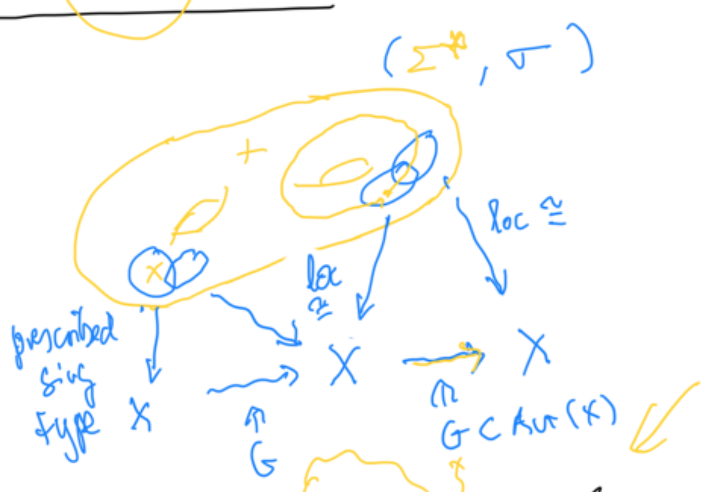
Singular geometric structures and isomonodromy

GABRIEL CALSAMIGLIA

UFF, BRASIL

Singular (G,X)-structure



Monodromy homo: $\rho_\sigma : \pi_1(\Sigma^*) \longrightarrow G$ up to G-conj.
$\gamma \longmapsto \rho(\gamma)$

THM: (Nascimento) Any representation can be attained. in the case $(\mathbb{C}P^1, PSL_2(\mathbb{C}))$

Isomonodromic deformation: $\{(\Sigma_t, \sigma_t)\}_{t\in[0,1)} \longrightarrow [\rho_{\sigma_t}]$ const.

MAIN QUESTIONS ON ISOMONODROMIC DEFORMATION SPACES:

1. TOPOLOGY AND GEOMETRY (LOCAL AND GLOBAL)
2. DESCRIBE LIMITS AND DEGENERATIONS
3. COMPACTIFY

BRANCHED TRANSLATION STRUCTURES on compact surfaces

$(\mathbb{C}, \mathrm{Transl}(\mathbb{C}))$ structure with sing of type $\boxed{z \to z^n}$ $n = 2, 3, 4, \ldots$

Monodromy of (Σ^*, σ) : Trivial local monodromy

$\rho_\sigma : H_1(\Sigma) \longrightarrow \mathbb{C}$

complex curve, holo 1-forme

$$\{(\Sigma_g, \sigma)\} \longleftrightarrow \Omega\mathcal{M}_g \ni (C, \omega)$$

$\omega = d\varphi$

(Σ_g, σ_t) ison $\rightsquigarrow (C_t, \omega_t)$ s.t $\boxed{\begin{array}{l} H_1(C_t) \to \mathbb{C} \\ \gamma \mapsto \int_\gamma \omega_t \\ \text{CONST.} \end{array}}$

Lemma: Isomonodromy induces a regular holo. foliation $\mathcal{F}_g$ of $\Omega\mathcal{M}_g$ of codim $2g$.

Thm: (C, Deroin, Francaviglia) Description of $\overline{L} \subset \Omega\mathcal{M}_g$ for every L leaf.
Description of $\overline{L} \subset \Omega\mathcal{M}_g$ where $\mathcal{F}_g$ is ergodic.
It is a real analytic set
$\overline{L} = L \iff L = \mathcal{H}_{g,d}(\mathbb{C}/\Lambda)$ (Hurwitz space)

Proof uses action of Mod_g on $H^1(\Sigma_g, \mathbb{C})$.

Thm: (Idem) L_p leaf associated to $p : H_1(\Sigma) \to \mathbb{C}$.

$$\exists\ \pi_1(L_p) \xrightarrow{\text{homo}} \mathrm{Stab}_{\mathrm{Aut}(H_1(\Sigma))}(p) \quad \text{if } \ker p \neq 0$$

When $g = 2$ $L \subset \mathcal{F}_g$ has dim $= 1$ and carries a $1/2$-translation structure $(\mathbb{C}, \{z \mapsto \pm z + c\})$-str.

Thm: (McMullen) Description of $1/2$-translation str for generic $L \subset \mathcal{F}_2$.

Deligne Mumford.

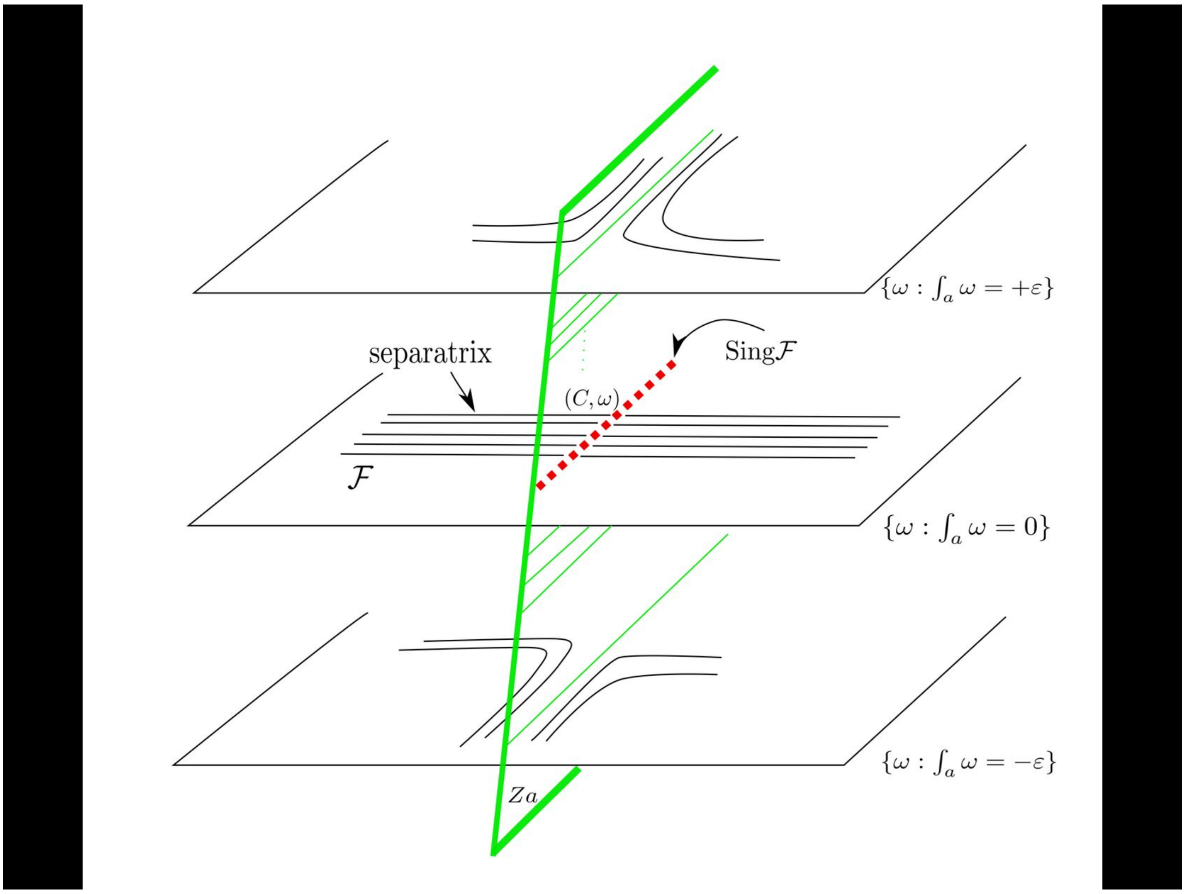
$$(\Omega^*\mathcal{M}_g, \mathcal{F}_g) \longleftrightarrow (\Omega\overline{\mathcal{M}}_g, \overline{\mathcal{F}_g})$$

regular   singular foliation?

. $\overline{\mathcal{F}_g}$ is regular on $\Omega^*\mathcal{M}_g^{ct}$. CLOSED LEAVES ARE ALGEBRAIC

<u>Problem:</u> Understand $\mathcal{F}_g$ around points of $\mathrm{Sing}(\mathcal{F}_g)$.

<u>Example:</u>  (C,ω) $\hat{\omega} \neq 0$ ~~holo~~ mero $\in \mathrm{Sing}(\mathcal{F}_g)$.



<u>Obs:</u> $\hat{\omega}$ might have two simple poles

Branched translation structures with simple poles

$(\mathbb{C}, Transl(\mathbb{C}))$- struct (Σ_g^*, σ) with singularities

of types $\quad z \longmapsto z^n \quad n=2,3,\dots$

$\quad\quad z \longmapsto a \log z \quad a \in \mathbb{C}^*$ (residue).

meromorphic form

Monodromy: $\rho_\sigma : H_1(\Sigma_g^*) \longrightarrow \mathbb{C}$ $a\mathbb{Z}$

closed curve.

$$\{(\Sigma_{g,n}^*, \sigma)\} \rightsquigarrow KM_g = (C, \omega)$$

[Use $Mod_g \cap H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$]

Similar results $n=2$ (w/ Deron)

THERE ARE CLOSED LEAVES THAT ARE NON ALGEBRAIC.

WORK IN PROGRESS:

Extension to any $n \geq 3$ (w/ ARZAKHOVA & DEROIN)

PROJECTIVE STRUCTURES ON $\Sigma_{0,5}$ WITH A SIMPLE BRANCH POINT

$(\mathbb{CP}^1, PSL_2(\mathbb{C}))$-str on $\Sigma_{0,5}$ having sing types:

$\quad z \longmapsto z^\alpha \quad \alpha \in \mathbb{C}^*$

$\quad z \longmapsto a \log z + \frac{1}{z^n} \quad n=1,2,\dots \quad a \in \mathbb{C}$

and at least one with $\alpha = 2$.

generalize mero. diff.

$$\{(\Sigma_{0,5}, \sigma)\}_{/iso} \xrightarrow{\text{Schwarzian}} (QM_{0,5}, \mathcal{F}_{0,5})$$

regular holomorphic foliation by curves

$\mathcal{F}_{0,5}$ is equivalent to the PAINLEVÉ VI foliation.

WORK IN PROGRESS WITH F. LORAY:

1. The leaves of $\mathcal{F}_{0,5}$ carry a natural projective structure. Describe it for closed leaves.

PAINLEVÉ VI :

② Towards compactifying

$$\boxed{(QM_{0,5}, \mathcal{F}_{0,5}) \hookrightarrow (\overline{QM}_{0,5}, \overline{\mathcal{F}_{0,5}})}$$

singular

+ Get information on $\mathcal{F}_{0,5}$ by analyzing $\overline{\mathcal{F}_{0,5}}$ over boundary divisors.

TOOLS

$$p: H_1(\Sigma_g) \longrightarrow \mathbb{C}/\mathbb{Z}$$

Im p discrete but not finite.

$$
P_{VI}\left\{\begin{array}{ll}
\frac{d^2q}{dt^2} & = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right) \\
 & \quad +\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{(\theta_\delta-1)^2}{2}-\frac{\theta_\alpha^2}{2}\frac{t}{q^2}+\frac{\theta_\beta^2}{2}\frac{t-1}{(q-1)^2}+\frac{1-\theta_\gamma^2}{2}\frac{t(t-1)}{(q-t)^2}\right).
\end{array}\right.
$$

the coefficients of which depend on complex parameters

$$
\theta=(\theta_\alpha,\theta_\beta,\theta_\gamma,\theta_\delta).
$$