non transverse intersection

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M smooth algebraic variety over $\mathbb{C}$,
$\mathcal{F}$ a foliation on M.

Problem

Understand algebraic subvarieties $V \subset M$ such that

$$\mathrm{codim}(\mathcal{F} \cap V) < \mathrm{codim}\mathcal{F} + \mathrm{codimV}$$

or

Understand formal curve in a leaf $\gamma \subset \mathcal{L}$ with $\dim \overline{\gamma}^z < \mathrm{codim}\mathcal{F} + 1$

Ax-Schanuel for exp :

If $\mathrm{trans.deg.}_{\mathbb{C}}\mathbb{C}(\hat{t}_1, \ldots \hat{t}_n, \exp(\hat{t}_1), \ldots \exp(\hat{t}_n)) < n + 1$ then $\sum n_i \hat{t}_i \in \mathbb{C}$

$\mathfrak{g} \subset \mathfrak{gl}_N(\mathbb{C})$ a Lie algebra.

$\mathcal{F}$ is a $\mathfrak{g}$-Lie foliation is there exists $\omega : TM \to \mathfrak{g}$ with

1. $d\omega = -\omega \wedge \omega$
2. $\mathrm{codim}\mathcal{F} = \dim \mathfrak{g}$
3. $\ker \omega = \mathcal{F}$

If $G \subset \mathrm{GL}_N(\mathbb{C})$ is a Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, the transversal structure of $\mathcal{F}$ is given by the set of local first integrals $Y : U \to G$ solving

$$dYY^{-1} = \omega$$

.

M, $\mathfrak{g}$, $\mathcal{F}$, $\mathcal{L}$

Proposition

If $\gamma \subset \mathcal{L}$ and $\dim \overline{\gamma}^z < 1 + \mathrm{codim}\mathcal{F}$ then $\exists\ \mathfrak{h} \subset \mathfrak{g}$ and $\mathcal{F}|_{\overline{\gamma}^z}$ is $\mathfrak{h}$-Lie.

Corollary

Consider an alg. subvariety $V \subset M$ with $\dim V < 1 + \mathrm{codim}\mathcal{F}$.
If there is no $\mathfrak{h}$ with $\dim \mathfrak{h} = \mathrm{codim}(\mathcal{F}|_V)$ the $\mathcal{F}|_V$ has a rational first integral.

Corollary

Assume $\dim \mathcal{F} = 1$, $\mathfrak{g}$ simple and leaves are Zariski dense.
Consider $\mathcal{F}^n$ on M^n and a leaf $\mathcal{L}^n$.
If $\gamma \subset \mathcal{L}^n$ with non constant projections and $\dim \overline{\gamma}^z < 1 + n \dim \mathfrak{g}$
then $\exists\ \ell < k$ and $X \subset M \times M$ a correspondence such that $pr_{\ell,k}(\gamma) \subset X$.

Example

Principal integrable connection on a bundle $P \to B$ with group G.

- $\mathcal{F}$ is locally defined by $dZ - \Omega Z = 0$
- is globally defined by $\omega = Z^{-1}dZ - Z^{-1}\Omega Z = 0$

Theorem

Let $\mathcal{F}$ be a G-principal connection on $P \to B$ with Zariski dense leaves. Let V be an algebraic subvariety of P, $\mathcal{L}$ is an horizontal leaf and $\mathcal{V}$ an irreducible component of $V \cap \mathcal{L}$. If $\dim V < \dim(\mathcal{V}) + \dim G$ then

1. the projection of $\mathcal{V}$ in B is contained in a proper algebraic subvariety $X \subset B$
2. Leaves of $\mathcal{F} \cap P|_X$ are not Zariski dense.

Ax-Schanuel for j

If $\mathrm{trans.deg.}_{\mathbb{C}}\mathbb{C}(\hat{t}_1, \ldots \hat{t}_n, j(\hat{t}_1), \ldots j''(\hat{t}_n)) < 1 + 3n$
then $\exists\ \ell < k$, $g \in PGL_2^+(\mathbb{Q})$ such that $\hat{t}_\ell = g(\hat{t}_k)$.