Weinstein meets Kupka

Poisson Structures

Definition [edit]

There are two main points of view to define Poisson structures: it is customary and convenient to switch between them, and we shall do so below.

As bracket [edit]

Let M be a smooth manifold and let $C^\infty(M)$ denote the real algebra of smooth real-valued functions on M, where the multiplication is defined pointwise. A **Poisson bracket** (or **Poisson structure**) on M is an $\mathbb{R}$-bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

defining a structure of Poisson algebra on $C^\infty(M)$, i.e. satisfying the following three conditions:

- Skew symmetry: $\{f,g\} = -\{g,f\}$.
- Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.
- Leibniz's Rule: $\{fg,h\} = f\{g,h\} + g\{f,h\}$.

The first two conditions ensure that $\{\cdot,\cdot\}$ defines a Lie-algebra structure on $C^\infty(M)$, while the third guarantees that, for each $f \in C^\infty(M)$, the linear map $X_f := \{f,\cdot\} : C^\infty(M) \to C^\infty(M)$ is a derivation of the algebra $C^\infty(M)$, i.e., it defines a vector field $X_f \in \mathfrak{X}(M)$ called the Hamiltonian vector field associated to f.

Choosing some local coordinates (U, x^i), any Poisson bracket is given by

$$\{f,g\}_{|U} = \sum_{i,j} \pi_{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

for $\pi_{ij} = \{x^i, x^j\}$ the Poisson bracket of the coordinate functions.

Lie algebra structure on $\mathcal{O}_X$ + Leibniz

As bivector [edit]

A **Poisson bivector** on a smooth manifold M is a bivector field $\pi \in \mathfrak{X}^2(M) := \Gamma(\wedge^2 TM)$ satisfying the non-linear partial differential equation $[\pi,\pi] = 0$, where

$$[\cdot,\cdot] : \mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q-1}(M)$$

denotes the Schouten–Nijenhuis bracket on multivector fields. Choosing some local coordinates (U, x^i), any Poisson bivector is given by

$$\boxed{\pi_{|U} = \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},}$$

for π_{ij} skew-symmetric smooth functions on U.

$$\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_i}$$

Examples

- Symplectic manifolds

Poisson manifolds generalize symplectic manifolds

- Two dimensional foliations with $\omega_{\mathcal{F}} = \mathcal{O}_X$

(e.g., degree two foliations on $\mathbb{P}^3$)

cod 1

$\det(T_{\mathcal{F}})^*$

- Products

Poisson distribution

$$\pi \in H^0(X, \bigwedge^2 T_X) \quad , \quad [\pi, \pi] = 0$$

$$\Omega^1_X \xrightarrow{i_\pi} T_X$$

Image of this morphism

is involutive $\Longrightarrow$ Defines a foliation $\mathcal{F}_\pi$

by symplectic leaves

Linear algebra $\Longrightarrow$ $\dim \mathcal{F}_\pi = 2k$

J. DIFFERENTIAL GEOMETRY
18 (1983) 523–557

THE LOCAL STRUCTURE OF POISSON MANIFOLDS

ALAN WEINSTEIN

$(X, x) = (\text{Symplectic}) \times (Y)$

Symplectic

Poisson with a zero

$\tilde{\pi}_Z$ vanishes at 0.

Theorem 2.1 (*Splitting theorem*). *Let x_0 be any point in a Poisson manifold P. Then there are a neighborhood U of x_0 in P and an isomorphism $\phi = \phi_S \times \phi_N$ from U to a product $S \times N$ such that S is symplectic and the rank of N at $\phi_N(x_0)$ is zero. The factors S and N are unique up to local isomorphism.*

$\pi \in H^0(X, \wedge^2 T_X)$

$x \in X$

$\operatorname{rank} \pi_x = 2k$

$\exists$ coordinates around x $(x_1, \dots, x_k, y_1 \dots, y_k, \underline{\underline{z_1, \dots z_m}})$

$$\pi = \left(\sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \right) + \tilde{\pi}_Z$$

A global Weinstein splitting theorem for holomorphic Poisson manifolds

Stéphane Druel[*] Jorge Vitório Pereira[†] Brent Pym[‡]
Frédéric Touzet[§]

$H^0(X, \wedge^2 T_X)$

Theorem 1.1. *Let* (X, π) *be a compact Kähler Poisson manifold, and suppose that* $\mathsf{L} \subset \mathsf{X}$ *is a compact symplectic leaf whose fundamental group is finite. Then there exist a compact Kähler Poisson manifold* Y*, and a finite étale Poisson morphism* $\widetilde{\mathsf{L}} \times \mathsf{Y} \to \mathsf{X}$*, where* $\widetilde{\mathsf{L}}$ *is the universal cover of* L*.*

compact leaf + $\#\pi_1(L) < \infty$

π restricts to a symplectic structure on L

$$\begin{array}{c} X \\ \big\downarrow {\scriptstyle \pi_1} \\ Y \end{array}$$

Poisson foliation, π_X

$$\tilde{L} \times Y$$

"trivial" only one leaf

actual foliation

THE SINGULARITIES OF INTEGRABLE STRUCTURALLY STABLE PFAFFIAN FORMS*

By Ivan Kupka

Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil

Communicated by S. Lefschetz, September 30, 1964

$\mathcal{F}$ foliation of $(\mathbb{C}^n,0)$

codim $\mathcal{F} = q$

If there exists

$\omega \in \bigwedge^{q} N^*_{\mathcal{F}}$

such that $d\omega(0) \neq 0$

then $\mathcal{F}$ is a product

$\times$

$\mathbb{C}^n,0$

$(\mathbb{C}^{q+1},0)$

foliation by curves

$(\mathbb{C}^{n-(q+1)},0)$

foliation with only one leaf

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then $\mathcal{F}$ is a product

Definition 5: A C^∞ function g on an open set $\Omega \subset M$ is called a *unit* on Ω if $g(x) \neq 0$ for $x \in \Omega$.

PROPOSITION 2. *If $w \in I(M)$ and g is a C^∞ function on an open set Ω, then $gw \in I(\Omega)$ and any integral manifold of w is integral manifold of gw. If g is a unit on Ω, then w and gw have the same integral manifolds in Ω.*

Proof: Trivial.

Definition 6: A form $w \in I(M)$ is called *structurally stable* if for every $\epsilon > 0$ there exists a neighborhood $\mathcal{V}_\epsilon$ of w in $I(M)$ with the C^1-topology such that: for any $w' \in \mathcal{V}_\epsilon$ there exists a homeomorphism $h: M \to M$ with the following properties: (1) $\text{dist}(x, h(x)) \leqslant \epsilon$ for all $x \in M$; (2) if D is an integral manifold of w, then $h(D)$ is an integral manifold of w'.

We can now announce our main result.

MAIN THEOREM. *If $w \in I(M)$ is structurally stable, the set of all singular points of w is the union of the following two sets: (1) A finite set $\Sigma_n(w)$ such that if $x_0 \in \Sigma_n(w)$, then x_0 is isolated in the set of all singular points of w and there exists a neighborhood N_0 of x_0 in M and two C^∞ functions $f, g: N_0 \to R$ such that: (i) g is a unit in N_0 and f admits in N_0 a unique singular point x_0 which is generic (in Morse's sense p. 172 of ref. 5) in N_0; (ii) $w = gdf$. The index of x_0 for f (in Morse's sense p. 143 of ref. 5) is called index of w. It determines the structure of the foliation defined by w in the neighborhood of x_0. (2) A finite union $\Sigma_2(w)$ of C^∞ compact manifolds of codimension 2, $W_1, \ldots, W_J$ such that if D is a C^∞ 2-dimensional cell transversal to W_i at a point x_0, the restriction of w to D (i.e., the intersection of the foliation defined by w with D) has at x_0 a generic singular point. More precisely, there exists for each $x_0 \in W_i$ a neighborhood N_0 of x_0 in M, a C^2 unit $g_0: N_0 \to R$, and a C^2 mapping $\varphi_0: N_0 \to R^2$ everywhere of rank 2 such that if (ξ, η) denotes the canonical system of coordinates in R^2, α_0 is the form $\eta d\xi - \lambda\, \xi d\eta$ ($\lambda \neq 0, 1$ scalar), and α_1 is the form $\xi d\xi + \eta d\eta + \mu\, (\eta d\xi - \xi d\eta)$ ($\mu \neq 0$ scalar), then either $w|N_0 = g_0\varphi_0{}^*\alpha_0$ or $w|N_0 = g_0\varphi_0{}^*\alpha_1$; λ or μ depend only on the manifold W_i, not on the point $x_0 \in W$. Except when λ is rational, φ_0 can be taken of class C^∞.* The proofs rely heavily on De Rham's lemma (p. 346 of ref. 6).

* This paper was written with the partial support of the Air Force Office of Scientific Research, AF OSR 62-334.

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Structural Stability of Integrable Differential Forms

by

Airton S. de Medeiros*

* This paper is essentially the author's doctoral dissertation prepared under the supervision of Professor C. Camacho at IMPA, Rio de Janeiro. This work was supported by CNPq and UFRN.

$\mathcal{F}$ foliation of $(\mathbb{C}^n,0)$

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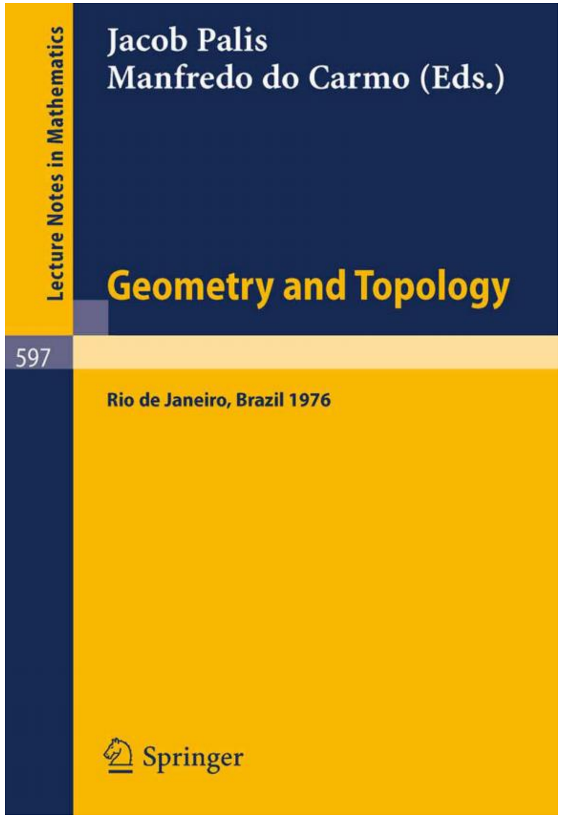

Project

Investigate when we can ~~...~~ globalize Kupka local decomposition.

$\mathcal{F}$ cod q on projective X

$S \subseteq \mathrm{Sing}(\mathcal{F})$ irreducible component

everywhere Kupka

- When S is a leaf of a foliation $\mathcal{G}$, $\dim \mathcal{G} = \dim S$
  ~~...~~ contained in $\mathcal{F}$?

Not automatic

- Examples on $\mathbb{P}^n$, $n \geq 3$ (algebraically integrable) with S simply-connected.

- Around S we have separatrices

separatrix

Ueda theory