Characterization of torus quotients

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Based on a joint work with P. Graf, H. Guenancia and P. Nauman.

Th (Yau, 1978): Let (X,ω) be a compact Kähler manifold.
If $c_1(X)\cdot\omega^{n-1} = c_2(X)\cdot\omega^{n-2} = 0$ then
$\exists\ T \xrightarrow[\text{étale}]{} X$ with $T = \mathbb{C}\times$ torus.

Natural question: characterize quotient $X = T/G$
with G = finite gp acting on T (NOT freely) ?
X singular

Th (Lu-Taji, 2016): Let X be projective and klt
such that $K_X \equiv 0$ ($c_1(X)=0$) and $\hat{c}_2(X)\cdot A^{n-2} = 0$
ample class
orbifold 2nd chern class
Then, $X \simeq T/G$.

klt condition: largest class of singularities stable under the elementary steps of the MMP.

Examples: ① quotient singularities $\mathbb{C}^n/G$, $G < GL_n(\mathbb{C})$ finite

② $0 \in (f=0) \subset \mathbb{C}^{n+1}$ is klt $\Leftrightarrow$ $k \leq n$
$(k = \mathrm{mult}_0(f))$

Why "X projective" in Lu-Taji ?

(i) hyperplane argument $\Rightarrow$ reduction to the surface case.

(ii) existence of a maximally quasi-étale cover

Def: A quasi-étale cover $\pi : Y \to X$ is a finite hol. map between normal $\mathbb{C}x$ spaces that is étale over X_{reg} (it is know from its restict° over X_{reg})

Example: T 2-dim torus $X = T/(\pm 1)$ Kummer

$T \to X$ maximally quasi-étale

① $\tilde{X} \to X$ minimal resolut° = K3 surface

$\pi_1(X) = 1 \Rightarrow$ no non trivial étale cover

② $1 \to \underbrace{\pi_1(T)}_{\mathbb{Z}^4} \to \pi_1(X_{reg}) \to \mathbb{Z}/2\mathbb{Z} \to 1.$

$\hookrightarrow$ a lot of quasi-étale cover

Th (Xu, Greb-Kebekus-Peternell 2016): Let X be a (quasi) projective klt variety. Then $\exists\, \tilde{X} \longrightarrow X$ quasi-étale such that $\widehat{\pi_1(\tilde{X}_{reg})} \simeq \widehat{\pi_1(\tilde{X})}$.

$\Longleftrightarrow$ on $\tilde{X}$, any flat vector bundle defined on $\tilde{X}_{reg}$ extends to the whole of $\tilde{X}$ (as a flat vb).

Th (CGGN, 2020): Let X be a klt Cx space that is either a germ or compact. Then $\exists\, Z \hookrightarrow X$ closed analytic with $\dim Z \leqslant \dim X_{sg} - 1$ st $X \setminus Z$ admits a maximally quasi-étale cover.

Tools from the proof:

- $\rightarrow$ topology of stratified spaces
- $\rightarrow$ isolated singularities are algebraic (Artin / Tougeron 1968).

Th (Characterization of torus quotients):

Let X be a klt $\mathbb{C}$x space such that $T_{X_{reg}}$ is flat.

(i) if X is a germ, then X is a quotient singularity.

(ii) if X is compact Kähler, then $X \simeq T/G$ is a torus quotient.

Proof:

① enough to prove that X admits a smooth quasi-étale cover.

② Prove ① by induct° on $\dim(X_{sg})$:

- $\dim(X_{sg}) = 0 \Rightarrow Z = \emptyset$ and $\tilde{X} \to X$ max q-étale cover and $T_{\tilde{X}_{reg}}$ is flat $\Rightarrow T_{\tilde{X}}$ is a flat vb in part. locally free (Lipman-Zariski) $\Rightarrow \tilde{X}$ smooth

- in general $\underset{\text{m.q-étal.}}{Y^\circ \longrightarrow X \setminus Z}$ + previous step Y° is smooth.

$$\begin{tikzcd}
& Y^\circ \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
X_{reg} \subset & X \setminus Z \arrow[r, hook] & X
\end{tikzcd}$$

$\dim(Y_{sg}) \leq \dim(Z) < \dim(X_{sg})$

Done by induct° !