ARITHMETICAL AND GEOMETRICAL ASPECTS OF
HOMOGENEOUS DIOPHANTINE APPROXIMATION BY
ALGEBRAIC NUMBERS IN A GIVEN NUMBER FIELD

FRANÇOIS MAUCOURANT

The aim of this talk is twofold. The first is to introduce the reader to some old results on Markoff and Lagrange spectra, which are not so well known. Particularly, we tried to put the emphasis on the idea that both spectra share strong common properties and that minima of quadratic forms and homogeneous diophantine approximation in one variable are deeply linked. The second is to show that natural generalizations of such objects have natural interpretations in geometrical terms, and that this geometry can be used to recover and generalize some of the results of the first part. We tried to keep the algebraic requirements to a minimum, so we explain what an ideal class group is and why it arises in a natural way in such a problem.

1. SOME ELEMENTARY PROPERTIES OF DIOPHANTINE APPROXIMATION AND MINIMA OF QUADRATIC FORMS

Most of the material presented here is taken from Cusick and Flahive's book [CS].

1.1. APPROXIMATION CONSTANTS OF REAL NUMBERS. For any irrational real number \( x \in \mathbb{R} \setminus \mathbb{Q} \), one defines the approximation constant of \( x \) by

\[
\nu_{\mathbb{Q}}(x) = \liminf_{q \to +\infty} \left| x - \frac{p}{q} \right| q^2,
\]

where \( p/q \) runs through the set of \( p/q \) with increasing denominators \( q \in \mathbb{N} \). For convenience, we shall set that rational numbers have approximation constant equal to zero. The famous Dirichlet Theorem in Diophantine approximation (whose proof is just an application of the pigeonhole principle) states

**Theorem 1.** (Dirichlet) For all \( x \in \mathbb{R} \), we have

\[
\nu_{\mathbb{Q}}(x) \leq 1.
\]

We can check that there exist indeed real numbers with positive approximation constant, although \( \nu_{\mathbb{Q}}(x) = 0 \) for Lebesgue-almost every \( x \). For example, let \( \phi \) be the golden ratio \( (1 + \sqrt{5})/2 \). We claim that

\[
\nu_{\mathbb{Q}}(\phi) = \frac{1}{\sqrt{5}}.
\]

Put \( \bar{\phi} = (1 - \sqrt{5})/2 \). Since both \( \phi \) and \( \bar{\phi} \) are irrational numbers, for any rational \( p/q \) we have

\[
\left( q^2 \left| \phi - \frac{p}{q} \right| \right) \left| \phi - \frac{p}{q} \right| \neq 0.
\]
Using the relationships $\phi \bar{\phi} = -1$, $\phi + \bar{\phi} = 1$, the left-hand side can be written as $|p^2 - q^2 - qp|$, so is a positive integer. This yields

$$q^2|\phi - \frac{p}{q}| \geq \frac{1}{|\phi - \frac{p}{q}|},$$

and when considering a sequence of rational numbers $p/q$ such that the left-hand side converges to $\nu_Q(\phi)$, $p/q$ converges to $\phi$ so the right-hand side tends to $1/(\phi - \bar{\phi})$, and we obtain

$$\nu_Q(\phi) \geq \frac{1}{\sqrt{5}}.$$

In order to obtain the converse inequality, consider the Fibonacci sequence $(F_n)_{n \geq 0}$, defined recursively by $F_{n+2} = F_{n+1} + F_n$, with $F_0 = F_1 = 1$. One can check, using the fact that $F_n = (\phi^n + \bar{\phi}^{n+1})/\sqrt{5}$, that

$$\lim_{n \to +\infty} F_{n-1}^2|\phi - \frac{F_n}{F_{n-1}}| = \frac{1}{\sqrt{5}},$$

so $\nu_Q(\phi) = 1/\sqrt{5}$.

Here, the golden ratio has continued fraction expansion

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \ldots}},$$

More generally, if an irrational $x > 0$ has continued fraction expansion

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}},$$

which we shall write $x = [a_0, a_1, ..., a_n, ...]$, then $\nu_Q(x) > 0$ if and only if the sequence $(a_i)_{i \geq 0}$ is bounded. In fact, the approximation constant is given by the following formula, due to Perron:

$$\nu_Q(x) = \liminf_{n \to +\infty} ([a_{n+1}, a_{n+2}, ...] + [0, a_n, a_{n-1}, ..., a_1])^{-1}.$$

This formula can be used to recover easily the fact $\nu_Q(\phi) = 1/\sqrt{5}$.

1.2. Minima of quadratic forms. In the "hand-made" computation of $\nu_Q(\phi)$, a quadratic form $(p, q) \mapsto p^2 - q^2 - pq$, whose minimum at integer points of $\mathbb{Z}^2 - \{(0, 0)\}$ was important for the computation of the approximation constant of the golden ratio. Thus, more generally, consider a binary quadratic form $Q$ with real coefficients $a, b, c$:

$$Q(x, y) = ax^2 + bxy + cy^2,$$

which is required to be indefinite and nondegenerate, that is

$$\Delta(Q) = b^2 - 4ac > 0,$$

with $\Delta(Q)$ its discriminant. Now define its normalized infimum as

$$\mu_Q(Q) = \inf_{(x, y) \in \mathbb{Z}^2 - \{(0, 0)\}} \frac{|Q(x, y)|}{\sqrt{\Delta(Q)}}.$$

Note the following elementary properties:
• The normalization insures that for any \( \lambda \in \mathbb{R} - \{0\} \), we have
\[
\mu_Q(\lambda Q) = \mu_Q(Q).
\]

• If \( Q \) has integer coefficients and \( \Delta(Q) \) is not a square, then \( Q \) does not vanish on \( \mathbb{Z}^2 - \{(0, 0)\} \), and
\[
\mu_Q(Q) \geq \frac{1}{\sqrt{\Delta(Q)}} > 0.
\]

• Except in the case when \( a = 0 \), we can write
\[
Q(x, y) = \lambda(x - \alpha y)(x - \beta y),
\]
where \( \lambda \) is a nonzero real number, and \( \alpha \neq \beta \) are the roots of the polynomial \( aX^2 + bX + c \). So to a such a quadratic form \( Q \) one can associate a couple \((\alpha, \beta)\) of distinct elements of \( \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\} \). In the case where \( a = 0 \), one can write
\[
Q(x, y) = \lambda y(x - \alpha y),
\]
and the associated couple will be \((\alpha, \infty)\). In any case, this couple is nothing else than the slopes of the two lines which are the isotropic cone of \( Q \).

• If \( p_n, q_n \) is a sequence such that \( q_n \to +\infty \) and
\[
\lim_{n \to +\infty} \left| \alpha - \frac{p_n}{q_n} \right| q_n^2 = \nu_Q(\alpha),
\]
then,
\[
Q(p_n, q_n) = \lambda q_n^2 \left( \frac{p_n}{q_n} - \alpha \right) \left( \frac{p_n}{q_n} - \beta \right),
\]
so
\[
\lim_{n \to +\infty} |Q(p_n, q_n)| = \lambda \nu_Q(\alpha) |\alpha - \beta|.
\]
Since \( \Delta(Q) = \lambda^2 (\alpha - \beta)^2 \), this shows that
\[
\mu_Q(Q) \leq \nu_Q(\alpha),
\]
and of course the same formula holds also for \( \beta \). Conversely, it can be checked that if the infimum of \( Q \) on the lattice points is not a minimum, then \( \mu_Q(Q) \) is equal to the minimum of \( \nu_Q(\alpha) \) and \( \nu_Q(\beta) \), but in general if the infimum is attained, it can be strictly lower than the two approximation constants of the slopes of the isotropic lines of \( Q \).

There is also a formula for \( \mu_Q(Q) \), although one has to assume that \( \alpha > 1 \) and \( \beta \in (-1, 0) \). In this case, write:
\[
\alpha = [a_0, a_1, \ldots, a_n, \ldots],
\]
and
\[
\beta = -[a_{-1}, a_{-2}, \ldots, a_{-n}, \ldots].
\]
Then Perron proved that
\[
\mu_Q(Q) = \inf_{i \in \mathbb{Z}} \left( [a_i, a_{i+1}, \ldots] + [0, a_{i-1}, a_{i-2}, \ldots] \right)^{-1}.
\]
1.3. The spectra. Let us define the Lagrange spectrum $L_Q$ (sometimes also called the Hurwitz spectrum) as the set of values of $\nu_Q(x)$ for all $x \in \mathbb{R}$, and the Markoff spectrum $M_Q$ as the set of values of $\mu_Q(Q)$ for all indefinite, nondegenerate binary quadratic forms $Q$ with real coefficients. The first thing to say about these two sets is that they are not easily described! In order to have a picture in mind, we can state the following properties.

**Theorem 2.**
- (Tornheim) We have $L_Q \subset M_Q$ and (Freiman) the inclusion is strict.
- (Markoff) The upper part of both spectra is equal and discrete:
  \[ L_Q \cap \left[ \frac{1}{3}, +\infty \right) = M_Q \cap \left[ \frac{1}{3}, +\infty \right) = \{ \nu_i \}_{i \geq 0}, \]
  where $(\nu_i)_{i \geq 0}$ is a decreasing sequence having limit $1/3$. We have $\nu_0 = 1/\sqrt{5}$, $\nu_1 = 1/2\sqrt{2}$, $\nu_2 = 5/\sqrt{21}$, $\nu_3 = 13/\sqrt{1517}$,... Moreover $\nu_Q^{-1}(\nu_i)$ is a finite union of orbits for the projective action of $SL(2,\mathbb{Z})$ on $\mathbb{R} \cup \{ \infty \}$.
- (Hall) The lower part of Markoff and Lagrange spectra contains an interval, called Hall’s ray: there exists $c > 0$ such that
  \[ [0, c] \subset L_Q \cap M_Q. \]
- (Freiman) The largest $c$ such that $[0, c] \subset L_Q$ is also the largest $c$ such that $[0, c] \subset M_Q$, and is equal to
  \[ c = \left( [4, 3, 2, 1, 1, 3, 1, 3, 1, 2, 1] + [0, 4, 2, 2, 3, 1, 3, 1, 2, 1] \right)^{-1}, \]
  where the overlined part indicates the period of the asymptotically periodic continued fraction expansion.
- (Cusick) If $Q$ is the set of real quadratic numbers, and $S$ the set of indefinite, nondegenerate quadratic forms $Q$ such that the equation $Q(x, 1) = 0$ have its two solutions in $Q$, then
  \[ L_Q = \{ \nu_Q(x) : x \in Q \}, \]
  and
  \[ M_Q = \{ \mu_Q(Q) : Q \in S \}. \]
  In particular, both spectra are compact.

2. Diophantine approximation by algebraic numbers

2.1. About the arithmetic of number fields. We collect and explain here classical results on number fields, which can be found in Borevitch and Chafarevitch’s book [BS], or Samuel’s [Sa].

Let $K$ be a number field (i.e. a finite extension of $\mathbb{Q}$). Its signature is the couple $(r, s)$ of integers where $r$ is the number of distinct embeddings $K \to \mathbb{R}$, and $s$ is the number of distinct, non conjugate, nonreal, embeddings $K \to \mathbb{C}$. We have $r + 2s = [K : \mathbb{Q}]$. For example, the signature of $\mathbb{Q}(i)$ is $(0, 1)$, whereas the signature of $\mathbb{Q}(\sqrt{2})$ is $(2, 0)$, and the one of $\mathbb{Q}(\sqrt{3})$ is $(1, 1)$. There is a natural notion of integers in this setup: let $\mathcal{O}$ be the set of $x \in K$ such that there is some unitary polynomial $P$ with integer coefficients such that $P(x) = 0$. Then it is a fact that $\mathcal{O}$ is a ring, and $K = \text{Frac}(\mathcal{O})$.

For example, $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{2}]$ are the integer rings of $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ respectively. Beware that the integer ring might be a bit different from what one could think at
The first guess, for example the integer ring of $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}[(1 + \sqrt{5})/2]$, not $\mathbb{Z}[\sqrt{5}]$.

The invertible elements for the multiplicative structure are given by the following.

**Theorem 3.** (Dirichlet Units’ theorem) We have an isomorphism

$$\mathcal{O}^* = G \times \mathbb{Z}^{r+s-1},$$

where $G$ is the finite subgroup of roots of unity lying in $K$.

For example, the units in $\mathbb{Z}[i]$ are $\{1, -1, i, -i\}$, those in $\mathbb{Z}[\sqrt{2}]$ are $\{1, -1\} \times \langle 1 + \sqrt{2} \rangle$.

Up to now, everything is all right and quite similar to the case of classical integers. But a problem appears when one considers the factorization into product of irreducible elements. For example, in the ring $\mathbb{Z}[\sqrt{5}]$, we have

$$6 = 2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5}),$$

but these four elements are irreducible (in the sense that they admit no other divisors than units and themselves times units), and are not associated. In particular, the unique factorization Theorem fails here. One way to avoid this problem is to work with ideals rather than elements. Let two ideals of $\mathcal{O}$, $I$ and $J$, be equivalent if there are some $a, b \in (\mathcal{O} - \{0\})$ such that $aI = bJ$. The ideals are ”not too far” from being principal, because of the following Theorem which is (again !) due to Dirichlet:

**Theorem 4.** Let $C(K)$ be the set of ideals classes. Then $C(K)$ is a finite group under multiplication of ideals. Its cardinal $h(K)$ is called the class number of $\mathcal{O}$.

Note that the neutral element for this group is the set of principal ideals. Returning to our example, write $I = \langle 2, 1 + i\sqrt{5} \rangle$ the ideal generated by 2 and $1 + i\sqrt{5}$, $J = \langle 3, 1 + i\sqrt{5} \rangle$, $\bar{J}$ the complex conjugate of $J$. These are prime ideals. It can be checked that $2\mathcal{O} = I^2$, $3\mathcal{O} = J\bar{J}$ and $(1 + i\sqrt{5})\mathcal{O} = IJ$, $(1 - i\sqrt{5})\mathcal{O} = I\bar{J}$. Thus, one can write our decomposition of 6 the following way:

$$6\mathcal{O} = I^2J\bar{J},$$

this time in a unique fashion!

Note that the ideal class group parametrizes the partition of $\mathcal{O}^2 - \{(0,0)\}$ into $SL(2, \mathcal{O})$-orbits:

$$\mathcal{O}^2 - \{(0,0)\} = \coprod_{I \in C(\mathbb{K})} (\mathcal{O}^2)_I,$$

where $(p, q) \in (\mathcal{O}^2)_I$ iff $(p, q) \in I$.

Also, one has a partition of the field $\mathbb{K}$ in a similar way, but for convenience later it will be useful to add the infinity to the field:

$$\mathbb{K} \cup \{\infty\} = \coprod_{I \in C(\mathbb{K})} \mathbb{K}_I,$$

where $\infty \in \mathbb{K}_\mathcal{O}$, and $p/q \in \mathbb{K}_I$ iff $(p, q) \in I$.

This time, this is a partition into orbits according to the projective action of $SL(2, \mathcal{O})$

$$g.z = \frac{az + b}{cz + d}, \text{ when } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. $$
Note that in the case when $O$ is a principal domain, all this is trivial. This occurs for imaginary quadratic fields $\mathbb{Q}(i\sqrt{d})$ only when $d = 1, 2, 3, 7, 11, 19, 43, 67, 163,$ and it is a (still-opened) conjecture due to Gauss that infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$ have principal integer rings.

### 2.2. Diophantine approximation by elements of $K$

Let $E$ be the product ring $\mathbb{R}^r \times \mathbb{C}^s$. There is an embedding of $K$ into the ring $E$, given diagonally by the $r$ real embeddings of $K$ and $s$ nonconjugate complex embeddings. This embedding will be thought as an identification. Then the subgroup $O$ is a lattice in $E$. On the ring $E$, one can define a so-called "norm" (which is not a norm in the sense of norm over a vector space),

$$N : E \to \mathbb{R},$$

$$x = (x_1, \ldots, x_{r+s}) \mapsto \prod_{i=1}^r |x_i| \prod_{j=r+1}^{r+s} |x_j|^2.$$ 

Then $N$ satisfies $N(xy) = N(x)N(y)$, and moreover, if $x \in O$ then $N(x)$ is an integer, and if $x \in K$ has zero norm, then $x = 0$. Equipped with these definitions, we are now interested in how close elements of $E$ can be approached by elements of $K$. Since we have a canonical finite partition of $K$ given by the ideal class group, we might wish to view separately how elements of $E$ are approached by fractions of $K$ that are different in nature, i.e. in different sets $K_I$.

Let $I \in \mathcal{C}(K)$, we define the approximation constant of $x \in E - K$ by elements of $K_I$:

$$\nu_I(x) = \left( \liminf_{p/q \in K_I, p/q \to x} N(q)^2 N(x - p/q) \right)^{1/[K : \mathbb{Q}]}.$$ 

Note that we require that $p/q$ tends to $x$ in the topological sense. The exponent $1/[K : \mathbb{Q}]$ is present in order to have nice properties when considering field extensions, which we will not detail, and to recover the usual imaginary quadratic case. This idea of considering only part of $K$ already appeared in the work of Swan [Sw].

The global approximation constant can be defined:

$$\nu_K(x) = \inf_{I \in \mathcal{C}(K)} \nu_I(x).$$

Similar definitions appeared in a paper of Quême [Q], and also one of Burger [B]. Both articles proved a Dirichlet-type Theorem, which in our case can be written:

**Theorem 5.** (Quême [Q]) For all $x \in E - K$, we have

$$\nu_K^{[K : \mathbb{Q}]}(x) \leq \left( \frac{4}{7} \right)^{2s} \frac{[K : \mathbb{Q}]^2 |\Delta_K|}{|\Delta_K|^{2s}} |\Delta_K|,$$

where $\Delta_K$ is the discriminant of the field $K$, and $(r, s)$ its signature.

We define the Lagrange spectra $L_I$ and $L_K$ as the images of $\nu_I$ and $\nu_K$.

### 2.3. Quadratic forms

As before, we have to expect that diophantine approximation have deep links with minima of certain quadratic forms. Let $Q$ be a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$, where $a, b, c$ are elements of $E$, and such that the discriminant

$$\Delta(Q) = b^2 - 4ac,$$
have positive real components and nonzero complex components. Such quadratic forms are in spirit, our "indefinite nondegenerate" quadratic forms in this case, and that is how we shall call them. Then we define the normalized infimum over \((O^2)_I\), as follows:

\[ \mu_I(Q) = \inf_{(x,y) \in (O^2)_I} \frac{N(Q(x,y))}{N(\Delta(Q))^{1/2}}, \]

and the global, normalized, infimum:

\[ \mu_K(Q) = \inf_{I \in C(K)} \mu_I(Q). \]

This last definition also appears in the work of Ferte [F]. Needless to say, the Markoff spectra \(M_I\) and \(M_K\) are defined as images of \(\mu_I\), \(\mu_K\), for all the indefinite, nondegenerate quadratic forms.

Now, we can state some general properties of these spectra.

**Theorem 6.** [M1]  
- For all \(I \in C(K)\), we have \(L_I \subset M_I\), and both sets are bounded. We have also \(L_K \subset M_K\).
- The sets \(M_I\) are closed, and so is \(M_K\).
- We have that \(\sup L_K = \sup M_K\), and both supremum are in fact maximum.

### 2.4. The case of imaginary quadratic fields

This case deserves a separate treatment, let alone the fact it had been studied by numerous authors (A.L.Schmidt, L.Y.Vulakh, S.Hersonsky and F.Paulin, and others). The big difference with the general case is that here \(N\) is the square of the usual absolute value, and \(E = \mathbb{C}\) is an integral domain. Define the Hurwitz constant by \(C_K = \sup L_K\). For the imaginary quadratic fields \(K = \mathbb{Q}(i\sqrt{d})\), some values are known, for example:

<table>
<thead>
<tr>
<th>(d)</th>
<th>(C_K)</th>
<th>(1/\sqrt{3})</th>
<th>(1/\sqrt{2})</th>
<th>(1/\sqrt{13})</th>
<th>(1/\sqrt{11})</th>
<th>(1/\sqrt{7})</th>
<th>(1/\sqrt{5})</th>
<th>(1)</th>
</tr>
</thead>
</table>

See Hersonsky-Paulin [HP] for references about these results. It is nontrivial, but not very hard to prove that \(L_Q \subset L_K\), and also that \(M_Q \subset M_K\). Here, it is known that the Lagrange spectrum is closed:

**Theorem 7.** [M2] Assume \(K\) is a imaginary quadratic field. Let \(I \in C(K)\), and denote by \(Q\) the set of quadratic numbers over \(K\). We have \(L_I = \{\nu_I(x) : x \in Q\}\), and in particular it is a closed set.

The Markoff spectrum can also be described in a analogous way (see [M2]).

### 3. Geometrical interpretation

#### 3.1. The geodesic flow on Weyl chambers

The approximation constant of an irrational number, like the normalized infimum of a quadratic form, have nice geometrical interpretations in term of hyperbolic geometry; this generalizes to this case, as we now explain.
Let $\mathcal{H}^2$, $\mathcal{H}^3$ be the hyperbolic plane and the hyperbolic space of dimension 3 respectively, and $T^1\mathcal{H}^2$, $T^1\mathcal{H}^3$ their unitary tangent bundle. We consider the action of $A = \mathbb{R}^{r+s}$, on 

$$
\tilde{V} = (T^1\mathcal{H}^2)^r \times (T^1\mathcal{H}^3)^s,
$$

where the $i$-th component acts on the $i$-component by geodesic flow. Let $\Gamma_K$ be the usual embedding of $SL(2, \mathbb{O})$ into $SL(2, \mathbb{R})^r \times SL(2, \mathbb{C})^s$. Since this last product acts by isometries on $(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$, we can consider the quotient 

$$
V = \Gamma_K \backslash \tilde{V},
$$

and the action of $A$ is compatible with the action of $\Gamma_K$, i.e. $A$ now acts on $V$. The space $V$ is called the manifold of Weyl chambers of the Hilbert manifold over $K$, and the action of $A$ is the geodesic flow on the Weyl chambers. We will also consider the action of the semigroup $A^+ = (\mathbb{R}^+)^{r+s}$.

It is a classical result that the manifold $V$ can be decomposed into a compact part $K$, and a finite set of noncompact, connected ends $\mathcal{E}_\mathcal{I}$, parametrized by the ideal class group.

### 3.2. From $E$ to $V$.

Now we describe briefly the link between the previous objects and these new, geometric, ones. First select a basepoint $p = (p_1, \ldots, p_{r+s})$ in $(\mathcal{H}^2)^r \times (\mathcal{H}^3)^s$. To a $x = (x_1, \ldots, x_{r+s})$ in $E$, one can associate $w(x) \in \tilde{V}$ which is the projection on $V$ of the $(r+s)$-uples of vectors $(v_1, \ldots, v_{r+s})$, where $v_i$ is the vector in $T^1\mathcal{H}^2$ or $T^1\mathcal{H}^3$ of base point $p_i$, and pointing toward $x_i$, when $\partial \mathcal{H}^2$ and $\partial \mathcal{H}^3$ are identified with $\mathbb{R} \cup \{\infty\}$ and $\mathbb{C} \cup \{\infty\}$ respectively.

If $Q$ is an indefinite, nondegenerate quadratic form with coefficients in $E$, decomposing it in each component gives $Q = (Q_i)_{i=1, \ldots, r+s}$. The requirements on its discriminant $\Delta(Q)$ implies that there are exactly 2 solutions $(x_i, y_i) \in \mathbb{R} \cup \{\infty\}$ and $\mathbb{C} \cup \{\infty\}$ respectively, to the equation $Q_i(x, 1) = 0$, with still the convention that if $Q_i(0, 1) = 0$ then one of the two solutions is infinite. Now, consider the point $m_i$ on the geodesic $(x_i, y_i)$ closest to $p_i$, and a unitary vector $v_i$ based at $m_i$ tangent to $(x_i, y_i)$ (what follows will be independent of this choice). Then we define $w(Q)$ as the projection of $V$ of $(v_1, \ldots, v_{r+s}) \in V$. One can prove:

**Theorem 8.** [M1] For each $\mathcal{I} \in C(K)$, there exists a continuous and proper map $f_\mathcal{I}$ from $V$ to $\mathbb{R}$ such that, for all indefinite, nondegenerate quadratic form $Q$, we have 

$$
\mu_\mathcal{I}(Q) = \inf_{a \in A} \exp(-f_\mathcal{I}(a.w(Q))),
$$

and for all $x \in E - K$,

$$
\nu_\mathcal{I}(x) = \sup_{a \in A^+} \left( \inf_{b \in A^+} \exp(-f_\mathcal{I}((a + b).w(x))) \right).
$$

A similar statement holds for $\nu_K$, $\nu_K$, with the map $f_K = \inf_{\mathcal{I} \in C(K)} f_\mathcal{I}$. In geometrical terms, the maps $f_\mathcal{I}$ depend only on the base point of a vector, and are proportional to some Busemann functions. More precisely, $f_\mathcal{I}$ tends to $+\infty$ in the noncompact end $\mathcal{E}_\mathcal{I}$, and to $-\infty$ in the other noncompact ends $\mathcal{E}_J$ for $J \neq \mathcal{I}$. Geometrically, for the quadratic forms, this simply means that the further the $A$-orbit of $w(Q)$ goes into the noncompact end labeled by $\mathcal{I}$, the smaller the normalized infimum of $Q$ on $K_\mathcal{I}$ is.
3.3. **Indication of proofs.** Now, we are in better position to analyze these spectra, since we have gained a topology (on $V$) and continuity properties (on $f_I$).

With the help of Theorem 8, it can be shown that $M_I$ and $L_I$ can be described as the set of infima of $\exp(-f_I)$ on all $A$-invariant, closed, sets, and on all $A^+$-omega limit sets respectively. Since $A^+$-omega limit sets are $A$-invariant and closed, we have $L_I \subset M_I$. The fact that $M_I$ is a closed set is now an easy exercise of topological dynamics, and the statement that $\sup L_I = \sup M_I$ is a consequence of the fact that $f_I$ is bounded from below. A compactness argument then shows that both supremum are in fact maximum. The boundedness of $M_I$ (and so $L_I$) is a consequence of the geometrical fact there is a compact set in $V$ which intersects every $A$-orbits.

The proof of Theorem 7 relies on the fact that in the case $r+s=1$, the action of the geodesic flow is an Anosov flow (it is an Anosov action in general, but it would not be sufficient here to conclude), and is based on the Anosov closing lemma. See [M2] for details.

4. **Concluding remarks and open questions**

Among the numerous things I did not mention, there exists a beautiful link between simple geodesics on specific surface and the upper part of the (rational) Markoff spectrum, which was described by Beardon-Lehner-Sheingorn [BLS], and Haas [H]. Numerous results are know about the Markoff and Lagrange spectra of some imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$, but all these results are restricted to specific values of $d$. On the other hand, almost nothing is known when $r+s \geq 2$. Let us explain why.

Ferte proved in [F] that the Furstenberg-Margulis conjecture for the Hilbert manifold over real quadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{d})$, which says that any orbit of the diagonal group on $\Gamma \mathbb{K} \backslash \text{PSL}(2, \mathbb{R})^2$ is either closed of dense, implies the following conjecture:

$$\text{(C)} \quad \text{Every indefinite, nondegenerate quadratic form } Q \text{ with coefficients in } \mathbb{R}^2 \text{ such that } \mu_\mathbb{K}(Q) > 0 \text{ is in fact proportional to a quadratic form with coefficients in } \mathbb{K}.$$  

Despite its formal resemblance with the conjecture of Oppenheim, which was solved by Margulis [Mar], it is in some sense more linked with the still open Littlewood conjecture, which says that for every real numbers $\alpha, \beta$,

$$\inf_{n>0} n\{na\}\{n\beta\} = 0,$$

where $\{x\}$ stands for the fractional value of $x$.

This suggests a very big difference between the rational and imaginary quadratic case ($r+s=1$) on the one hand, and the higher rank situation on the other hand ($r+s \geq 2$). If $r+s \geq 2$, one can also ask if for any $\epsilon > 0$, there is finitely many $\text{SL}(2, \mathbb{O})$-orbits of quadratic forms such that $\mu_\mathbb{K}(Q) > \epsilon$. In the case of a positive answer to this question and if (C) is also true, that would say that in the higher rank situation, the Markoff spectrum (and hence the Lagrange spectrum) is not a very interesting object, since it is only a sequence tending to zero.
References


