The monodromy of meromorphic projective structures

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August 27, 2021

CAPES-COFECUB workshop

S an infinitely differentiable compact real oriented surface.

Definition

A **(complex) projective structure** $\mathfrak{P}$ **on** S is

a maximal atlas $(U_i, \varphi_i : U_i \to V_i)$ on S, with $V_i \subset \mathbb{CP}^1$

such that
$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} =$ a restriction of a $g \in \mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}(2, \mathbb{C})$.

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➥ The atlas induces a complex structure on S, denoted C.

Let $\mathfrak{P}$ be a projective structure. Then, $\varphi_{ik} \circ \varphi_{kj} = \varphi_{ij}$.

We define cocycles

$$\begin{aligned} g_{ij} : U_i \cap U_j &\longrightarrow \mathrm{PGL}(2, \mathbb{C}) \\ x &\longmapsto g_{ij}(x) \text{ whose restriction equals } \varphi_{ij} \end{aligned}$$

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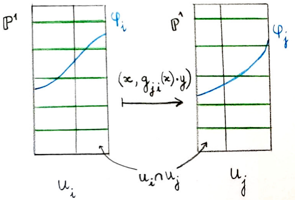

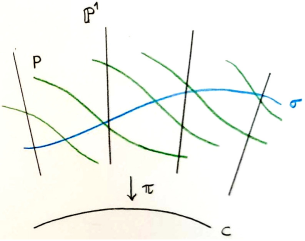

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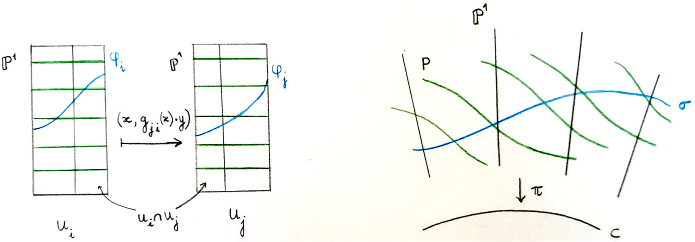

This defines **a triple** $(\pi : P \to C, \mathcal{F}, \sigma)$, where

- $\pi : P \to C$ is a holomorphic $\mathbb{P}^1$-bundle,
- $\mathcal{F}$ is a Riccati foliation on P,
- $\sigma : C \to P$ is a holomorphic section of π, transverse to $\mathcal{F}$.

And vice versa.

Definition: The monodromy map

$$\begin{array}{ccc} \mathcal{P}_S & \xrightarrow{\mathrm{Mon}_S} & \mathcal{R}_S := {}^{\textstyle \mathrm{Hom}(\pi_1(S), \mathrm{PGL}(2,\mathbb{C}))}\big/_{\textstyle \mathrm{PGL}(2,\mathbb{C})} \\ \downarrow & & \\ \mathcal{T}_S & & \end{array}$$

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$\mathcal{P}_S$: the set of isomorphism classes of **marked** projective structures on S.

$\mathfrak{P}_1 \sim \mathfrak{P}_2 \Leftrightarrow \exists \Phi : S \to S$ a $\mathcal{C}^\infty$-diffeomorphism isotopic to id_S such that $\forall \varphi_1 \; \exists g \in \mathsf{PGL}(2,\mathbb{C})$ such that $g \circ \varphi_2 = \varphi_1 \circ \Phi$

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Complex structures on the domain and range of Mon_S**?**

Fact

Fibers of $\mathcal{P}_S \to \mathcal{T}_S$ are affine spaces for the vector space $H^0(C, T^*C^{\otimes 2})$ of **quadratic differentials**.

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z_1, z_2 corresponding projective coordinates, $\psi := z_2 \circ z_1^{-1}$

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($\leftarrow$) $\mathfrak{P}_1$ projective structure on a fixed C
$\phi_{z_1} := \frac{q(z_1)}{2} dz_1^{\otimes 2}$

$\mathfrak{P}_2 := \mathfrak{P}_1 + \phi$: charts are the solutions of $\mathcal{S}_{z_1}(\varphi) = q(z_1)$

$\Leftrightarrow$ charts are the quotients of independent solutions of $y'' + \dfrac{q(z_1)}{2} y = 0$.

Case where S of genus $g \geq 2$

- $\mathcal{P}_C \overset{\text{Fact}}{=} H^0(C, T^*C^{\otimes 2}) \simeq \mathbb{C}^{3g-3}$ (Gunning 1967);
- $\dim_{\mathbb{C}} \mathcal{T}_S = 3g - 3$.

$\Rightarrow$ A complex structure of dimension $6g-6$ is brought on $\mathcal{P}_S$ via its identification with $\mathcal{Q}_S$, the holomorphic cotangent bundle of the Teichmüller space of S, with fiber space $H^0(C, K_C^{\otimes 2})$ (Hubbard 1981).

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If $g \geq 2$, $\mathrm{Mon}_S : \mathcal{P}_S \to \mathcal{R}_S^{\text{irr}}$ is a **local biholomorphism**.

Our aim is to generalise this result for projective structures with poles.

C a complex curve.

Definition

A **meromorphic projective structure on** C is a projective structure $\mathfrak{P}^*$ on the complement $C^* = C \setminus \Sigma$ of a finite subset $\Sigma \subset C$, such that given a holomorphic projective structure $\mathfrak{P}_0$ on C, the quadratic differential $\phi = \mathfrak{P}^* - \mathfrak{P}_{0|C^*}$ on C^* extends to a meromorphic quadratic differential on C.

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➥ Pole orders are well defined (does not depend on $\mathfrak{P}_0$).

Example

Every **hypergeometric equation**

$$z(z-1)y'' + [(\alpha+\beta+1)z-\gamma]y' + \alpha\beta y = 0, \text{ with } \alpha,\beta,\gamma \in \mathbb{C}$$

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Projective charts take the form

$$\varphi(x) = f(x)^\theta \text{ or } \varphi(x) = f(x)^\theta + \log(f(x))$$

with $\theta \in \mathbb{C}$ and f a local coordinate around a singularity x_0, $f(x_0) = 0$.

Theorem (Hejhal 1975, Earle, Hubbard 1981)
If $g \geq 2$, $\mathrm{Mon}_S : \mathcal{P}_S \to \mathcal{R}_S^{\mathrm{irr}}$ is a **local biholomorphism**.

Idea of the proof (local injectivity, case without poles)

$\mathcal{M}_S = \{(\pi : P \to C, \mathcal{F})\}$ moduli space of holomorphic connections.
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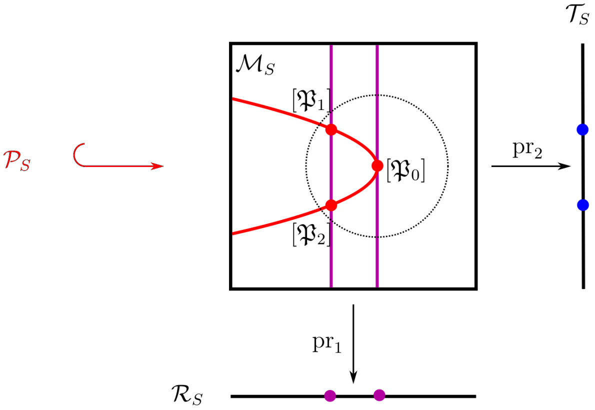

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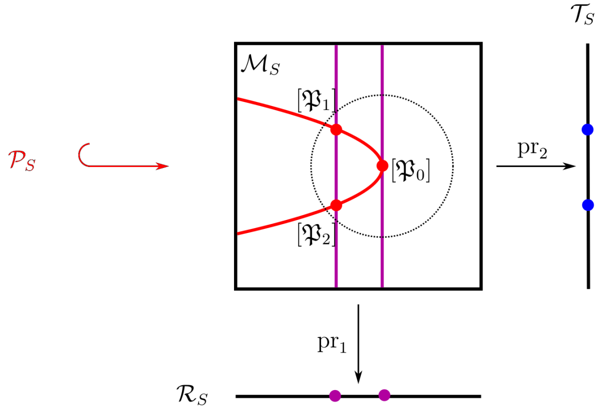
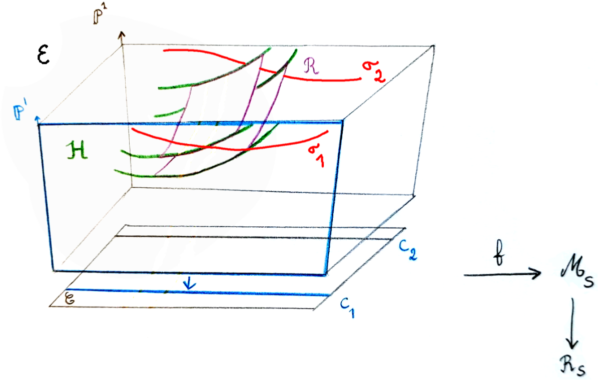
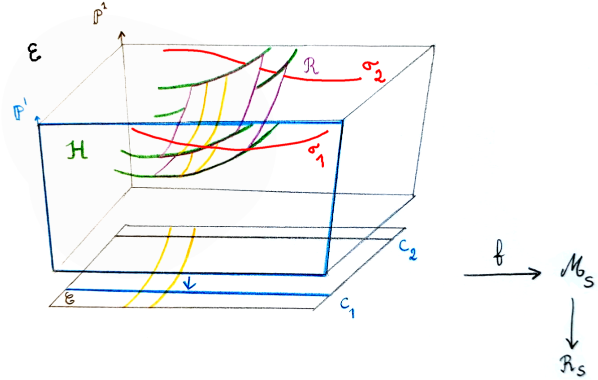


Local injectivity of Mon_S $\Leftrightarrow$ **transversality** of $\mathcal{P}_S$ with respect to the fibers of the projection pr_1 on $\mathcal{R}_S$.



Along isomonodromic deformations, there exists **a codimension one foliation** $\mathcal{R}$ on $\mathcal{E}$ (Heu, 2010).

This allows to lift the $\mathcal{C}^\infty$ trivialization of the family $\mathcal{C}$ of curves.



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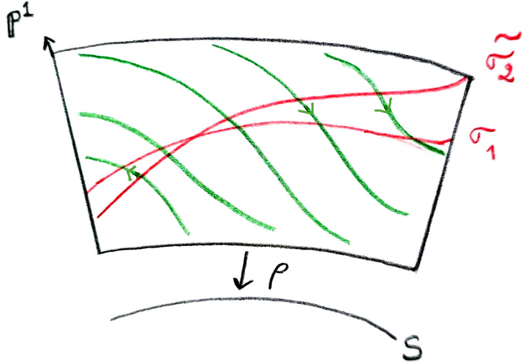

"$\Phi = \rho \circ h_{\text{tub}} \circ \tilde{\sigma_2}$" (modulo the trivialization above).

Proof in the meromorphic case

- Spaces corresponding to $\mathcal{P}_S$ and $\mathcal{R}_S$ where defined by (Allegretti/Bridgeland 2020);
- The space $\mathcal{M}_S^{\text{mero}}$ of meromorphic connexions exists and is a complex manifold (Inaba 2016, 2021);
- The foliation $\mathcal{R}$ still exists (Heu 2010).

S an oriented smooth compact real surface of genus g.

$\Sigma = \{p_i\}_{1 \leq i \leq n}$ a finite subset of S. $\{n_i\}_{1 \leq i \leq n}$ a collection of integers.

Statement (work with Frank)

If $N = 3g - 3 + \sum_{i=1}^{n} \lceil \frac{n_i}{2} \rceil > 0$, then the **generalised monodromy map** of (Allegretti/Bridgeland 2020)

$$\text{Mon} : \mathcal{P}(S, \{n_i\}) \longrightarrow \mathcal{R}(S, \{n_i\})$$

is **a local biholomorphism**.

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The statement is already known when

- all poles have order ≤ 2 with loxodromic local monodromy (Luo 1993);
- all poles have order ≤ 2 with parabolic local monodromy and some specific residues (Hussenot Desenonges 2019);
- all poles have order ≥ 3 (Gupta/Mj 2020).

Thank you for your attention!