Numerical characterization of some toric fiber bundle

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June 25, 2021

Joint with F. Lo Bianco

(X, D) log smooth complex analytic space ($n := \dim X$)

$T_X(-\log D) \subseteq T_X = \mathrm{Der}_{\mathbb{C}}(\mathscr{O}_X)$ (*logarithmic tangent sheaf*) subsheaf consisting of those derivations that preserve the ideal sheaf $\mathscr{O}_X(-D) \rightsquigarrow T_X(-\log D)$ is a locally free sheaf of Lie subalgebras of T_X

If D is defined at x by the equation $x_1 \cdots x_k = 0$, then a local basis of $T_X(-\log D)$

$$x_1 \frac{\partial}{\partial x_1}, \ldots, x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}$$

$T_X(-\log D)$ can be identified with the subsheaf of T_X containing those vector fields that are tangent to D at smooth points of D

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Suppose X compact Kähler (e.g. X projective) and $T_X(-\log D) \cong \mathscr{O}_X^{\oplus n}$

Example 1: $(T, 0)$ where T is a complex torus

Example 2: pairs (X, D) where X is a smooth toric variety with boundary divisor D
$X \setminus D \cong (\mathbb{C}^*)^n$ and $(\mathbb{C}^*)^n$ acts on X
$\rightsquigarrow$ $\mathrm{Lie}((\mathbb{C}^*)^n) \subseteq H^0(X, T_X(-\log D))$ and $\mathrm{Lie}((\mathbb{C}^*)^n) \otimes \mathscr{O}_X \cong T_X(-\log D)$

In general (Wang 1954 if $D = 0$, Winkelmann 2004), $G := \mathrm{Aut}(X, D)^0$ is a connected complex Lie group of dimension n which is an extension

$$1 \to (\mathbb{C}^*)^d \to G \to T \to 1$$

where T is a complex torus

The Albanese map $X \to \mathrm{Alb}(X)$ is a smooth locally trivial fibration with typical fiber F being a toric variety with boundary divisor $D_{|F}$.

$\rightsquigarrow$ toric fiber bundles over abelian varieties

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From the point of view of birational classification of complex (projective algebraic) manifolds, it is more natural to consider the case where the logarithmic tangent bundle $T_X(-\log D)$ is *numerically flat*.

Definition

A vector bundle $\mathscr{E}$ of positive rank is called numerically flat if $\mathscr{E}$ and $\mathscr{E}^*$ are nef.

If $D = 0$, then X is covered by a complex torus, as a classical consequence of Yau's theorem on the existence of a Kähler-Einstein metric.

Definition

A vector bundle $\mathscr{E}$ of positive rank r is called R-flat if $\nu^*\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^1}^{\oplus r}$ for all $\nu\colon \mathbb{P}^1 \to X$.

A numerically flat vector bundle is R-flat. The converse holds true if X is rationally chain connected.

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Theorem

Let (X, D) be a log smooth reduced pair with X projective. Suppose that $K_X + D \equiv 0$ and that $T_X(-\log D)$ is R-flat.

There exist a smooth morphism with connected fibers $a\colon X \to T$ onto a smooth projective variety T such that:

1. $K_T \equiv 0$;
2. *The fibration $(X, D) \to T$ is locally trivial for the analytic topology and any fiber F of the map a is a smooth toric variety with boundary divisor $D_{|F}$;*
3. *T contains no rational curve.*

Conversely, let $(X, D) \to T$ is a toric fiber bundle over a projective manifold T with $K_T \equiv 0$. Suppose in addition that T contains no rational curve. Then $T_X(-\log D)$ is obviously R-flat. Moreover, $K_X + D \equiv 0$ ($\exists$ finite étale quasi-sections).

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Remark

In the setup of the theorem, we expect T to be a torus quotient $\rightsquigarrow$ *numerical characterization of toric fiber bundles over torus quotients*

Indeed, by the Beauville-Bogomolov decomposition theorem, T admits a finite étale cover that decomposes into the product of an abelian variety and a simply-connected Calabi-Yau manifold. On the other hand, a folklore conjecture asserts that any projective Calabi-Yau manifold contains a rational curve.

Corollary

Let (X, D) be a log smooth reduced pair with X projective. Suppose that $T_X(-\log D)$ is numerically flat.

Then there is a smooth morphism $a\colon X \to T$ with connected fibers onto a torus quotient T. The fibration $(X, D) \to T$ is locally trivial for the analytic topology and any fiber F of the map a is a smooth toric variety with boundary divisor $D_{|F}$.

$T_X(-\log D)$ numerically flat $\Rightarrow$ $T_X(-\log D)$ R-flat and $K_X + D \equiv 0$

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Definition

A vector bundle $\mathscr{E}$ of positive rank r is called R-flat if $\nu^*\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^1}^{\oplus r}$ for all $\nu\colon \mathbb{P}^1 \to X$.

Lemma

Let $f\colon X \to Y$ be a morphism of smooth projective varieties with rationally chain connected general fibers, and let $\mathscr{E}$ be a locally free sheaf on Y. Then $f^*\mathscr{E}$ is R-flat if and only if so is $\mathscr{E}$.

Remark

If $f\colon X \to \mathbb{P}^1$ contains no rational curve, then $f^*\mathscr{E}$ is R-flat for any $\mathscr{E}$.
E.g.: C hyperelliptic curve with hyperelliptic involution i, A abelian variety and $X := C \times A / < i > \to C / < i > \cong \mathbb{P}^1$ where i acts on A by translation by a 2-torsion point.

Lemma (Biswas - dos Santos 2009 – D. - Lo Bianco 2021)

Let X be a projective reduced space, not necessarily irreducible. Suppose X is rationally chain connected. Then any locally free, R-flat sheaf on X is numerically flat.

R-flat vector bundles

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Let X be a projective reduced space, not necessarily irreducible. Suppose X is rationally chain connected. Then any locally free, R-flat sheaf on X is numerically flat.

Let $f\colon X \to Y$ be a Mori extremal contraction with X projective klt. By the cone theorem, a line bundle on X of degree zero on every contracted rational curve comes from Y.

Theorem (Greb-Kebekus-Peternell-Taji 2019 – D. - Lo Bianco 2021)

Let $f\colon X \to Y$ be a projective morphism with connected fibers of normal, quasi-projective varieties. Suppose that there is an effective $\mathbb{Q}$-divisor D on X such that the pair (X, D) is klt and $-(K_X + D)$ is f-ample.

Let $\mathscr{E}$ be a locally free, f-relatively R-flat sheaf on X.

Then there exists a locally free sheaf $\mathscr{G}$ on Y such that $\mathscr{E} \cong f^*\mathscr{G}$.

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A canonical bundle formula

Let $f\colon X \to Y$ be a surjective morphism with connected fibers of normal, projective varieties. Let D be a $\mathbb{Q}$-divisor on X such that (X, D) is log canonical over the generic point of Y.

Definition

The *discriminant divisor* of (f, D) is the $\mathbb{Q}$-divisor $B = \sum_P b_P P$ on Y, where P runs through all prime divisor on Y, and

$$1 - b_P := \sup\{t \in \mathbb{R} \,|\, (X, D + tf^*P) \text{ is log canonical over the generic point of } P\}.$$

The discriminant divisor measures the singularities of special fibers.

Theorem

Suppose that D is integral and that it is effective in a neighbourhood of a general fiber of f. Suppose in addition that there exists a Cartier divisor C on Y such that

$$K_X + D \sim_{\mathbb{Q}} f^*C.$$

If (f, D) is generically isotrivial, then $C \sim_{\mathbb{Q}} K_Y + B$, where B denotes the discriminant divisor of (f, D).

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Recall that Winkelmann's theorem holds for compact Kähler manifold.

Question

Is the theorem true in the compact Kähler setting?

Recall that T is expected to be an abelian variety. In particular, if $\dim T > 0$, then $\pi_1(X)$ should be infinite.

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Is X a smooth toric variety with boundary divisor D?

$T_X(-\log D)$ R-flat $\Rightarrow K_X + D$ nef (Cone theorem for lc pairs)

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Can we say something if $T_X(-\log D)$ R-flat and $\nu(K_X + D) \geqslant 1$?

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