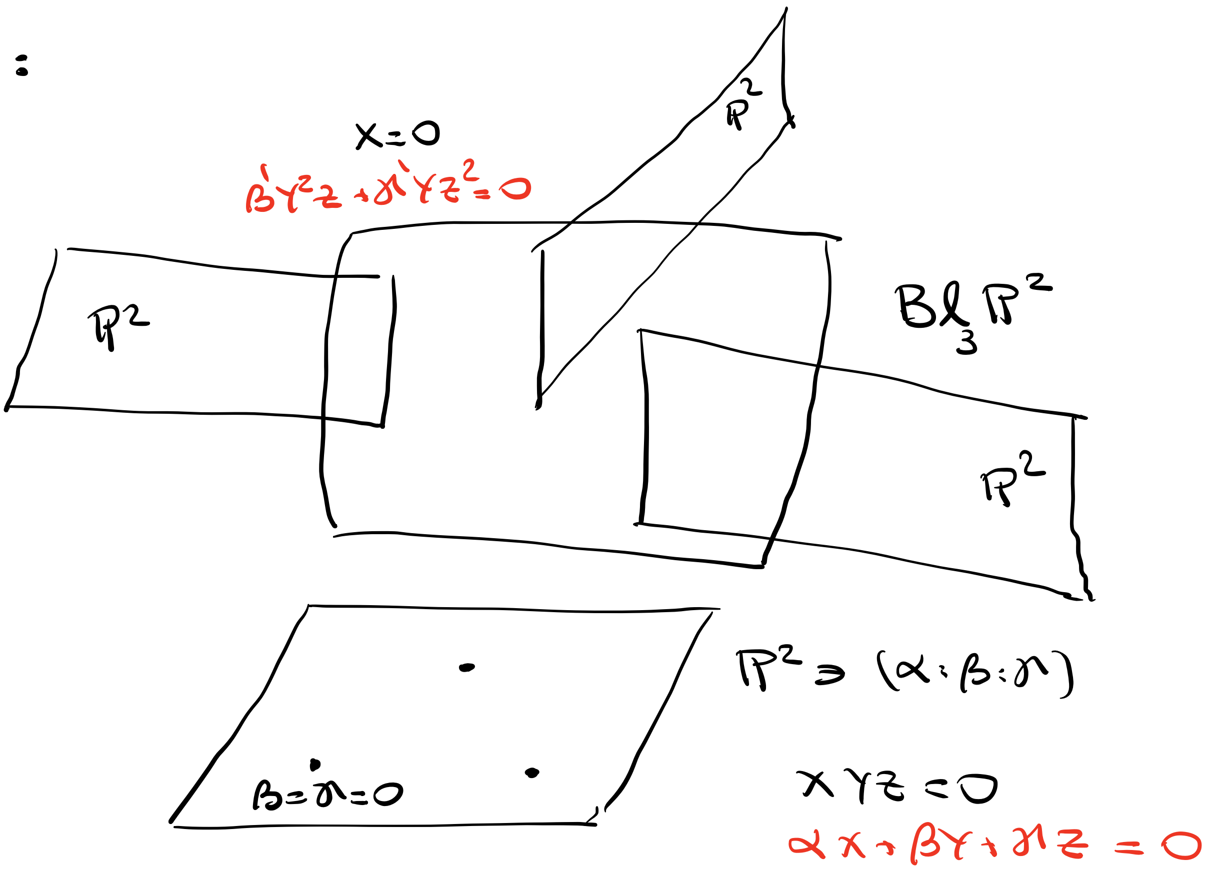
We fixed the line and let the curve vary, we fixed the curve and let the line vary.

Now:

$$\begin{matrix} X-\beta(t)tY-\gamma(t)tZ=0 \\ XYZ+tF=0 \end{matrix} \quad \xrightarrow[t\to 0]{} \quad \begin{matrix} X=0 \\ (\beta(0)Y+\gamma(0)Z)YZ+F=0 \end{matrix}$$

We obtain a net of divisors on $X=0$ cut out by the system of cubics $\langle Y^2Z, YZ^2, F\rangle$.

Picture:



Several questions?

1) Are these all limits of divisors?

2) What is this surface?

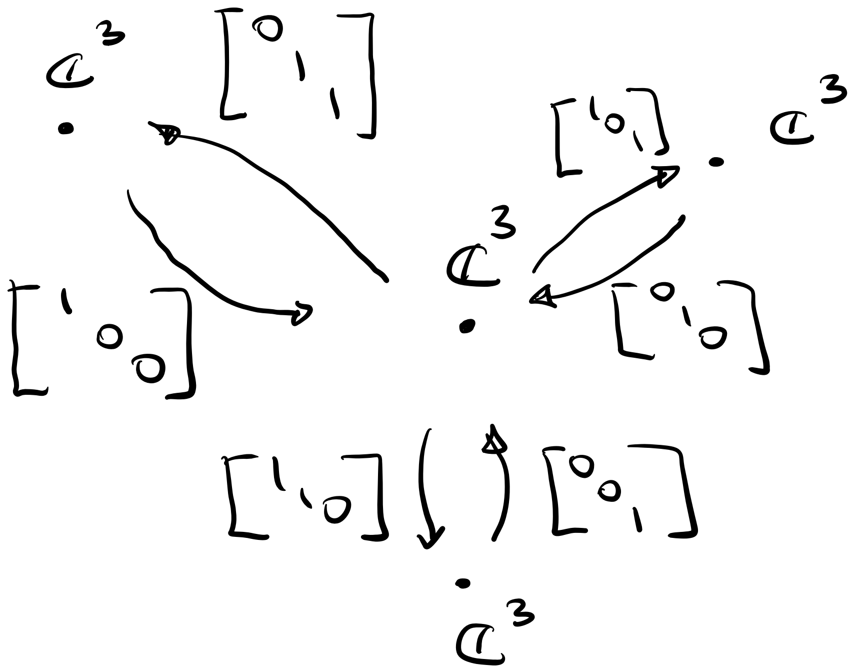
3) A $\mathbb{P}^n$ of divisors gives rise to a map to $\mathbb{P}^n$?
Do you see $XYZ=0$ on the surface above?

4) Let's rephrase:

A $\mathbb{P}^n$ of divisors gives rise to a map to $\check{\mathbb{P}}^n$?

What is the dual to the above surface??

5) Is there a general picture?



This is a quiver.

And we have a 3-dim'l representation g of it.

The associated quiver Grassmannian of 1-dim'l subspaces, $\mathbb{LP}(g)$, is the surface we have drawn!

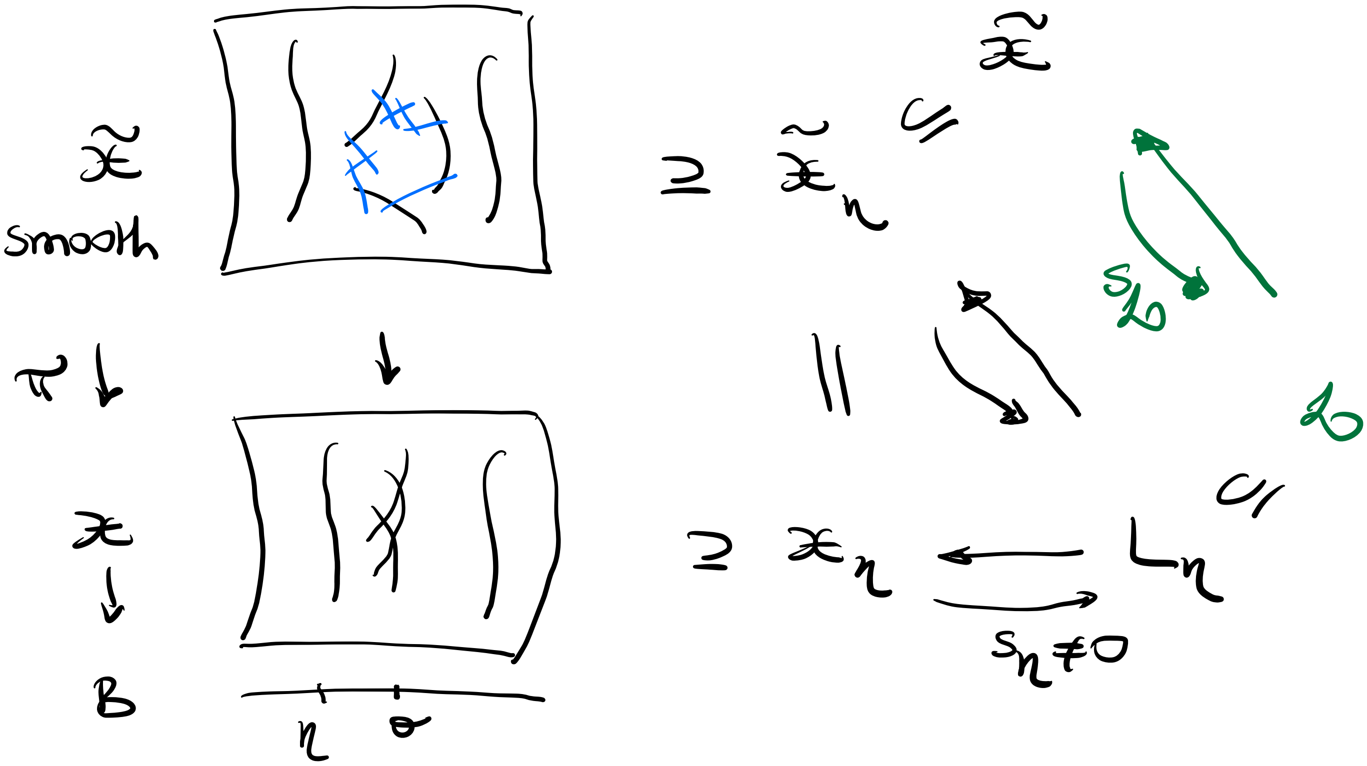
Its dual?

$\mathbb{LP}(g^*)$!

Is it isomorphic to $\mathbb{LP}(g)$?

We'll see ...

<u>General theory</u> !

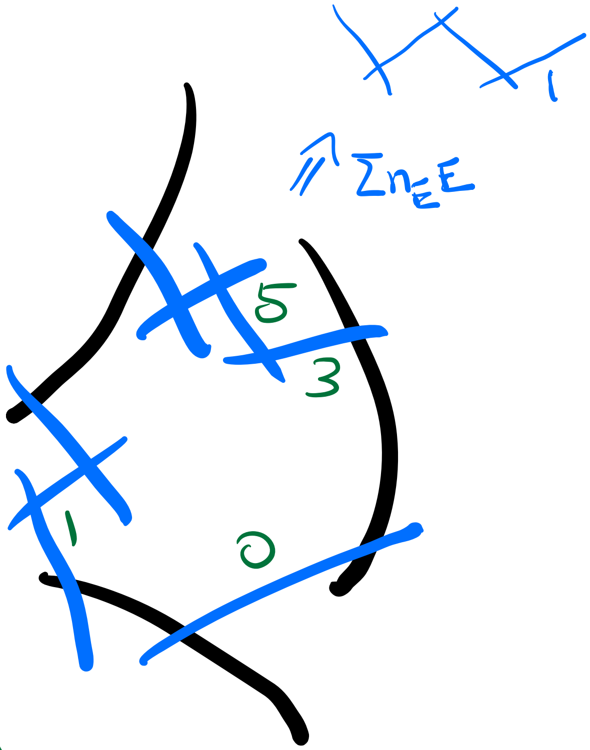


- There are infinitely many extensions $\mathcal{L}$ if $X := \mathfrak{X}(0)$ is reducible. An extension $\mathcal{L}$ is determined by:

$$\underline{\deg}\,(\mathcal{L}|_{\tilde{X}}) \qquad \text{where } \tilde{X} := \tilde{\mathfrak{X}}(0).$$

- Given s_η there is a unique extension $\mathcal{L}$ such that $Z(s_{\mathcal{L}}|_{\tilde{X}})$ is finite : it's the limit <u>on $\tilde{X}$</u> of $Z(s_\eta)$.

$\underline{\deg}(\mathcal{L}|_{\tilde{X}})$:



$$\begin{tikzcd}
\mathcal{O}_{\tilde{X}} \arrow[r, "s_{\mathcal{L}}"] \arrow[rr, bend left, "s_{\mathcal{L}'}"] & \mathcal{L} \arrow[r, "\sum n_E E"] & \mathcal{L}'
\end{tikzcd}$$

where $\underline{\deg}(\mathcal{L}'|_{\tilde{X}})$ is admissible.

$$\begin{tikzcd}
\Rightarrow \quad \mathcal{O}_{\mathcal{X}} = \pi_* \mathcal{O}_{\tilde{\mathcal{X}}} \arrow[r, "\pi_* s_{\mathcal{L}'}"] \arrow[d] & \pi_* \mathcal{L}' \arrow[d] \\
\mathcal{O}_X \arrow[r, hook, "s_I"] & I := \pi_* \mathcal{L}'|_X
\end{tikzcd}$$

flat family of torsion-free, rank-1 sheaves

Sections of torsion-free rank-1 sheaves correspond to certain subschemes of X called <u>pseudo-divisors</u> (Hartshorne)

$$\mathcal{O}_X \hookrightarrow I \quad \overset{\text{dual}}{\Longrightarrow} \quad I^* \hookrightarrow \mathcal{O}_X$$ sheaf of ideals of $Z(s_I)$.

The structure of the I: (Amini, –)

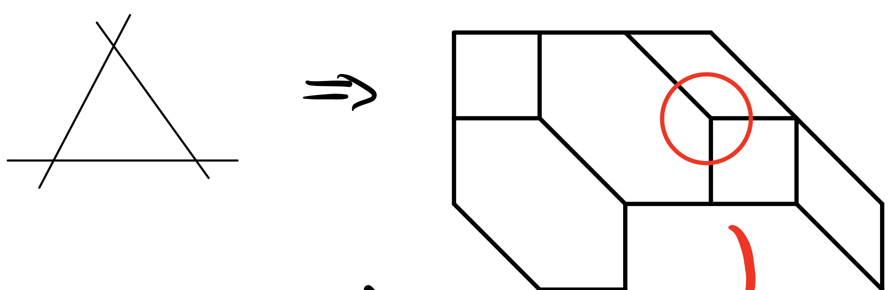
$$V := \{\text{irreducible components of } X\}$$

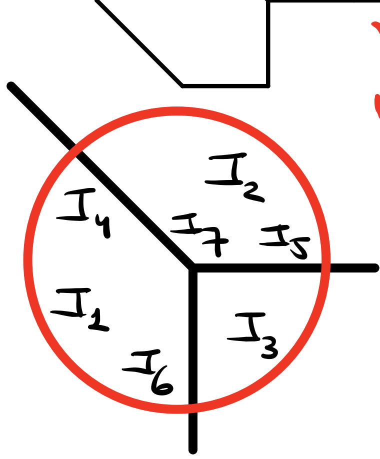
$$\begin{array}{ccccc} H_{d,V} & \longrightarrow & \mathbb{R}^V & \xrightarrow{\deg} & \mathbb{R} \\ \shortparallel & & f & \longmapsto & \sum_{v \in V} f(v) \\ \deg^{-1}(d) & & & & \end{array}$$

$d = \deg L_\eta$.

There is a decomposition of $H_{d,V}$ in polytopes each corresponding to one I.

Example:





The sheaves I_1, I_2, I_3 don't decompose as sums (they are simple!) but I_4, I_5, I_6, I_7 do: they are degenerations of I_1, I_2, I_3.

Simple sheaves are easier to parameterize!

But we could have that I is not simple!

Then the polytope of I is in the boundary of a certain number of maximal dimensional polytopes, corresponding to certain simple sheaves $I_2, \dots, I_m$.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "s_I"] \arrow[rr, bend right, "s_j"'] & I \arrow[r] & I_j \end{tikzcd}$$

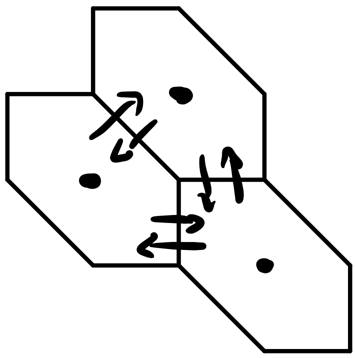
$$I_2^* \oplus \cdots \oplus I_m^* \xrightarrow{\varphi} I^* \hookrightarrow \mathcal{O}_X$$

<u>Claim:</u> φ is surjective!

(at least if $\# V = 2$ (Santana) or $\# V = 3$ (—))

<u>Cor:</u> $Z(s_I) = \bigcap_{j=2}^{m} Z(s_j)$

There is a quiver Q associated to the maximal dimensional polytopes



A system of sections $V_n \subseteq H^0(\mathfrak{X}(n), L_n)$ degenerates to a quiver representation g.

And a section to a subrepresentation $h \subset g$ of dimension 1.

The quiver Q and representation g are special.

Teo: (—, Santos, Vital): Assume $\mathfrak{X}$ smooth, $\# V = 3$

Given g and $h \leq g$:

a) h is generated by a triangle

b) $\mathbb{LP}(g)$ is Cohen-Macaulay

Maps

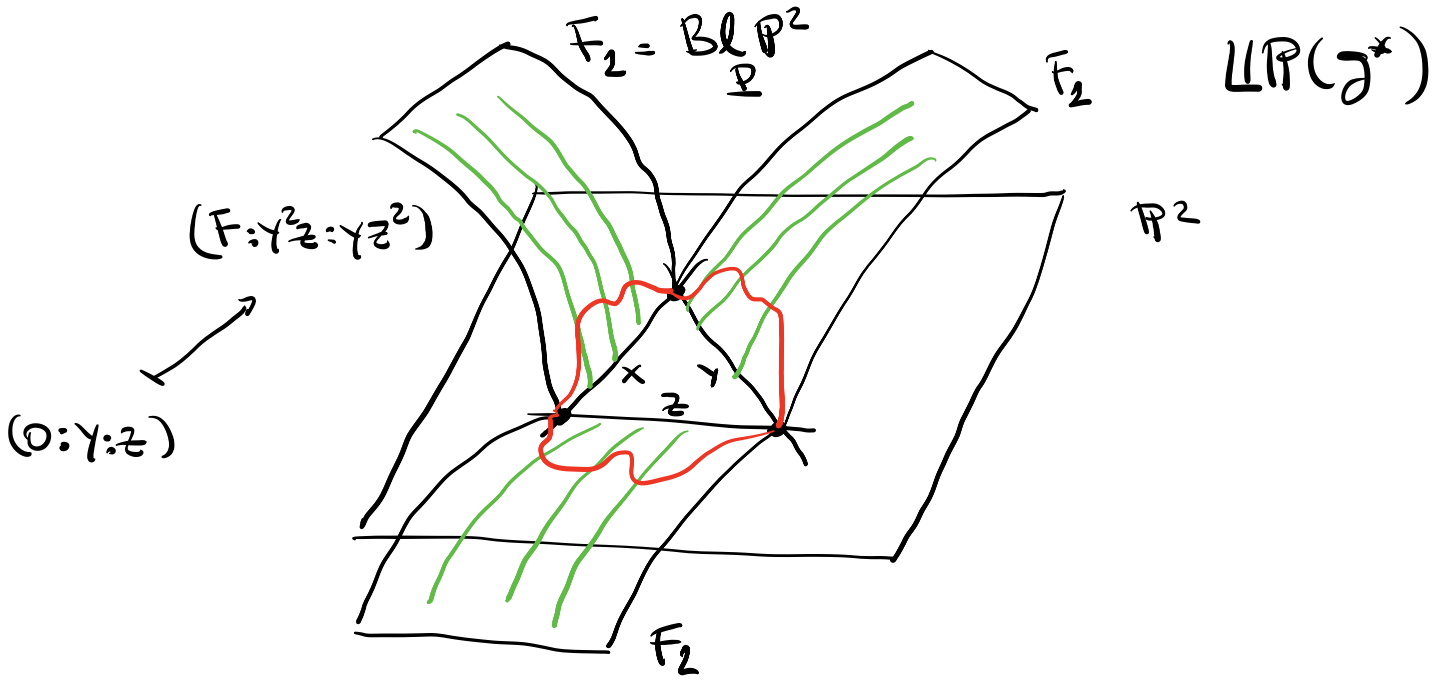
<u>Teo</u>: (–, Rodriguez). Assume $\mathfrak{X}$ smooth, $\# V = 2$.

Given g:

a) $\amalg \mathbb{P}(g^*)$ is Cohen-Macaulay

b) $X \dashrightarrow \amalg \mathbb{P}(g^*)$

<u>Example</u>:



In the example you don't see the limits of flexes just by looking at the triangle embedded in $\mathbb{P}^2$,

But you see them with the map to $\sqcup \mathbb{P}(\mathfrak{g}^*)$!

(Medeiros, —): The limits of the flexes are the three inflection points on each component of the triangle of the map to the corresponding F_1.