Regular foliations on Rationally Connected Manifolds

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Outline:

I. Introduction

II. MMP for Foliations

III. Nef reduction map

IV. References

Introduction

Working with regular foliations on surfaces, Brunella showed:

Theorem (Brunella)

Let S be a rational surface. If $\mathcal{F}$ is a regular foliation on S with rank 1, then $\mathcal{F}$ is algebraically integrable, with rational leaves.

Thus, it is natural to make the following conjecture:

Conjecture (Touzet)

Let X be a rationally connected manifold. If $\mathcal{F}$ is a regular foliation on X, then $\mathcal{F}$ is algebraically integrable, with rationally connected leaves.

To recall: a projective manifold X is rationally connected if $\forall x, y \in X$, $\exists\, C \subset X$ a rational curve, s.t. $x, y \in C$.

For instance, any rational manifold is rationally connected.
But there are examples of rationally connected manifolds which are not rational.

Example: Let X be a conic bundle over a rational surface. Then X is not necessarily rational.

Let $\mathcal{F}$ be a regular foliation of corank 1

Suppose the fibers of the conic bundle $X \xrightarrow{\pi} S$ defining X are tangent to $\mathcal{F}$. Then there exists a regular foliation $\mathcal{G}$ with rank 1 on S such that

$$\mathcal{F} = \pi^* \mathcal{G}$$

By Brunella's Theorem, $\mathcal{G}$ is algebraically integrable with rational leaves.

Thus $\mathcal{F}$ is algebraically integrable with rational leaves.

Another class of examples are the Fano manifolds (X is Fano if $-K_X$ is ample)

Theorem (Campana, Kollar-Miyaoka-Mori)

If X is a Fano manifold, then X is rationally connected.

Example: If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d. Then $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(n+1-d)$ and thus X is Fano iff $d \leq n$.

More generally, a manifold X is weak Fano if $-K_X$ is nef and big.

Recall: A divisor D is nef if $D \cdot C \geq 0$, for all curves $C \subset X$.

A divisor D is big if the rational map associated to mD is birational for big m.

$$\boxed{\text{A weak Fano manifold is also rationally connected.}}$$

It turns out that Touzet's conjecture is true for weak Fano manifolds.

Theorem (Druel)

Let X be a weak Fano manifold. If $\mathcal{F}$ is a regular foliation on X, then $\mathcal{F}$ is algebraically integrable with weak Fano leaves.

A bigger class of manifolds are manifolds X which are rationally connected and $-K_X$ is nef.

In this talk, I will show:

Theorem

Let X be a rationally connected manifold with $\dim X = 3$ and $-K_X$ nef. Let $\mathcal{F}$ be a regular foliation on X with $\operatorname{corank}(\mathcal{F}) = 1$. Then $\mathcal{F}$ is algebraically integrable with rational leaves.

MMP for Foliations

One of the tools used in the proof is:

Theorem (Spicer)
Let X be a smooth projective 3-fold and $\mathcal{F}$ a regular foliation on X with $\operatorname{corank} \mathcal{F} = 1$. Then there exists a sequence of smooth blow-downs

$$X \to X_1 \to X_2 \to \cdots \to X_n$$

s.t. the non-trivial fibers of $X_i \to X_{i+1}$ are tangent to $\mathcal{F}_i$, for every i, where $\mathcal{F}_i$ is the foliation induced by $\mathcal{F}$ on X_i, and s.t. $\mathcal{F}_i$ is regular for every i, and

(i) either $K_{\mathcal{F}_n}$ is nef;

(ii) or there exists a Mori Fiber space structure $X_n \to B$, with fibers tangent to $\mathcal{F}_n$ and $K_{\mathcal{F}_n}$-negative.

Definition: A Mori Fiber Space is a fibration $\pi : X \to Y$, st. $\rho(X/Y) = 1$ and the general fiber of π is $-K_X$-positive.

This allows us to reduce our problem to foliation with nef canonical bundle and foliations with non-pseff canonical bundle.

Suppose first that $K_{\mathcal{F}}$ is not pseff.
Then by Spicer's theorem there exists a morphism $\pi : X \to Y$ and a regular foliation $\mathcal{G}$ on Y st. $\mathcal{F} = \pi^* \mathcal{G}$ and Y is a smooth MFS with fibers tangent to $\mathcal{G}$.
By Mori's classification, Y is either a del Pezzo fibration or a conic bundle.
In the first case, we conclude that $\mathcal{G}$ is induced by a del Pezzo fibration

In particular $\mathcal{G}$ (and thus $\mathcal{F}$) is algebraically integrable with rational leaves.

In the second case, there is a conic bundle $\varphi : Y \to S$ to a smooth rational surface and a rank 1 foliation $\mathcal{H}$ on S, such that:

$$\mathcal{G} = \varphi^* \mathcal{H}.$$

Let us show $\mathcal{H}$ is regular:

Let $p \in S$ be any point and ω a local 1-form defining $\mathcal{H}$ in an analytic neighborhood of p in S.

If ω vanishes at p (i.e. $p \in \operatorname{sing}(\mathcal{F})$) then $\varphi^*\omega$ vanishes at every point of $\varphi^{-1}(p)$. Moreover, $\varphi^*\omega$ vanishes (possibly) at points where φ is not smooth. By Mori's classification, this set has codimension at least 2. Thus $\varphi^*\omega$ vanishes in a set of codimension at least 2. Thus $\varphi^*\omega$ defines $\mathcal{G}$ in a neighborhood of $\varphi^{-1}(p)$.

But if $\mathcal{G}$ is regular, then no 1-form defining $\mathcal{G}$ can vanish in a set of codimension at least 2.

Thus ω cannot vanish at p.

We conclude $\mathcal{H}$ is regular. By Brunella's theorem, $\mathcal{H}$ is algebraically integrable with rational leaves.

Therefore $\mathcal{G}$ (and thus $\mathcal{F}$) is algebraically integrable with rational leaves.

What about the case $K_{\mathcal{F}}$ pseff?
By Spicer's theorem, there exists a smooth morphism

$$\pi : X \to Y$$

and a regular foliation $\mathcal{G}$ on Y, with $K_{\mathcal{G}}$ nef, such that

$$\mathcal{F} = \pi^* \mathcal{G}$$

We will show that this case is impossible.

Nef reduction map

We will use now the hypothesis

$$-K_X \text{ nef}$$

In this case there is a nef reduction map associated to $-K_X$:

Theorem (Bauer–Peternell)
Let X be a smooth projective 3-fold with $-K_X$ nef. Then there exists a morphism $f: X \to B$ of projective varieties s.t.

1. $-K_X$ is numerically trivial on all fibers of f;
2. If $x \in X$ is general and $C \subset X$ is an irreducible curve passing through x s.t. $\dim f(C) > 0$, then $-K_X \cdot C > 0$.

Keeping the notation:

$$\pi : X \to Y, \qquad \mathcal{F} = \pi^* \mathcal{G}, \qquad K_{\mathcal{G}} \text{ nef}$$

let us show that $-K_Y$ is nef.

Indeed by the exact sequence

$$0 \to T_{\mathcal{G}} \to T_Y \to N_{\mathcal{G}} \to 0$$

we have $N_{\mathcal{G}} = K_{\mathcal{G}} - K_Y$.

Theorem (Bott's vanishing)

If $\mathcal{G}$ is a regular foliation with corank $= q$ then $N_{\mathcal{G}}^{q+1} \equiv 0$.

Thus, in our case

$$N_g^2 \equiv 0$$

$$\Rightarrow (K_g - K_Y)^2 \equiv 0$$

$$\Rightarrow K_g^2 - 2K_g \cdot K_Y + K_Y^2 \equiv 0$$

If $C \subset X$ is any curve, then

$$K_g^2 \cdot C - 2K_g \cdot K_Y \cdot C + K_Y^2 \cdot C = 0$$

Since K_g is nef, and $-K_Y$ pseff these three numbers are ≥ 0. Thus each of them is zero.

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We conclude

$$K_g^2 \equiv K_g \cdot K_y \equiv K_y^2 \equiv 0$$

By Hodge theory we conclude that K_g and K_y are proportional. Since $K_y \not\equiv 0$ we have $-K_y \equiv \alpha K_g$ for some $\alpha \neq 0$. Since $-K_y$ is pseff we must have $\alpha > 0$. Thus $-K_y$ is nef.

We use the following:

Theorem (Bauer–Peternell)

Let X be a smooth projective 3-fold which is rationally connected and s.t. $-K_X$ is nef. Suppose that either X admits a divisorial contraction or a conic bundle structure. If $K_X^2 \equiv 0$, then the nef reduction $f : X \to B$ of $-K_X$ is a K3-fibration, $B \simeq \mathbb{P}^1$ and $-K_X \simeq f^* \mathcal{O}_{\mathbb{P}^1}(1)$.

In our case, if Y does not admit a divisorial contraction or conic bundle structure, then Y is either Fano or a del Pezzo fibration, with relative Picard one.

The first case cannot happen because $K_Y^2 \equiv 0$.

Suppose $\varphi: Y \to \mathbb{P}^1$ is a del Pezzo fibration with $\rho(Y/\mathbb{P}^1) = 1$. Let F be a general fiber of φ.

Since $\rho(Y/\mathbb{P}^1)=1$, we have:

(a) either $N_g|_F$ ample;

(b) or $-N_g|_F$ ample;

(c) or $N_g|_F \equiv 0$

Since $N_g^2 \equiv 0$ only (c) can happen.

Thus $K_F = K_X|_F = (K_g - N_g)|_F \equiv K_g|_F$

and this is nef, a contradiction to the fact that F is rational.

Thus Y cannot be a del Pezzo

fibration with relative Picard one.

We conclude that Y has a divisorial contraction or a conic bundle structure. Thus by Bauer–Peternell, the nef reduction map $f: Y \to \mathbb{P}^1$ of $-K_Y$ is a K3-fibration, with $-K_Y = f^*\mathcal{O}_{\mathbb{P}^1}(1)$.

Now we show that G is induced by f.

Since $-K_Y = \alpha K_{\mathcal{G}}$, we have

$$N_{\mathcal{G}} = K_{\mathcal{G}} - K_Y = -\left(\frac{1}{\alpha}+1\right) K_Y$$

Thus if F is a general fiber of f, then $N_{\mathcal{G}} \cdot F = -\left(\frac{1}{\alpha}+1\right) K_Y \cdot F = \left(\frac{1}{\alpha}+1\right) F \cdot F \equiv 0$

Thus if $\omega \in H^0(Y, \Omega^1_Y \otimes N_{\mathcal{G}})$ defines $\mathcal{G}$, then $\omega|_F \in H^0(F, \Omega^1_F \otimes N_{\mathcal{G}}|_F)$ and since F is $K3$

$$N_{\mathcal{G}}|_F \equiv 0 \Rightarrow N_{\mathcal{G}}|_F \sim 0$$

$$\Rightarrow \omega|_F \in H^0(F, \Omega^1_F) = 0 \quad (\pi_1(F) = 0)$$

This implies that F is tangent to $\mathcal{G}$, for general F, and thus $\mathcal{G}$ is induced by f.

Now using the fact that $-K_Y = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ and the possible cases of Mori contractions $g: Y \to Z$ given by Mori's classification, we can show that f is a smooth fibration.

Finally we apply:

Theorem (Viehweg–Zuo)
If $f: Y \to \mathbb{P}^1$ is a smooth fibration such that the general fiber F has a minimal model F' with $K_{F'}$ semi-ample, then f is birationally isotrivial.

Theorem (Oguiso–Viehweg) Let $f : Y \to \mathbb{P}^1$ be a smooth projective family of minimal surfaces with non-negative Kodaira dimension. Then f is birationally isotrivial iff $Y \simeq F \times \mathbb{P}^1$, for F a fiber of f.

Thus since Y is rationally connected, F is rational, a contradiction.

With this we conclude that $K_{\mathcal{F}}$ cannot be pseff. ///

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