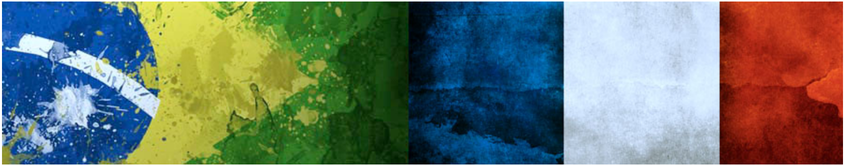
The Cremona group and foliations

Carolina Araujo (IMPA)

CAPES/Cofecub - 25 June 2021



The Cremona Group

$$\mathrm{Bir}(\mathbb{P}^n) \ := \ \left\{ \varphi : \mathbb{P}^n \ \overset{\sim}{\dashrightarrow} \ \mathbb{P}^n \ \text{ birational self-map } \right\}$$

The Cremona Group of the plane

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Example (The standard quadratic transformation)

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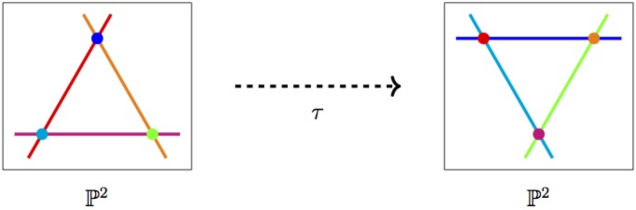

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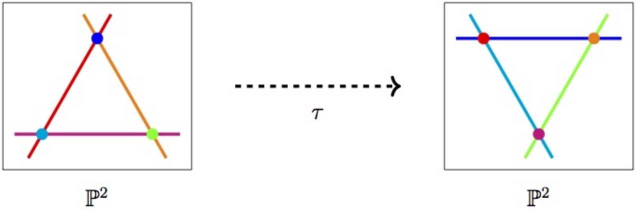

Theorem (Noether-Castelnuovo 1870-1901)

$$\mathrm{Bir}(\mathbb{P}^2) = \langle \, \mathrm{Aut}(\mathbb{P}^2) \, , \, \tau \, \rangle$$

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$\mathrm{Bir}(\mathbb{P}^2)$ is not a simple group.

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- Theorem holds over any base field (Lonjou 2016).
- Any quotient of $\mathrm{Bir}(\mathbb{P}^2)$ is infinite and non-abelian.

The Cremona Group in higher dimensions

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(Weak) Factorization theorem for Cremona transformations.

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(Weak) Factorization theorem for Cremona transformations.

Theorem (Blanc-Lamy-Zimmermann 2019)

For $n \geq 3$:

$$\mathrm{Bir}(\mathbb{P}^n) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}.$$

The Sarkisov program

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$$\psi$$

$$\mathbb{P}^n = X_0 \overset{\psi_1}{\dashrightarrow} X_1 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{n-1}}{\dashrightarrow} X_{n-1} \overset{\psi_n}{\dashrightarrow} X_n = \mathbb{P}^n$$

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The ψ_i's are elementary links

The X_i's are Mori fiber spaces (possible outcomes of the MMP)

The Sarkisov program

$$\begin{array}{ccccccccc} & & & & \overset{\psi}{\curvearrowright} & & & & \\ \mathbb{P}^n = X_0 & \overset{\psi_1}{\dashrightarrow} & X_1 & \overset{\psi_2}{\dashrightarrow} & \cdots & \overset{\psi_{n-1}}{\dashrightarrow} & X_{n-1} & \overset{\psi_n}{\dashrightarrow} & X_n = \mathbb{P}^n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \mathrm{pt} & & Y_1 & & & & Y_{n-1} & & \mathrm{pt} \end{array}$$

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The X_i's are Mori fiber spaces (possible outcomes of the MMP)

The surface case

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The Mori fiber spaces are:

- $\mathbb{P}^2 \to \mathrm{pt}$
- $\mathbb{F}_m \to \mathbb{P}^1$ ($\mathbb{P}^1$-bundle)

- $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$

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- $\mathbb{F}_1 \cong Bl_P \mathbb{P}^2$

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Type 1

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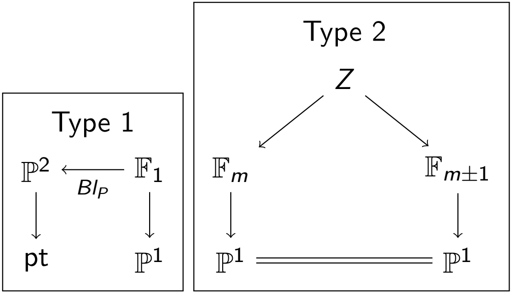

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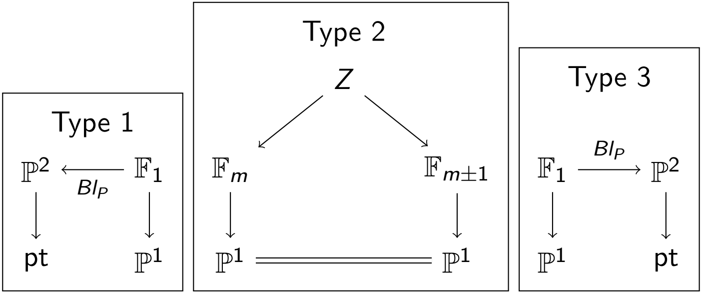

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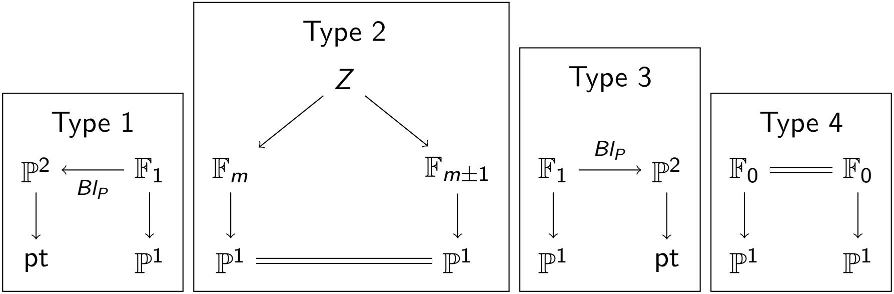
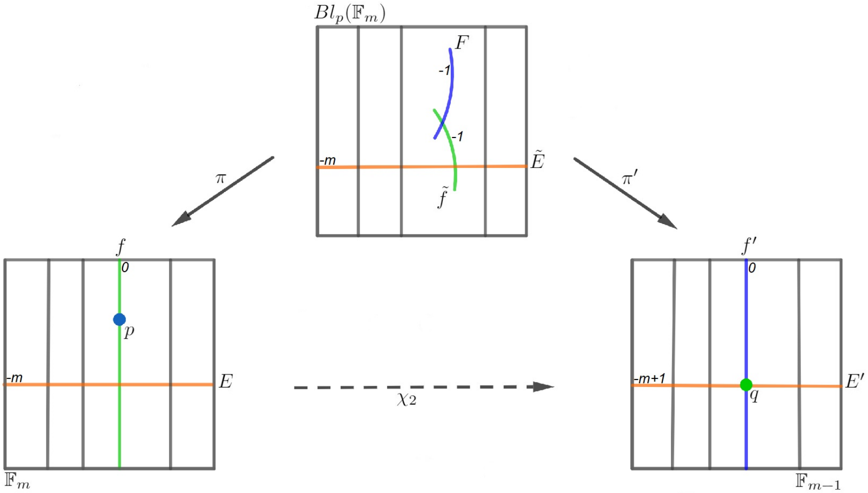
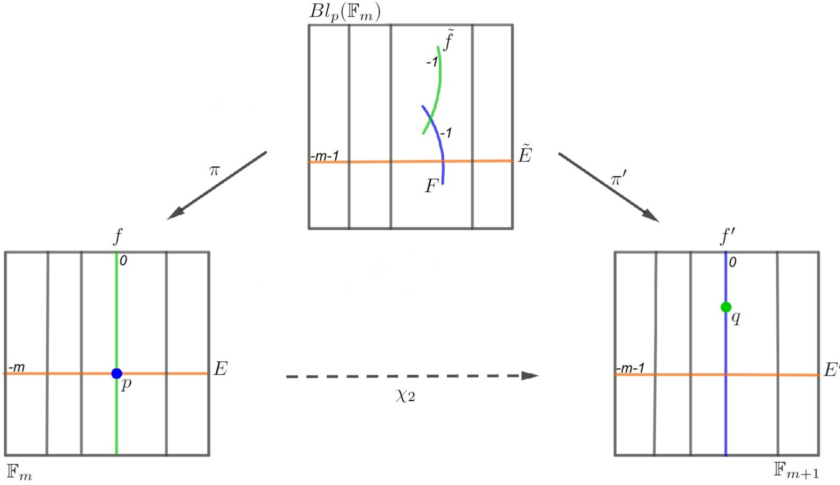




$Bl_p(\mathbb{F}_m)$
F
-1
-1
-m
$\tilde{E}$
$\tilde{f}$
π
π'
f
0
p
-m
E
$\mathbb{F}_m$
χ_2
f'
0
-m+1
E'
q
$\mathbb{F}_{m-1}$



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π
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q
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$\mathbb{F}_{m+1}$

Elementary links in higher dimensions

$\boxed{Type 1}$

Surfaces

$$\begin{array}{ccc} \mathbb{P}^2 & \xleftarrow[Bl_P]{} & \mathbb{F}_1 \\ \downarrow & & \downarrow \\ \text{pt} & \longleftarrow & \mathbb{P}^1 \end{array}$$

Elementary links in higher dimensions

*Type*1

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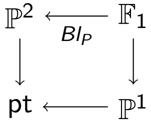
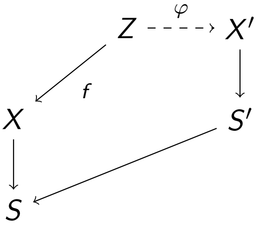


Higher dimensions



Volume preserving Cremona transformations

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$\frac{dx}{x} \wedge \frac{dy}{y}$ - meromorphic volume form on $\mathbb{P}^2$ with simple poles along Δ

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Theorem (Blanc 2013)

$$\mathrm{Bir}\left(\mathbb{P}^2, \frac{dx}{x} \wedge \frac{dy}{y}\right) = \left\langle \underbrace{(\mathbb{C}^*)^2 \,,\, \mathrm{SL}(2,\mathbb{Z})}_{\text{preserve the torus } (\mathbb{C}^*)^2} \,,\, (x,y) \mapsto \left(y, \frac{1+y}{x}\right) \right\rangle$$

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (x,y) \mapsto (\, x^a y^b \ , \ x^c y^d \,)$$

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Problem

To describe $\mathrm{Bir}\left(\mathbb{P}^n, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}\right)$

Definition (Calabi-Yau pair)

(X, D) such that

- X terminal projective variety
- D is a hypersurface $\sim -K_X$
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$(\mathbb{P}^n, D)$ where $D \subset \mathbb{P}^n$ is a hypersurface of degree $n + 1$

Remark

(X, D) Calabi-Yau pair $\rightsquigarrow$ $\exists$ ω_D (unique up to scaling)

$$\operatorname{div}(\omega_D) = -D$$

Remark

If (X, D) has canonical singularities, then

$$\mathrm{Bir}(X, \omega_D) \;=\; \left\{ \varphi \in \mathrm{Bir}(X) \;\middle|\; \varphi_* D = D \right\}$$

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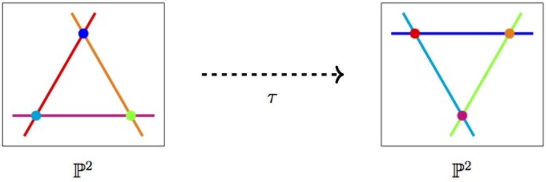
Example (Canonicity is necessary)



Problem (Joint with A. Corti and A. Massarenti)

Given a Calabi-Yau pair (X, D), *to determine* $\mathrm{Bir}(X, \omega_D)$.

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Theorem A

If (X, D) is terminal with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$ and $\mathrm{Pic}(D) = \mathbb{Z} \cdot \left(H_{|D}\right)$, then

$$\mathrm{Bir}(X, D) \; = \; \mathrm{Aut}(X, D).$$

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Corollary

If $D \subset \mathbb{P}^n$ is a very general hypersurface of degree $n+1$ ($n \geq 3$), then

$$\mathrm{Bir}\left(\mathbb{P}^n, D\right) \;=\; \mathrm{Aut}\left(\mathbb{P}^n, D\right).$$

Theorem B

If $D \subset \mathbb{P}^3$ is a general quartic surface with one singular point, then

$$\mathrm{Bir}(\mathbb{P}^3, D) \;\cong\; \mathbb{G} \rtimes \mathbb{Z}/2\mathbb{Z}$$

$\mathbb{G}$ is a form of $\mathbb{G}_m$ over $\mathbb{C}(x, y)$

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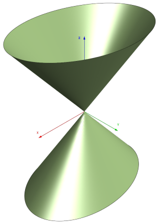
$$x_0^2 A_2(x_1, x_2, x_3) + x_0 B_3(x_1, x_2, x_3) + C_4(x_1, x_2, x_3) = 0$$



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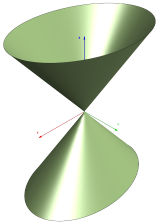

$$\mathbb{G} = \left\{ \left[(AG - BF)x_0 - CF : A(Fx_0 + G)x_1 : A(Fx_0 + G)x_2 : A(Fx_0 + G)x_3 \right] \right\}$$

$\big(F, G \in \mathbb{C}[x_1, x_2, x_3]$ homogeneous with $\deg(G) = \deg(F) + 1\big)$

Definition (Calabi-Yau pair)

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Definition

(X, D_X) and (Y, D_Y) Calabi-Yau pairs

$f : X \dashrightarrow Y$ birational map $\rightsquigarrow$ $f_* : \Omega^n_{\mathbb{C}(X)/\mathbb{C}} \to \Omega^n_{\mathbb{C}(Y)/\mathbb{C}}$

If $f_*\omega_{D_X} = \omega_{D_Y}$ (up to scaling) then we say that

$$f : (X, D_X) \dashrightarrow (Y, D_Y) \text{ is volume preserving}$$

Volume Preserving Sarkisov Program

Theorem (Corti-Kaloghiros 2016)

A volume preserving birational map between Mori fibered Calabi-Yau pairs is a composition of volume preserving Sarkisov links .

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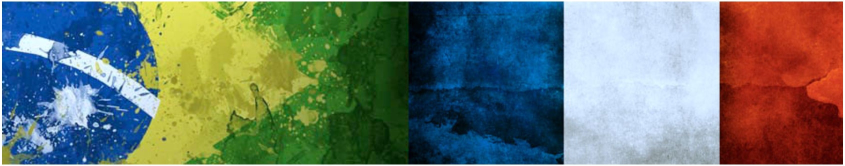
$$\psi : (X_0, D_0) \dashrightarrow (X_n, D_n)$$

$$\begin{array}{ccccccccc}
(X_0, D_0) & \overset{\psi_1}{\dashrightarrow} & (X_1, D_1) & \overset{\psi_2}{\dashrightarrow} & \cdots & \overset{\psi_{n-1}}{\dashrightarrow} & (X_{n-1}, D_{n-1}) & \overset{\psi_n}{\dashrightarrow} & (X_n, D_n) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
Y_0 & & Y_1 & & & & Y_{n-1} & & Y_n
\end{array}$$

Thank you! [1]



[1] Thanks to Santiago Arango, Daniela Paiva and Wikipedia for the nice pictures