Local foliations with closed leaves

Capes - Cofecub Workshop

August 27th

Introduction

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Fundamental tool: Holonomy group of a leaf.

Let $\mathcal{L}_P$ be a leaf of a holomorphic foliation

Let $\gamma : [0,1] \to \mathcal{L}_P$ a path with $\gamma(0) = \gamma(1) = P$.

Hol_γ is defined in (T, P) where $T = \pi^{-1}(P)$.

We have a morphism $\pi(\mathcal{L}_P, P) \overset{\mathrm{Hol}}{\to} \mathrm{Diff}(T, P)$ of groups defined by $[\gamma] \mapsto \mathrm{Hol}_\gamma$.

Leaves with finite holonomy groups

Theorem (Reeb local stability theorem)

Let $\mathcal{L}$ be a compact leaf of a C^1 foliation with finite holonomy group. Then there exists a fundamental system of saturated neighborhoods of $\mathcal{L}$.

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It is easy to construct a first integral defined in a neighborhood of $\mathcal{L}$.

Foliations with closed leaves

Consider a (singular) holomorphic foliation $\mathcal{F}$ of dimension p.

Definition

We say that $\mathcal{F}$ has *closed leaves* if $\overline{\mathcal{L}}$ is an analytic variety of dimension p for any leaf $\mathcal{L}$.

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Example: Radial foliation in $\mathbb{C}^2$ or $\mathbf{CP(2)}$.

Natural questions

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Answer: No and no.

Groups of local biholomorphisms

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The holonomy group can be identified with a finitely generated subgroup of $\mathrm{Diff}(\mathbb{C}^q, 0)$.

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$\pi^2 = \mathbb{C}/\langle z+1, z+i\rangle$

$S = \mathbb{C}^3/\sim$ where

$(z,x,y) \sim (z+1, x, x+y)$ and $(z,x,y) \sim (z+i, x, y)$.

$\mathcal{F}$ is a suspension. The leaves of $\mathcal{F}$ are the images of $\mathbb{C} \times \{(x_0, y_0)\}$ by $\pi : \mathbb{C}^3 \to \mathbb{C}^3/\sim$.

$\mathcal{F}$ has closed leaves in the neighborhood of the zero section because $(x,y) \mapsto (x, x+y)$ has finite orbits in a neighborhood of the origin.

nth iterate of $(x, x+y) = (x, x+ny)$

Natural questions

Stability $\implies$ existence of first integrals? No

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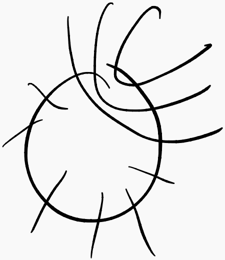
Suzuki's example

$(\mathbb{C}^2, 0)$



$y \cdot e^{y/x}$

Natural questions

Stability $\implies$ existence of first integrals? No

Suzuki's example

There are no meromorphic first integrals

Local case

Theorem (Mattei-Moussu '80)

Let $\mathcal{F}$ be a local holomorphic foliation in $(\mathbb{C}^n, 0)$ of codimension 1 with closed leaves. Suppose that there are finitely many leaves whose closure contains the origin. Then $\mathcal{F}$ has a holomorphic first integral.

closed leaves $\Leftrightarrow$ finite holonomy groups

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It is a topological criterium for the existence of holomorphic first integrals

Global case

Theorem (Edwards-Millet-Sullivan ’77)

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Theorem (Gómez Mont ’89)

Let $\mathcal{F}$ be a holomorphic foliation (of dimension p) with closed leaves in a projective manifold. Suppose $\dim(\mathrm{Sing}(\mathcal{F})) < p$. Then the foliation is stable and there is a meromorphic first integral.

Finite holonomy groups

Theorem (Pinheiro-Reis)

Let $\mathcal{F}$ be a singular holomorphic foliation of dimension 1 defined in $(\mathbb{C}^3, 0)$. Suppose that it has two independent holomorphic first integrals. Suppose that $\tilde{\mathcal{F}}$, the transform of $\mathcal{F}$ by the punctual blow-up centered at the origin, has only isolated singularities which, in addition, are simple. Then either $\dim(\mathrm{S}ing(\mathcal{F})) \geq 1$ or it has infinitely many separatrices.

Generalization (Cano, Ravara-Vaz, X)

→ Finite holonomy

$\exists\, X \in \mathcal{H}(\mathbb{C}^3, 0)$ with

- finitely many separatrices
- $Sing(X) \subset \{(0,0,0)\}$
- Every leaf has a finite holonomy group

$\Longrightarrow$ X is a regular vector field

The fixed point curve theorem

Theorem (Lisboa-Ribón)

Let $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ with finite orbits. Then there exists $k \in \mathbb{N}$ such that $\dim(\mathrm{Fix}(\phi^k), 0) \geq 1$.

$$F(x,y) = (x, x+y)$$

$$\mathrm{Fix}(F) = \{x = 0\}$$

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Corollary (Lisboa-Ribón)

Let $\phi \in \text{Diff}(\mathbb{C}^n, 0)$ with finite orbits. Then at least one of the eigenvalues of $\text{spec}(D_0\phi)$ is a root of unity.

Finite orbits and linear part

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There are examples of $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ that has finite orbits and $\mathrm{spec}(D_0F)$ contains exactly a root of unity eigenvalue.

Bad set

Consider a regular foliation with compact leaves on a compact manifold.

The bad set is the set of leaves in which the volume function is not locally bounded.

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Let $\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$. The bad set is the set of periodic orbits.

Theorem

The connected components of the bad set are analytic.

General result

Theorem (Lisboa-Ribón)

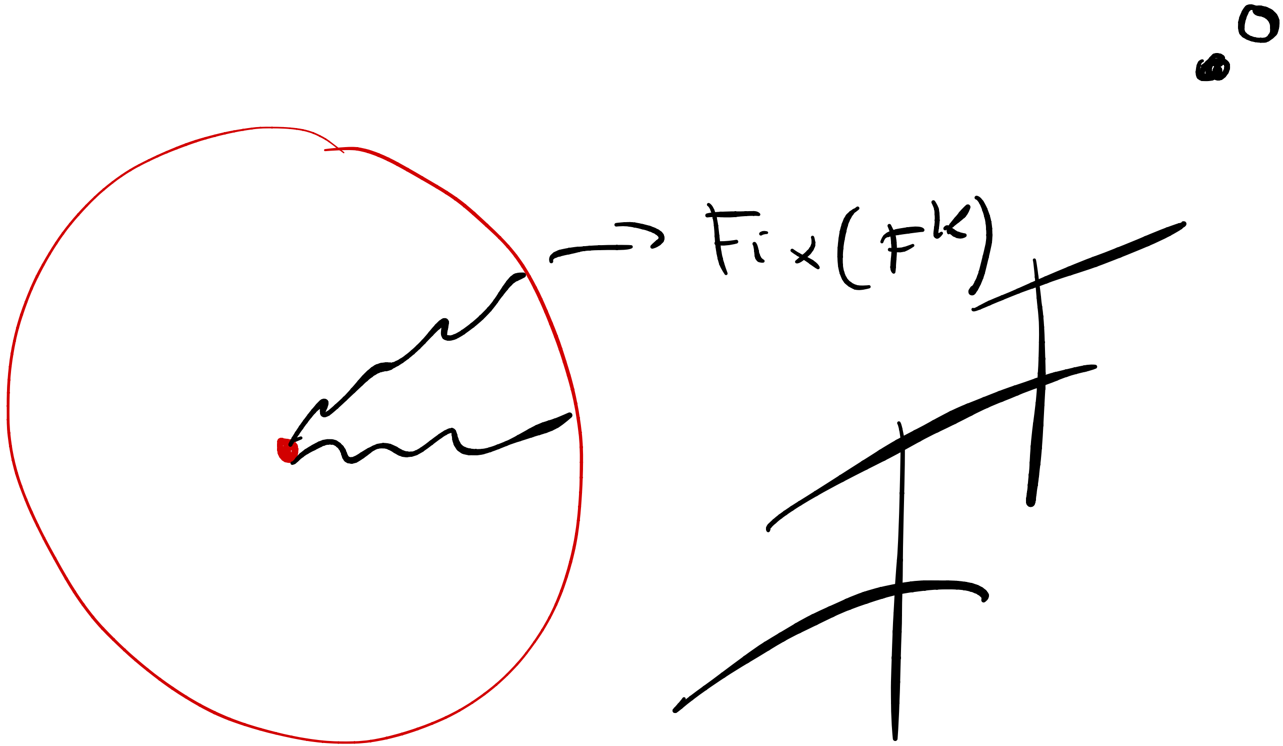
Let G be a finitely generated subgroup of $\mathrm{Diff}(\mathbb{C}^n, 0)$ with finite orbits. Then there exists a finite index normal subgroup J of G such that $\dim(\mathrm{Fix}(J), 0) \geq 1$.

Theorem (L.-R., Stability in intermediate dimension)

Let $\mathcal{F}$ be a codimension p regular holomorphic foliation defined in a neighborhood of a compact leaf $\mathcal{L}$. Assume that $\mathcal{F}$ has closed leaves in a neighborhood of $\mathcal{L}$. Then there exists an analytic set S of dimension greater than $n - p$ such that

- S is $\mathcal{F}$-invariant and contains $\mathcal{L}$.
- There exists a fundamental system of saturated neighborhoods of $\mathcal{L}$ in S consisting of compact leaves.

$n = 2$


Fix(F^k)

Thank you